Hamilton-Jacobi-Bellman approach for optimal control problems of sweeping processes

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Abstract

This paper is concerned with a state constrained optimal control problem governed by a Moreau's sweeping process with a controlled drift. The focus of this work is on the Bellman approach for an infinite horizon problem. In particular, we focus on the regularity of the value function and on the Hamilton-Jacobi-Bellman equation it satisfies. We discuss a uniqueness result and we make a comparison with standard state constrained optimal control problems to highlight a regularizing effect that the sweeping process induces on the value function.

Keywords: State constraints, Infinite horizon problems, Sweeping processes, Hamilton-Jacobi-Bellman equations, Optimal control

1 Introduction

In this paper, we are concerned with infinite horizon optimal control problems of trajectories that satisfy a time-dependent state constraint and whose dynamics is governed by a sweeping process with a controlled drift. Our goal is to characterize the value function of the associated problem as the unique solution to a Hamilton-Jacobi-Bellman (HJB) equation in a viscosity sense.

To be more precise, given $(\tau, x) \in \mathbb{R} \times \mathbb{R}^N$, a running cost $\ell\colon \mathbb{R}^N \times \mathbb{R}^m \to \mathbb{R}$ and $\lambda > 0$, we are interested in the value function defined via the formula

$$\vartheta(\tau, x) := \inf_{\alpha \in \mathcal{A}} \int_\tau^\infty e^{\lambda(\tau - t)} \ell(y^\alpha_{\tau,x}(t), \alpha(t)) dt, \tag{1}$$

where $\mathcal{A}$ stands for the set of Borel measurable functions $\alpha\colon \mathbb{R} \to A$ with values in a nonempty set $A \subset \mathbb{R}^m$ and $y^\alpha_{\tau,x}(\cdot)$ is an arc that solves the controlled sweeping process

$$\begin{cases} \dot{y}(t) \in f(y(t), \alpha(t)) - \mathcal{N}_{\mathbf{C}(t)}(y(t)), & \text{for a.e. } t > \tau, \\ y(\tau) = x, \quad \text{with } x \in \mathbf{C}(\tau). \end{cases} \tag{2}$$

Here $f\colon \mathbb{R}^N \times \mathbb{R}^m \to \mathbb{R}^N$ is the controlled part of the dynamics, the set-valued map $\mathbf{C}\colon \mathbb{R} \rightrightarrows \mathbb{R}^N$ corresponds to the moving set (or time-dependent state constraint), and $\mathcal{N}_S(x)$ stands for the *proximal normal* cone to a set $S \subset \mathbb{R}^n$ at a point $x \in S$ (see definition below).

In the autonomous case, when $\mathbf{C} \equiv \mathbb{R}^N$ and under standard hypotheses on the data, it is well known that $\vartheta(\cdot)$ is a uniformly continuous function which can be characterized as the unique viscosity solution to an HJB equation in that class of functions; see, e.g., [2, Chapter 3] (see also [5] for an extension to the non-autonomous setting). Furthermore, it is not difficult to see that the mapping $\tau \mapsto \vartheta(\tau, x) =: \vartheta(x)$ is constant for any $x \in \mathbb{R}^N$ fixed. Thus, in this particular setting, the HJB equation takes the following form

$$\lambda\vartheta(x) + H(x, \nabla\vartheta(x)) = 0, \qquad x \in \mathbb{R}^N, \tag{3}$$

where $H(x, \zeta) := \sup\{-\langle f(x, a), \zeta\rangle - \ell(x, a) \colon a \in A\}$ for any $x, \zeta \in \mathbb{R}^N$.

When the control problem is in the presence of state constraints ($\mathbf{C} \neq \mathbb{R}^N$), a constrained HJB equation can be associated with the value function as done in [22]. In this case, it is well-known that the value function satisfies (3) in the constrained viscosity sense, meaning that $\vartheta(\cdot)$ is a viscosity subsolution on $\operatorname{int}(\mathbf{C})$ and a viscosity supersolution on $\mathbf{C}$. However, it is troublesome to prove the uniqueness of the solution to (3). The main difficulty comes from the fact that the HJB equation may admit several solutions (in the constrained viscosity sense) if the behavior of the solution on the boundary is not taken into account (see, e.g., the discussion in [6, 18]). Another key point in the analysis is that the value function of a classical optimal control problem with state constraints (no sweeping process involved) is likely to be merely lower semicontinuous (see, for instance, [2, Example IV.5.3]) and undefined in some regions of the state-constraint set; essentially because the viability domain may be strictly contained in the state-constraint set. One of the main contributions of this paper is to show that the sweeping process induces a regularizing effect on the value function (it turns out to be a continuous function). This outcome is due to the fact that trajectories are allowed to slide along the boundary of the state-constraint set.

