Hamilton-Jacobi-Bellman equations for optimal control processes with convex state constraints

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Abstract

This work aims at studying some optimal control problems with convex state constraint sets. It is known that for state constrained problems, and when the state constraint set coincides with the closure of its interior, the value function satisfies a Hamilton-Jacobi equation in the *constrained viscosity sense*. This notion of solution has been introduced by H.M. Soner (1986) and provides a characterization of the value functions in many situations where an *inward pointing condition* (IPC) is satisfied. Here, we first identify a class of control problems where the constrained viscosity notion is still suitable to characterize the value function without requiring the IPC. Moreover, we generalize the notion of constrained viscosity solutions to some situations where the state constraint set has an empty interior.

Keywords: State constraint sets, Optimal control problems, Convex constraints, HJB equations, Viscosity solutions.

1. introduction.

This work is concerned with optimal control problems governed by a linear system and subject to state constraints. For a given finite horizon $T > 0$, we consider the state constrained linear system:

$$\begin{cases} \dot{y}(t) = A(t)y(t) + B(t)u(t), & \text{a.e. } t \in [0, T], \\ u(t) \in U(t) \subseteq \mathbb{R}^m, & \text{a.e. } t \in [0, T], \\ y(t) \in \mathcal{K} \subseteq \mathbb{R}^N, & \text{for any } t \in [0, T]. \end{cases} \tag{1}$$

where $A(t)$ and $B(t)$ are time-dependent matrices of dimension $N \times N$ and $N \times m$, respectively. Here $\mathcal{K}$ and $U(t)$ are nonempty closed sets. We center our attention on the Bolza problem

$$\text{Minimize } \Psi(y(T)) + \int_0^T L(t, y(t), u(t))dt \;\text{ over all } y : [0, T] \to \mathbb{R}^N \text{ satisfying (1).} \tag{2}$$

It is known that when the interior of $\mathcal{K}$ is nonempty, the value function associated with the Bolza problem is a constrained viscosity solution ([13, 9, 12, 15]) to the Hamilton-Jacobi-Bellman (HJB) equation:

$$-\partial_t \vartheta(t, x) + H(t, x, \nabla_x \vartheta(t, x)) = 0, \qquad \text{on } (0, T) \times \mathcal{K},$$

with $\vartheta(T, x) = \Psi(x)$ on $\mathcal{K}$, and where $H(t, x, p) := \sup\{-\langle p, A(t)x + B(t)u\rangle - L(t, x, u) \mid u \in U\}$.

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In this paper we show that the notion of constrained viscosity solution is a convenient notion for characterizing the value function of Bolza problems governed by state constrained linear systems, which satisfy an *interior equilibrium condition* of the type:

$$\exists u_0 : [0,T] \to U \text{ measurable such that } A(t)x_0 + B(t)u_0(t) = 0 \quad \text{for a.e. } t \in [0,T],$$

for some x_0 in the relative interior of $\mathcal{K}$. This assumption says that there is an interior point in $\mathcal{K}$ that is in addition an equilibrium for the linear system. This result provides a new qualification condition for the characterization of the value function as a unique constrained viscosity solution to the HJB equation.

As pointed out earlier, the notion of constrained viscosity solution was introduced by Soner [22, 23]. It turns out to be suitable for characterizing the value function when some controllability conditions are satisfied. In particular, the so-called *inward pointing condition* (IPC), or related outward pointing condition (OPC) condition turn out to be a sufficient requirements to guarantee Lipschitz continuity of the value function and its characterization as the constrained viscosity solution to the HJB equation, see [17, 9, 20, 12] and the references therein. From the point of view of the dynamical system, either the IPC or the OPC insure the existence of the so-called *neighboring feasible trajectories* (NFT) which makes it possible to approximate any trajectory hitting the boundary by a sequence of arcs which remain in the interior of $\mathcal{K}$; see for instance [13, 3, 4]. We refer to [6, 18, 19, 12, 10] for weaker inward pointing assumptions. Let us point out that [10] appears in the context of non-degenerate and normal forms of the Maximum principle.

When the IPC is not satisfied, the HJB equation may admit several solutions and then it needs to be completed by additional boundary conditions in order to single out the value function as the unique solution, see for example [17, 5, 16, 14, 15].

In the general case where $\mathcal{K}$ is assumed to be any closed subset of $\mathbb{R}^N$, and under some convexity assumptions on the dynamics, the value function is lower semicontinuous (l.s.c.) and it can be characterized as the smallest supersolution to the HJB equation; see [7] for more details. In [1], it has been shown that the epigraph of the value function can always be described by an auxiliary unconstrained optimal control problem for which the value function is Lipschitz continuous and characterized, with no further assumptions, as the unique viscosity solution to a HJB equation. This approach leads to a constructive way for determining the epigraph of the value function and to its numerical approximation. It can also be extended to more general situations of time-dependent state constraint sets [11].

In this work, we extend the notion of constrained viscosity solution in an appropriate way to deal with situations where $\mathcal{K}$ may have an empty interior. Furthermore, the analysis we provide is slightly more general and allows to treat cases beyond linear systems. The proofs of these results are based on the observation that the value function can be characterized as constrained viscosity solution if any admissible trajectory can be approximated by a sequence of admissible trajectories that are lying in the relative interior of the set $\mathcal{K}$ (and when the latter is dense on $\mathcal{K}$). This property, required by any IPC approach, turns out to be also satisfied for a large class of problems with some convex properties.

1.1. Notation.

Throughout this paper, $\mathbb{R}$ denotes the sets of real numbers, $|\cdot|$ is the Euclidean norm and $\langle\cdot,\cdot\rangle$ is the Euclidean inner product on $\mathbb{R}^N$, $\mathbb{B}$ the unit open ball $\{x \in \mathbb{R}^N : \ |x| < 1\}$ and $\mathbb{B}(x,r) = x + r\mathbb{B}$. For a set $S \subseteq \mathbb{R}^N$, $\text{int}(S)$ and $\overline{S}$ denote its interior and closure, respectively. Also for S convex we denote by $\text{ri}(S)$ its relative interior. The distance function to $\mathcal{S}$ is $\text{dist}_{\mathcal{S}}(x) = \inf\{|x-y| : \ y \in \mathcal{S}\}$. Let S_1 and S_2 be two compact sets. Then the Hausdorff distance is given by

$$d_H(S_1,S_2) = \max\left\{\sup_{x\in S_2} \text{dist}_{S_1}(x), \sup_{x\in S_1} \text{dist}_{S_2}(x)\right\}.$$

We adopt the convention that $d_H(\emptyset,\emptyset) = 0$ and $d_H(\emptyset,S) = +\infty$ if $S \neq \emptyset$. For a given function $v : \mathbb{R}^N \to \mathbb{R}\cup\{+\infty\}$, the effective domain of v is given by $\text{dom}(v) = \{x \in \mathbb{R}^N \mid \ v(x) \in \mathbb{R}\}$.

If Γ is a set-valued map, then $\mathrm{dom}(\Gamma)$ is the set of points for which $\Gamma(x) \neq \emptyset$. For an embedded manifold of $\mathbb{R}^N$, the tangent space to $\mathcal{M}$ at x is $\mathcal{T}_{\mathcal{M}}(x)$.

Let $\mathcal{AC}[a,b]$ stand for the set of absolutely continuous arcs $y : [a,b] \to \mathbb{R}^N$ and $\mathbb{M}_{n\times m}(\mathbb{R})$ the space of matrices of dimension $n \times m$. For any matrix-valued map $t \mapsto A(t)$ defined on $[0,T]$ we set

$$\|A\|_\infty := \sup_{t\in[0,T]} \max_{i=1,\dots,n} \max_{j=1,\dots,m} |A_{i,j}(t)|.$$

2. Preliminary results

2.1. General controlled systems.

Consider a differential inclusion in $\mathbb{R}^N$ for a given initial time $\tau \in [0,T]$:

$$\begin{cases} \dot{y}(t) \in F(t, y(t)), & \text{for a.e. } t \in (\tau, T) \\ y(\tau) = x, \end{cases} \tag{3}$$

where $F : [0,T] \times \mathbb{R}^N \rightrightarrows \mathbb{R}^N$ is a set-valued map satisfying:

$$\begin{cases} i) & F \text{ is a continuous multifunction, with nonempty compact and convex images.} \\ ii) & \text{There is } L_F > 0 \text{ such that} \\ & \qquad d_H(F(t,x), F(t,y)) \leq L_F|x-y|, \ \forall x, y \in \mathbb{R}^N, \ \forall t \in [0,T]. \\ iii) & \text{There is } c_F > 0 \text{ such that} \\ & \qquad \max\{|v| \mid \ v \in F(t,x)\} \leq c_F(1+|x|), \ \forall x \in \mathbb{R}^N, \ \forall t \in [0,T]. \end{cases} \tag{H_F}$$

Remark 2.1. *Let us point out that the linear system (1) can be reformulated as a differential inclusion having the form of (3). Indeed, it is enough to set*

$$F(t,x) := \{A(t)x + B(t)u \mid \ u \in U(t)\}, \quad \forall x \in \mathbb{R}^N, \ \forall t \in [0,T].$$

Furthermore, (H_F) is satisfied whenever the data of the linear systems satisfy

$$\begin{cases} i) & A : [0,T] \longmapsto \mathbb{M}_{N\times N}(\mathbb{R}) \text{ and } B : [0,T] \longmapsto \mathbb{M}_{N\times m}(\mathbb{R}) \text{ are continuous mappings.} \\ ii) & U \text{ is a continuous multifunction, with nonempty compact and convex images.} \end{cases} \tag{H_F'}$$

Here we must take $L_F := \|A\|_\infty$ *and* $c_F := \max\{\|A\|_\infty, \|B\|_\infty \max\{|u| \mid \ u \in U(t), t \in [0,T]\}\}$

Remark 2.2. *Let us point out that we have imposed F to have nonempty images on the whole space only for sake of simplicity. However, for our purposes it would have been enough for F to be defined only on $\mathcal{K}$. Indeed, since later we are assuming that $\mathcal{K}$ is convex (see (H_0)), the projection over $\mathcal{K}$, denoted $\mathrm{proj}_{\mathcal{K}}(\cdot)$, is a non-expansive map and so $x \mapsto F(t, \mathrm{proj}_{\mathcal{K}}(x))$ is a suitable extension for F to the whole space, and moreover (H_F) holds as well for this extension.*

Under assumption (H_F), the dynamical system (3) admits a solution that belongs to $\mathcal{AC}[\tau, T]$. To emphasis the reliance on the initial data, we reserve the notation $y_{\tau,x}(\cdot)$ for such a trajectory.

Remark 2.3. *By Gronwall's Lemma ([24, Lemma 2.4.4]) and (H_F), each solution to (3) associated with $(\tau, x) \in [0,T] \times \mathbb{R}^N$ satisfies:*

$$|y_{\tau,x}(t) - x| \leq (1+|x|)\left(e^{c_F(t-\tau)} - 1\right), \quad \forall t \in [\tau, T]. \tag{4}$$

In many practical applications, the trajectories are required to stay in a given closed set $\mathcal{K}$:

$$y_{\tau,x}(t) \in \mathcal{K}, \qquad \forall t \in [\tau, T].$$

In the sequel, for any time interval $[a,b] \subset [0,T]$ and any initial position $x \in \mathcal{K}$, the set of *admissible trajectories* on $[a,b]$ is defined by

$$\mathbb{S}_{[a,b]}(x) := \{y \in \mathcal{AC}[a,b] \mid y(\cdot) \text{ satisfies (3) and } y(s) \in \mathcal{K},\ \forall s \in [a,b]\},$$

Under (H_F), for any $x \in \mathcal{K}$ and any $\tau \in [0,T]$, the set $\mathbb{S}_{[\tau,T]}(x)$ is a compact subset of $\mathcal{AC}[\tau,T]$ endowed with the appropriate topology. For any set $\Omega \subset \mathcal{K}$, we shall also use the notation $\mathbb{S}^{\Omega}_{[\tau,T]}(x)$ for the collection of trajectories:

$$\mathbb{S}^{\Omega}_{[a,b]}(x) := \left\{y \in \mathbb{S}_{[a,b]} \mid y(t) \in \Omega,\ \forall t \in [a,b]\right\},$$

2.2. Optimal control problems

Consider a general optimal control problem in Bolza form:

$$(\tau, x) \mapsto \inf\left\{\Psi(y(T)) + \int_{\tau}^{T} \ell(t, y(t), \dot{y}(t))\, dt \;\middle|\; y \in \mathbb{S}_{[\tau,T]}(x)\right\}, \tag{5}$$

with the convention that $\inf \emptyset = +\infty$, and with $\Psi : \mathbb{R}^N \to \mathbb{R}$ (final cost) and $\ell : [0,T] \times \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ (distributed cost) being functions satisfying the following:

$$\Psi : \mathbb{R}^N \to \mathbb{R} \text{ is locally Lipschitz continuous on } \mathcal{K}. \tag{H_Ψ}$$

$$\begin{cases} i) \;\; \ell : [0,T] \times \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R} \text{ is continuous.} \\ ii) \text{ There is } L_\ell > 0 \text{ such that} \\ \qquad |\ell(t,x,q) - \ell(t,y,p)| \le L_\ell(|x-y| + |q-p|),\ \forall x,y,p,q \in \mathbb{R}^N,\ \forall t \in [0,T]. \end{cases} \tag{H_L}$$

Remark 2.4. *Note that if the optimal control problem at hand is governed by a linear system as in Remark 2.1 and the distributed cost is $L(t,x,u)$, that is, it depends explicit on the control, we can cover this case by setting*

$$\ell(t,x,q) := \min_{u \in U(t)} \{L(t,x,u) \mid \; q = A(t)x + B(t)u\}$$

By using appropriate measurable selection theorems, it can be proved that the Bolza problems (2) and (5) are equivalent.

The value function of the Bolza problem is denoted by $\vartheta(\tau,x)$ and is the mapping that associates any initial time τ and any initial position x with the optimal cost of (5). Hence, this map is defined on $[0,T] \times \mathcal{K}$ and may take unbounded values, that is, $\vartheta : [0,T] \times \mathcal{K} \to \mathbb{R} \cup \{+\infty\}$.

For any $(t,x) \in [0,T] \times \mathcal{K}$, by compactness of the set $\mathbb{S}_{[\tau,T]}(x)$ and taking into account assumption (H_Ψ), we can easily show that when $\vartheta(\tau,x) < +\infty$, the control problem (5) admits a minimizer. Moreover, the function ϑ is l.s.c. on $[0,T] \times \mathcal{K}$.

2.3. Optimality principles

A fundamental tool in the analysis of the value function is the well known Dynamic Programming Principle, which may be stated as follows:

$$\vartheta(\tau,x) = \inf\left\{\vartheta(s, y(s)) + \int_t^s \ell(t, y(t), \dot{y}(t))\, dt \mid \quad y \in \mathbb{S}_{[\tau,s]}(x)\right\}, \tag{6}$$

for every $x \in \mathcal{K}$ and $s \in [\tau, T]$. The two important aspects of the Dynamic Programming Principle are that the value function is constant along optimal trajectories and is not decreasing along non-optimal ones. This remark motivates the next definition.

Definition 2.1. *Let Ω be a subset of $\mathcal{K}$, and let $\mathfrak{S} = \{\mathfrak{S}_\tau\}_{\tau\in[0,T]}$ be a collection of set-valued maps defined on Ω, where $\mathfrak{S}_\tau : \Omega \rightrightarrows \mathcal{AC}[\tau, T]$ for each $\tau \in [0, T]$. A function $\varphi : [0, T] \times \mathcal{K} \to \mathbb{R} \cup \{+\infty\}$ is said:*

i) weakly decreasing along trajectories of $\mathfrak{S}$ *provided for all $(\tau, x) \in dom(\varphi)$ with $x \in \Omega$ there exists a curve $y \in \mathfrak{S}_\tau(x)$ so that $\varphi(s, y(s)) + \displaystyle\int_\tau^s \ell(t, y(t), \dot{y}(t))\, dt \leq \varphi(\tau, x)$ for all $s \in [\tau, T]$.*

ii) strongly increasing along trajectories of $\mathfrak{S}$ *if for each $(\tau, x) \in [0, T] \times \Omega$ and each $y \in \mathfrak{S}_\tau(x)$ satisfies $\varphi(s, y(s)) + \displaystyle\int_\tau^s \ell(t, y(t), \dot{y}(t))\, dt \geq \varphi(\tau, x)$ for all $s \in [\tau, T]$.*

The notions introduced in Definition 2.1 are closely related to the classical optimality principles known in the literature. However, standard optimality principle are usually stated for the family of all admissible trajectories, while here we state it for a sub-family of admissible trajectories. We will see that the choice of the family $\mathfrak{S}$ with respect to which the optimality is met will determine the HJB equation that characterizes the value function. For this reason we introduce the following concept.

Definition 2.2. *Let Ω be a subset of $\mathcal{K}$, and let $\mathfrak{S} = \{\mathfrak{S}_\tau\}_{\tau\in[0,T]}$ be a collection of set-valued maps defined on Ω, where $\mathfrak{S}_\tau : \Omega \rightrightarrows \mathcal{AC}[\tau, T]$ for each $\tau \in [0, T]$. We say that $\mathfrak{S}$ is a* suboptimal collection of trajectories (for the control problem (5)) *if the following conditions are verified:*

1. $\mathfrak{S}_\tau(x) \subseteq \mathbb{S}_{[\tau,T]}(x)$ *for each $x \in \Omega$ and for every $\tau \in [0, T]$.*
2. *For any $(\tau, x) \in dom(\vartheta)$ and $\varepsilon > 0$ we can find $x_\varepsilon \in \mathbb{B}(x, \varepsilon) \cap \Omega$, $\tau_\varepsilon \in (\tau - \varepsilon, \tau + \varepsilon) \cap [0, T]$ and $y_\varepsilon \in \mathfrak{S}_{\tau_\varepsilon}(x_\varepsilon)$ so that $\Psi(y_\varepsilon(T)) + \displaystyle\int_{\tau_\varepsilon}^T \ell(t, y_\varepsilon(t), \dot{y}_\varepsilon(t))\, dt \leq \vartheta(\tau, x) + \varepsilon$.*

Remark 2.5. *If $\mathfrak{S}$ is a suboptimal collection of trajectories on $\Omega \subseteq \mathcal{K}$, then we must have $dom(\vartheta) \subseteq [0, T] \times \overline{\Omega}$. This is not restrictive for our purposes because in general we take Ω to be dense on $\mathcal{K}$.*

It is clear that the collection $\mathfrak{S} = \{\mathbb{S}_{[\tau,T]}(x)\}_{\tau\in[0,T]}$, containing all trajectories of the control system, is a suboptimal collection of trajectories. We can also go to the other extreme and set $\mathfrak{S}$ as the set of minimizers. Clearly, in one case there are too many trajectories -some of them with a wild structure difficult to handle- and in the other case there are too few which makes the HJB approach useless. Hence, it will be convenient to work with a suboptimal collection of trajectories that lies in between the situations described earlier, meaning that it is large enough so that it is not too complicated to be constructed, and sufficiently small so that it contains only somehow *well-behaved* trajectories.

In any case, the preceding notion will prove its utility in the next statement.

Lemma 2.1. *Let $\varphi : [0, T] \times \mathcal{K} \to \mathbb{R} \cup \{+\infty\}$ be an l.s.c. function satisfying $\varphi(T, x) = \Psi(x)$ for all $x \in \mathcal{K}$. Consider also $\mathfrak{S}$ a suboptimal collection of trajectories defined on $\Omega \subset \mathcal{K}$.*

1. *If φ is weakly decreasing along trajectories of $\mathfrak{S}$, then $\vartheta(\tau, x) \leq \varphi(\tau, x)$ on $[0, T] \times \Omega$.*
2. *If φ is strongly increasing along trajectories of $\mathfrak{S}$, then $\vartheta(\tau, x) \geq \varphi(\tau, x)$ on $[0, T] \times \mathcal{K}$.*

Proof. 1. The case $\varphi(\tau, x) = +\infty$ is trivial, so assume $\varphi(\tau, x) < \infty$. By definition, there exists a trajectory $y \in \mathfrak{S}_\tau(x) \subseteq \mathbb{S}_{[\tau,T]}(x)$ such that

$$\varphi(\tau, x) \geq \varphi(T, y(T)) + \int_\tau^T \ell(t, y(t), \dot{y}(t))\, dt = \Psi(y(T)) + \int_\tau^T \ell(t, y(t), \dot{y}(t))\, dt \geq \vartheta(\tau, x),$$

the last equality being a consequence of the definition of the value function.

2. Let $(\tau, x) \in [0,T] \times \mathcal{K}$, if $\mathbb{S}_{[\tau,T]}(x) = \emptyset$ then $\vartheta(\tau, x) = +\infty$ and the conclusion follows easily. Otherwise, take $\{\varepsilon_n\} \subseteq (0,1)$ so that $\varepsilon_n \to 0$. By Definition 2.2, there exist sequences $(x_n)_n \subset \Omega$, $(\tau_n)_n \subset [0,T]$ and $(y_n)_n \subset \mathbb{S}_{[\tau_n,T]}(x_n)$ with $x_n \to x$, $\tau_n \to \tau$ and

$$\Psi(y_n(T)) + \int_{\tau_n}^{T} \ell(t, y_n(t), \dot{y}_n(t))\, dt \leq \vartheta(\tau, x) + \varepsilon_n.$$

The strongly increasing property yields

$$\begin{aligned} \varphi(\tau_n, x_n) \leq \varphi(T, y_n(T)) + \int_{\tau_n}^{T} \ell(t, y_n(t), \dot{y}_n(t))\, dt &= \Psi(y_n(T)) + \int_{\tau_n}^{T} \ell(t, y_n(t), \dot{y}_n(t))\, dt \\ &\leq \vartheta(\tau, x) + \varepsilon_n, \quad \forall n \in \mathbb{N}. \end{aligned}$$

Since φ is l.s.c., by taking the limit inferior on the righthand side of the foregoing inequality, we conclude the proof.

□

The preceding lemma asserts that the value function is the unique function that is weakly decreasing and strongly increasing along trajectories of $\mathfrak{S} = \{\mathbb{S}_{[\tau,T]}(x)\}_{\tau \in [0,T]}$ at the same time. But, this result is even stronger because it indicates that we do not need all the trajectories of the control system to characterize the value function, just some of them that are almost-optimal. This is of particular utility when dealing with subsolution to the HJB equation.

2.4. Case when the NFT property is satisfied

The *neighboring feasible trajectory* property states that any admissible trajectory can be approximated by a sequence of trajectories lying in the interior of the state constraint set. Consequently, when the NFT property is satisfied, a suboptimal collection for the control problem is provided by the set of trajectories that remain on $\text{int}(\mathcal{K})$. In this case, the subsolution principle can be characterized using exclusively information on the interior of $\mathcal{K}$. An NFT theorem was first derived in [22], under an assumption of *inward pointing condition*, with a view to establishing the continuity of the value function. With the NFT property at hand, the notion of sub-solution for the HJB equation associated to the value function involves only an inequality on $\text{int}(\mathcal{K})$ (no information is required on the boundary of $\mathcal{K}$).

3. Convex problems

In this section we construct a particular suboptimal collection of trajectories, well suited for treating Bolza problems governed by linear systems with a convex state constraint set. We assume that $\mathcal{K}$ is a convex set of $\mathbb{R}^N$ whose interior may be empty, that is,

$$\mathcal{K} \text{ is a nonempty, closed and } \textit{convex} \text{ subset of } \mathbb{R}^N. \qquad (H_0)$$

We recall that $\text{ri}(\mathcal{K})$, the relative interior of $\mathcal{K}$, is always a non-empty set. Furthermore, we have that this set is always an embedded manifold of $\mathbb{R}^N$.

Proposition 3.1. *Suppose that (H_0) holds, then ri($\mathcal{K}$) is a $\mathcal{C}^\infty$-embedded manifold of $\mathbb{R}^N$.*

Proof. Let $\text{aff}(\mathcal{K})$ stand for the affine hull of $\mathcal{K}$, then there exists $v_1, \ldots, v_p \in \mathbb{R}^N$ linearly independent that are orthogonal to $\text{aff}(\mathcal{K}) - x_0$ for any $x_0 \in \mathcal{K}$. We set $h_i(x) = \langle v_i, x - x_0 \rangle$ for any $x \in \mathbb{R}^N$ then the function $h(x) = (h_1(x), \ldots, h_p(x))$ is a $\mathcal{C}^\infty$ submersion on $\mathbb{R}^N$. Moreover, for every $x \in \text{ri}(\mathcal{K})$ there exists an open set $\mathcal{O} \subseteq \mathbb{R}^N$ containing x such that

$$\text{ri}(\mathcal{K}) \cap \mathcal{O} = \text{aff}\, \mathcal{K} \cap \mathcal{O} = \{x \in \mathbb{R}^N \mid \ h(x) = 0\} \cap \mathcal{O}.$$

This proves that $\text{ri}(\mathcal{K})$ is a $\mathcal{C}^\infty$-embedded manifold of $\mathbb{R}^N$. □

The advantage of convex sets is that the Accessibility Lemma (quoted below) provides a simple way to approximate any feasible trajectory by trajectories lying in the relative interior of $\mathcal{K}$ as long as the dynamics has a particular structure.

Proposition 3.2 ([21, Theorem 6.1])**.** *For any convex subset $\mathcal{S} \subseteq X$, $x \in ri(\mathcal{S})$ and $\tilde{x} \in \overline{\mathcal{S}}$ we have*

$$\lambda x + (1-\lambda)\tilde{x} \in ri(\mathcal{S}), \quad \forall \lambda \in (0,1].$$

In particular, $\mathrm{ri}(\overline{\mathcal{S}}) = \mathrm{ri}(\mathcal{S})$ and $\overline{\mathcal{S}} = \overline{\mathrm{ri}(\mathcal{S})}$.

Standing Assumptions: In the rest of this paper, and unless otherwise stated, we always assume that (H_0), (H_F), (H_L) and (H_Ψ) hold.

Now, assume that the dynamics fulfills the ensuing qualification requirements:

$$\begin{cases} i) \quad F(t,\cdot) \text{ has convex graph for any } t \in [0,T], \text{ that is, } \forall x,y \in \mathcal{K}, \ \forall t \in [0,T] \\ \qquad \lambda v + (1-\lambda)w \in F(t, \lambda x + (1-\lambda)y), \ \forall v \in F(t,x), \ \forall w \in F(t,y). \\ ii) \ \exists x_0 \in \mathrm{ri}(\mathcal{K}), \exists y_0 \in \mathbb{S}_{[0,T]}(x_0) \text{ s.t. } y_0(t) \in \mathrm{ri}(\mathcal{K}) \ \forall t \in [0,T]. \end{cases} \tag{H_Q}$$

Remark 3.1. *Assumption (H_Q)-(i) is satisfied when considering linear systems as in Remark 2.1; note that the fact that $\mathcal{K}$ and $U(t)$ are convex sets is crucial for this.*

Moreover, the interior trajectory condition (H_Q)-(ii) can be satisfied if for instance F has a equilibrium point in the relative interior of the state constraint set, which means that

$$0 \in F(t,x_0) \quad \text{for some } x_0 \in ri(\mathcal{K}).$$

Note that in the case of linear systems as in Remark 2.1, this equilibrium point in the relative interior means that there is an admissible control $u_0 : [0,T] \to \mathbb{R}^m$ such that

$$u_0(t) \in U(t) \quad \text{and} \quad A(t)x_0 + B(t)u_0(t) = 0, \ \text{for a.e. } t \in [0,T].$$

Under assumption (H_Q), we shall prove that any admissible trajectory may be approximated by a sequence of trajectories lying in $\mathrm{ri}(\mathcal{K})$. This claim is stated in the next proposition.

Proposition 3.3. *Assume (H_Q) holds. Then, for every $(\tau,x) \in dom(\vartheta)$ and $\varepsilon > 0$ there exists $x_\varepsilon \in ri(\mathcal{K}) \cap \mathbb{B}(x,\varepsilon)$ and $y_\varepsilon \in \mathbb{S}_{[\tau,T]}(x_\varepsilon)$ such that*

$$\vartheta(\tau,x) + \varepsilon \geq \Psi(y_\varepsilon(T)) + \int_\tau^T \ell(t, y_\varepsilon(t), \dot{y}_\varepsilon(t))\, dt \quad \text{and} \quad y_\varepsilon(s) \in ri(\mathcal{K}), \ \forall s \in [\tau,T].$$

Proof. Let $(\tau,x) \in \mathrm{dom}(\vartheta)$, let y_0 be given by (H_Q)-(ii) and let $y^* \in \mathbb{S}_{[\tau,T]}(x)$ be a $\frac{\varepsilon}{2}$-suboptimal trajectory. Let

$$r := \max\left\{\|y_0\|_\infty, (1+|x|)e^{c_F T}\right\}$$

By definition and Gronwall's Lemma, it is clear that $y_0(t), y^*(t) \in \mathbb{B}(0,r)$ for any $t \in [\tau,T]$. Set $L_\Psi > 0$ for a Lipschitz constant for Ψ on $\mathbb{B}(0,r)$. Recall that $L_\ell > 0$ is a Lipschitz constant for $\ell(t,\cdot)$, and consider $\tilde{\varepsilon} \in (0,\varepsilon] \cap (0,1)$ satisfying

$$2r\tilde{\varepsilon}L_\Psi \leq \frac{\varepsilon}{4}, \qquad \text{and} \quad \tilde{\varepsilon}(2r + 2c_F(1+r))L_\ell T \leq \frac{\varepsilon}{4}.$$

Now, if we define on $[\tau,T]$ the function: $y_\varepsilon(t) = \tilde{\varepsilon}y_0(t) + (1-\tilde{\varepsilon})y^*(t)$, we get an absolutely continuous arc that remains in $\mathrm{ri}(\mathcal{K})$ (because of the Accessibility Lemma). Moreover,

$$\dot{y}_\varepsilon(t) = \tilde{\varepsilon}\dot{y}_0(t) + (1-\tilde{\varepsilon})\dot{y}^*(t) \in \tilde{\varepsilon}F(t,y_0(t)) + (1-\tilde{\varepsilon})F(t,y^*(t)).$$

By definition of F and convexity of the graph of $F(t,\cdot)$, $\tilde{\varepsilon}F(t,y_0(t))+(1-\tilde{\varepsilon})F(t,y^*(t)) \subseteq F(t,y_\varepsilon(t))$ a.e. on $[\tau,T]$, and consequently, y_ε is an admissible trajectory starting from the position $x_\varepsilon := \tilde{\varepsilon}y_0(\tau)+(1-\tilde{\varepsilon})y^*(\tau)$; i.e. $y_\varepsilon \in \mathbb{S}_{[\tau,T]}(x_\varepsilon)$. Furthermore, $y_\varepsilon(t) \in \mathbb{B}(0,r)$ for every $t \in [\tau,T]$, thus by using the definition of y_ε and (4), we deduce that:

$$\max_{t\in[\tau,T]} |y^*(t)-y_\varepsilon(t)| \leq 2r\tilde{\varepsilon}, \quad \text{and} \quad \text{ess}-\max_{t\in[\tau,T]} |\dot{y}^*(t)-\dot{y}_\varepsilon(t)| \leq 2\tilde{\varepsilon}c_F(1+r).$$

From here we conclude that

$$|\Psi(y^*(T))-\Psi(y_\varepsilon(T))| \leq 2r\tilde{\varepsilon}L_\Psi$$

and

$$\int_\tau^T |\ell(t,y^*(t),\dot{y}^*(t))-\ell(t,y_\varepsilon(t),\dot{y}_\varepsilon(t))|\,dt \leq \tilde{\varepsilon}L_\ell(T-\tau)(2r+2c_F(1+r)).$$

Note that, by the way how we have chosen $\tilde{\varepsilon}$, the right-hand sides in the last two inequalities are smaller than $\frac{\varepsilon}{4}$. Hence, gathering all the information, we get the desired result setting $x_\varepsilon = y_\varepsilon(\tau)$ and noticing that

$$\frac{\varepsilon}{2}+\vartheta(\tau,x) \geq \Psi(y^*(T))+\int_\tau^T \ell(t,y^*(t),\dot{y}^*(t))\,dt \geq \Psi(y_\varepsilon(T))+\int_\tau^T \ell(t,y_\varepsilon(t),\dot{y}_\varepsilon(t))dt-\frac{\varepsilon}{2}.$$

□

Remark 3.2. *From now on, we denote by $\mathfrak{S}_{ri} = \{\mathfrak{S}_{ri,\tau}\}_{[0,T]}$ the collection of set-valued maps defined on ri($\mathcal{K}$) by:*

$$\mathfrak{S}_{ri,\tau}(x) := \{y \in \mathbb{S}_{[\tau,T]} \mid \; y(t) \in ri(\mathcal{K}),\ \forall t \in [\tau,T]\}.$$

By proposition 3.3 it follows that that $\mathfrak{S}_{ri}$ is a suboptimal collection for the control problem (5). We shall use this information to characterize the value function ϑ as the unique solution -in a weak sense- of a suitable HJB equation.

3.1. Tangent dynamics and Hamiltonian

Let us introduce the set of tangent maps to the relative interior of $\mathcal{K}$

$$F^{\#}(t,x) = F(t,x) \cap \mathcal{T}_{\text{ri}(\mathcal{K})}(x), \quad \forall x \in \text{ri}(\mathcal{K}).$$

Recall that the affine hull of $\mathcal{K}$ can be identified with $x_0 + \ker P$, where P is a surjective matrix. So, $\mathcal{T}_{\text{ri}(\mathcal{K})}(x) = \ker P$ for every $x \in \text{ri}(\mathcal{K})$ and thereby

$$F^{\#}(t,x) = F(t,x) \bigcap \ker P, \quad \forall x \in \text{ri}(\mathcal{K}).$$

One assumption that plays a fundamental role on the exposition is that the tangent dynamics is a regular one in the following sense:

$$F^{\#}(t,.) \text{ is locally Lipschitz on ri}(\mathcal{K}) \text{ w.r.t. the Hausdorff distance.} \tag{H_1}$$

In the case when $\text{int}(\mathcal{K}) \neq \emptyset$, one can take $P = 0$ and (H_1) holds trivially with $F^{\#}(t,\cdot) \equiv F(t,\cdot)$ along $\text{ri}(\mathcal{K}) = \text{int}(\mathcal{K})$. Accordingly, the tangential Hamiltonian to ri($\mathcal{K}$) is defined by:

$$H_{\text{ri}}(t,x,\zeta) = \max_{q\in F^{\#}(t,x)} \big\{ -\langle q,\zeta\rangle - \ell(t,x,q)\big\}, \quad \forall (t,x,\zeta) \in [0,T]\times\mathbb{R}^N\times\mathbb{R}^N.$$

This Hamiltonian is continuous under assumptions (H_F), (H_1) and (H_L).

Proposition 3.4. *Suppose* (H_1) *holds Then for any* $t \in [0,T]$ *fixed,* $H_{ri}(t,\cdot)$ *is locally Lipschitz continuous on* $ri(\mathcal{K}) \times \mathbb{R}^N$*.*

Proof. Let $r > 0$ be fixed and let $L_F^r > 0$ be a Lipschitz constant of $F^\#(t,\cdot)$ on $\mathbb{B}(0,r)$ given by (H_1). Let L_ℓ be the Lipschitz constant of $\ell(t,\cdot)$ given by (H_L).

Fix $\zeta \in \mathbb{R}^N$ and take $x, y \in \mathbb{B}(0,r)$. Since $F(t,x)$ is compact, there exists $q_x \in F^\#(t,x)$ so that $H_{\rm ri}(x,\zeta) = -\langle \zeta, q_x\rangle - \ell(t,x,q_x)$. On the other hand, thanks to (H_1), there exists $q_y \in F^\#(t,y)$ for which $|q_x - q_y| \leq L_F^r|x-y|$. Gathering all the information we get that

$$\begin{aligned} H_{\rm ri}(t,x,\zeta) - H_{\rm ri}(t,y,\zeta) &\leq |\zeta||q_y - q_x| + |\ell(t,y,q_y) - \ell(t,x,q_x)| \\ &\leq (|\zeta| + L_\ell)(|x-y| + |q_x - q_y|) \\ &\leq (|\zeta| + L_\ell)(1 + L_F^r)|x-y|. \end{aligned}$$

Since, x and y are arbitrary, we can interchange their roles and get that $x \mapsto H_{\rm ri}(t,x,\zeta)$ is Lipschitz continuous for any $\zeta \in \mathbb{R}^N$ on $\mathbb{B}(0,r)$. Besides, using (H_F) , (H_L) and a similar argument as above, we get

$$|H_{\rm ri}(t,x,\zeta) - H_{\rm ri}(t,x,\xi)| \leq c_F(1+|x|)|\zeta - \xi|, \quad \forall x \in {\rm ri}(\mathcal{K}),\ \forall \zeta,\xi \in \mathbb{R}^N.$$

Finally, combining both partial Lipschitz estimations we get the result. □

Additionally, in agreement with what said earlier, if ${\rm int}(\mathcal{K}) \neq \emptyset$, then $H_{\rm ri}$ coincides with the usual Hamiltonian H. In particular, (H_L) can be weakened by requiring ℓ to be continuous and locally Lipschitz only with respect to the state.

In our setting, we are interested in l.s.c. value functions. When dealing with a Bolza problem, it is usual to introduce an augmented dynamics. Let $G : [0,T] \times \mathbb{R}^N \rightrightarrows \mathbb{R}^N \times \mathbb{R}$ defined by:

$$G(t,x) = \left\{ \begin{pmatrix} q \\ \ell(t,x,q) + r \end{pmatrix} \;\middle|\; \begin{matrix} q \in F(t,x), \\ 0 \leq r \leq \beta(t,x,q) \end{matrix} \right\}, \quad \forall (t,x) \in [0,T] \times \mathbb{R}^N,$$

where,

$$\beta(t,x,q) := \max_{v \in F(t,x)} \ell(t,x,v) - \ell(t,x,q) \qquad \forall (x,q) \in \mathbb{R}^N \times \mathbb{R}^N.$$

It is not difficult to see that by (H_L) and (H_F) this set-valued map has compact and non-empty images on a neighbourhood of $[0,T] \times \mathcal{K}$. Moreover, throughout the paper, we also suppose that

$$G(\cdot) \text{ has convex images on a neighborhood of } [0,T] \times \mathcal{K}. \tag{H_2}$$

Remark 3.3. *Let us point out that in the case of linear systems, that is,*

$$F(t,x) = \{A(t)x + B(t)u \mid \ u \in U(t)\}$$

the hypothesis (H_2) *is satisfied whenever the following holds*

$$q \mapsto \ell(t,x,q) \text{ is convex.} \tag{H_2'}$$

Definition 3.1. *Let* $\varphi : \mathbb{R}^d \to \mathbb{R} \cup \{+\infty\}$ *(for* $d \geq 1$*) be l.s.c.. A vector* $\zeta \in \mathbb{R}^d$ *is called a viscosity sub-gradient of* φ *at* $x \in dom(\varphi)$ *if and only if there exists a continuously differentiable function* g *so that*

$$\nabla g(x) = \zeta \text{ and } \varphi - g \text{ attains a local minimum at } x.$$

Furthermore, ζ *is called a proximal sub-gradient of* φ *at* x *if for some* $\eta > 0$,

$$g(y) := \langle \zeta, y - x\rangle - \eta|y-x|^2.$$

The sets of all viscosity and proximal sub-gradients at x *are denoted by* $\partial_V\varphi(x)$ *and* $\partial_P\varphi(x)$*, respectively.*

Since $\mathrm{ri}(\mathcal{K})$ is a manifold and H_{ri} is locally Lipschitz continuous, the classical viscosity theory and a routine adaptation of the arguments given in [8, Theorem 4.6.1 and 4.6.3] or [16, Proposition 5.1 and 5.2, Lemma 5.1] lead to the following result (see also [2, Theorem 3.22] or [15, Proposition 3.2 and 3.5] for similar results).

Lemma 3.1. *Suppose (H_1) and (H_2) hold. Consider an l.s.c. function $\omega : [0,T] \times \mathcal{K} \to \mathbb{R} \cup \{+\infty\}$ verifying $\omega(T,x) = \Psi(x)$ for all $x \in \mathcal{K}$. Then the following assertions hold:*

(i) ω is weakly decreasing along trajectories of $\mathbb{S}_{[0,T]}(x)$ if and only if

$$-\theta + H(t,x,\zeta) \geq 0 \quad \text{for all } (\theta,\zeta) \in \partial_V \omega(t,x), \quad \forall (t,x) \in [0,T) \times \mathcal{K}.$$

(ii) ω is strongly increasing along trajectories of $\mathbb{S}^{ri(\mathcal{K})}_{[0,T]}(x)$ if and only if

$$-\theta + H_{ri}(t,x,\zeta) \leq 0 \quad \text{for all } (\theta,\zeta) \in \partial_V \omega(t,x), \quad \forall (t,x) \in (0,T] \times ri(\mathcal{K}).$$

We are now in position to provide and prove the main result of this section.

Theorem 3.1. *Suppose that (H_1), (H_2) and (H_Q). Then the value function ϑ is the only l.s.c. function which is $+\infty$ on $\mathbb{R}^N \setminus \mathcal{K}$ and that satisfies $\vartheta(T,x) = \Psi(x)$ for any $x \in \mathcal{K}$ and that verifies:*

$$-\theta + H(t,x,\zeta) \geq 0 \qquad \forall t \in [0,T),\ \forall x \in \mathcal{K},\ \forall (\theta,\zeta) \in \partial_V \vartheta(t,x), \tag{7}$$

$$-\theta + H_{ri}(t,x,\zeta) \leq 0 \qquad \forall t \in (0,T],\ \forall x \in ri(\mathcal{K}),\ \forall (\theta,\zeta) \in \partial_V \vartheta(t,x), \tag{8}$$

Proof. By using similar arguments as [16, Proposition 3.1 and Proposition 3.2], one can easily show that $\vartheta(\cdot)$ is l.s.c.. The rest of the proof is divided into several steps.

1. We prove that $\vartheta(\cdot)$ verifies (7)-(8).
   Let $(t,x) \in \mathrm{dom}(\partial_P \vartheta) \cap ((0,T] \times \mathrm{ri}(\mathcal{K}))$. We start by showing the following inequality:

   $$-\theta - \langle \zeta, q \rangle - \ell(t,x,q) \leq 0, \qquad \forall \zeta \in \partial_P \vartheta(t,x), \quad \forall q \in F^{\#}(t,x). \tag{9}$$

   Let $q_{t,x} \in F^{\#}(t,x)$, since $F^{\#}$ has convex closed images and by (H_1) it is locally Lipschitz, by the Michael's Selection Theorem, there exists a continuous feedback $q(\cdot,\cdot)$ for which $q(t,x) = q_{t,x}$. Hence, there exists $\varepsilon > 0$ with $t - \varepsilon \geq 0$ and $y : (-\varepsilon, \varepsilon) \to \mathbb{R}^N$ a smooth arc that verifies

   $$\dot{y}(s) = q(s, y(s)),\ y(t) = x, \quad \text{and} \quad y(s) \in \mathrm{ri}(\mathcal{K}),\ \forall s \in (t-\varepsilon, t+\varepsilon).$$

   By (6), we have

   $$\vartheta(t,x) + \int_{t-\tau}^{t} \ell(s, y(s), q(s,y(s)))ds \geq \vartheta(t-\tau, y(t-\tau)), \quad \forall \tau \in (0,\varepsilon).$$

   Let (θ,ζ) be in $\partial_P \vartheta(t,x)$. Then, there exist $\sigma, \delta > 0$ such that

   $$\vartheta(s,z) \geq \vartheta(t,x) + \langle \zeta, z-x \rangle + \theta(s-t) - \sigma|y-x|^2 - \sigma(s-t)^2 \quad \forall z \in \mathbb{B}(x,\delta) \cap \mathcal{K}, \forall s \in [t-\delta, t].$$

   Take τ small enough, so that the proximal sub-gradient inequality is valid. Then

   $$\vartheta(t-\tau, y(t-\tau)) \geq \vartheta(t,x) + \langle \zeta, y(t-\tau) - x \rangle - \theta\tau - \sigma|y(t-\tau) - x|^2 - \sigma\tau^2.$$

   In particular,

   $$\frac{1}{\tau}\int_{t-\tau}^{t} \ell(y(s), q(s,y(s)))ds + \left\langle \zeta, \frac{x - y(t-\tau)}{\tau} \right\rangle + \theta \geq h(\tau),$$

   with $\lim_{\tau \to 0^+} h(\tau) = 0$. By virtue of the continuity of the feedback q and smoothness of y, passing to the limit in the last inequality we obtain (9).

Moreover, if $(\theta,\zeta) \in \partial_V \vartheta((t,x))$, by [8, Proposition 3.4.5] we can find two sequences $\{(t_n,x_n)\} \subseteq \mathrm{dom}(\vartheta)$ and $\{(\theta_n,\zeta_n)\} \subseteq \mathbb{R}\times\mathbb{R}^N$ such that $(t_n,x_n) \to (t,x)$, $\vartheta(t_n,x_n) \to \vartheta(t,x)$, $(\theta_n,\zeta_n) \in \partial_P \vartheta(t,x_n)$ and $(\theta_n,\zeta_n) \to (\theta,\zeta)$ for which

$$-\theta_n - \langle \zeta_n, q\rangle - \ell(t_n,x_n,q) \leq 0, \qquad \forall \zeta_n \in \partial_P \vartheta(t_n,x_n), \quad \forall q \in F^{\#}(t_n,x_n).$$

The preceding inequality is due to (9). Therefore, letting $n \to +\infty$, we get the inequality (8). On the other hand, let (t,x) be in $[0,T)\times\mathcal{K}$ with $\partial_P\vartheta(t,x) \neq \emptyset$. There exists $\bar{y} \in \mathcal{S}_{[t,t+\tau]}(x)$ such that:

$$\vartheta(t,x) = \vartheta(t+\tau, \bar{y}(t+\tau)) + \int_t^{t+\tau} \ell(s,\bar{y}(s),\dot{\bar{y}}(s))\,ds \quad \forall \tau \in [0, T-\tau].$$

For $(\theta,\zeta) \in \partial_P\vartheta(t,x)$ there exist $\delta,\sigma > 0$ such that for any $\tau \in (0,\delta)$, we have:

$$-\theta - \frac{1}{\tau}\int_t^{t+\tau} \langle \zeta, \dot{\bar{y}}(s)\rangle\,ds - \frac{1}{\tau}\int_t^{t+\tau} \ell(s,\bar{y}(s),\dot{\bar{y}}(s))\,ds \geq -\frac{\sigma}{\tau}|\bar{y}(t+\tau)-x|^2 - \sigma\tau.$$

This inequality implies:

$$-\theta + \frac{1}{\tau}\int_t^{t+\tau} H(s,\bar{y}(s),\zeta)\,ds \geq -\frac{\sigma}{\tau}\int_t^{t+\tau} |\dot{\bar{y}}(s)|^2\,ds - \sigma\tau.$$

Consequently by the continuity of H, dividing by $t > 0$ and letting $t \to 0$ we get

$$-\theta + H(t,x,\zeta) \geq 0 \qquad \forall (\theta,\zeta) \in \partial_P\vartheta(t,x).$$

By using the same arguments as above, the last inequality holds true for the viscosity subdifferential as well, and so, the value function ϑ verifies (7)-(8).

2. Uniqueness:
Let $\omega : \mathcal{K} \to \mathbb{R}\cup\{+\infty\}$ be another l.s.c. function that verifies $\omega(T,x) = \Psi(x)$ and (7). In the light of Lemma 3.1-(i), ω is weakly decreasing along trajectories of $\mathbb{S}_{[0,T]}(x)$, and by Lemma 2.1 we have that $\vartheta \leq \omega$ over $\mathcal{K}$.
On the other hand, if ω satisfies (8), then by Lemma 3.1-(ii), ω is strongly increasing along trajectories of $\mathfrak{S}_{\mathrm{ri}}$, which by Lemma 2.1 proves that $\vartheta \geq \omega$. So, $\vartheta = \omega$ over $\mathcal{K}$ and the proof is complete.

□

4. Conclusion

In this paper we have provided a characterization of the value function for a Bolza problem governed by linear systems with convex state constraint sets. We have seen that our framework covers situations involving an accumulative cost that is convex in the velocity. This follows from Remark 2.1, Remark 2.4 and Remark 3.1.

The preceding analysis is based on a hypothesis (H_Q) (concerning properties of the data of the problem), that does not impose any a priori conditions on behavior of the dynamics on the boundary of the state constraint set. In consequence, our main result does not depend on the constituent constraints that combine to define the overall state constraint set, and so permits treatment of problems with higher order index constraints. Indeed, in the simplest cases, it can be reduced to find an equilibrium point of the dynamical system in the relative interior of the state constraint set. An assumption of this kind suits well for dynamical systems arising from a mechanical models. For example, consider the so-called double integrator system:

$$\dot{y}_1(t) = y_2(t), \quad \dot{y}_2(t) = u(t), \quad u(t) \in [-1,1] \quad \text{for a.e. } t \in [0,T].$$

It is straightforward that, if the state constraint set contains the origin, then (H_Q) is immediately satisfied. However, the existence of vector fields pointing to the interior (or exterior) of the state constraint set depend on the sign of $y_2(t)$, and so, the IPC is unlikely satisfied; for example the rather simple case $\mathcal{K} = [-1,1]\times\mathbb{R}$ doesn't satisfy the IPC nor any of its variations.

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