To the best of our knowledge, there are very few works on this topic in the setting of controlled sweeping processes. For instance, in [21] the author deals with the Mayer problem in an autonomous setting, while in [11], the authors study the minimum time problem to reach a prescribed target. In both papers, suitable HJB inequalities are introduced and used to characterize the value function as the unique solution in an appropriate sense. It is worth mentioning that both works require the value function to be continuous; this is essentially due to the dissipative character of the sweeping processes, which does not allow to work with backward solutions in time.

The paper is organized as follows. After some mathematical preliminaries, in Section 3, we prove the continuity of the value function and the existence of solutions for the controlled sweeping process. Section 4 is dedicated to studying monotonicity along trajectories and invariance results. Section 5 provides weakly decreasing principles for the controlled sweeping process. The paper ends with a discussion section and an example illustrating the regularizing effect of the sweeping process on the value function.

1.1 Main contributions of the paper

The main novelty of this work is that we are able, for the first time, to write down proper Hamilton-Jacobi-Bellman inequalities, tested on the sub/super-differentials of the value function. Such a result was achieved in [21, 11] just in the case in which the constraint of the sweeping process does not dependent on time. Let us stress that this is not a straightforward result since the sweeping process is a naturally non-autonomous dynamics, merely upper semicontinuous with respect to time. Such a feature makes the Hamiltonian of the problem discontinuous in time and space, and makes the value function continuous, but not locally Lipschitz continuous with respect to time. Furthermore, Corollary 5.1 and Theorem 5.2 provide an equivalence result between different notions of solutions for the Hamilton-Jacobi-Bellman equation related to an optimal control problem governed by a Sweeping Process dynamics. This is a well-known fact when the Hamiltonian is continuous with respect to the state and the time, but it is not true in general for Hamilton-Jacobi equations with discontinuous dynamics. In this respect, the notion of viscosity solution presented in our main result (see Theorem 3.3) agrees, under suitable conditions, with the notion introduced by Ishii in [17] for boundary value problem with discontinuous Hamiltonian; see Remark 3.2. We refer to [4] and the references therein for a recent up-to-date reference on the topic of optimal control problems with discontinuities and Hamilton-Jacobi equations.

Moreover, by means of an appropriate Relaxation Theorem, we are able to prove our Uniqueness Theorem without requiring any convexity assumptions on the dynamics nor on the running cost, as, for instance, done in [16] for problems with standard optimal control problems with state constraints or in [21, 11] for optimal control problems of sweeping processes.

2 Mathematical Preliminaries and Assumptions

Throughout this paper $\langle\cdot,\cdot\rangle$ stands for the Euclidean inner product of $\mathbb{R}^N$ with associated norm $\|x\| = \sqrt{\langle x,x\rangle}$. The closed unit ball will be denoted by $\mathbb{B}$ and $\mathbb{B}(x,r)$ denotes the closed ball centered at $x \in \mathbb{R}^N$ of radius $r > 0$.

Let S be a nonempty closed subset of $\mathbb{R}^N$ and $x \in \mathbb{R}^N$. The distance of x to S, denoted by $\operatorname{dist}_S(x)$, is defined by $\operatorname{dist}_S(x) := \inf\{\|x-y\| : y \in S\}$.

The domain and the epigraph of an extended real-valued function $f : \mathbb{R}^N \to \mathbb{R} \cup \{+\infty\}$ are defined, respectively, as

$$\operatorname{dom}(f) := \{x \in \mathbb{R}^N : f(x) < +\infty\} \text{ and } \operatorname{epi}(f) := \{(x,\lambda) \in \mathbb{R}^N \times \mathbb{R} : f(x) \le \lambda\}$$

Given a set $S \subset \mathbb{R}^N$ and $x \in S$, we say that a vector $\eta \in \mathbb{R}^n$ belongs to the *proximal normal cone* of S at $x \in S$, denoted by $\mathcal{N}_S(x)$, if there is $\sigma = \sigma(x,\eta) \ge 0$ such that $\langle \eta, y - x\rangle \le \sigma\|x-y\|^2$ for all $y \in S$. A vector $\eta \in \mathbb{R}^N$ is called a *proximal subgradient* of a lower semicontinuous function $f : \mathbb{R}^N \to \mathbb{R} \cup \{+\infty\}$ at $x \in \operatorname{dom}(f)$ provided that

$$(\eta, -1) \in \mathcal{N}_{\operatorname{epi}(f)}(x, f(x)).$$

The set of all such η is the *proximal subdifferential*, which is denoted by $\partial_P f(x)$. We refer to [10] for more details.

For any nonempty closed subset S of $\mathbb{R}^N$ and any $x \in S$, one has

$$\partial_P \operatorname{dist}_S(x) = \mathcal{N}_S(x) \cap \mathbb{B}. \tag{4}$$

Let $C \subset \mathbb{R}^N$ be a closed set and $\rho > 0$. We say that C is ρ *uniformly prox-regular* (see [12] for a survey) if

$$\langle \eta, y - x\rangle \le \frac{1}{2\rho}\|\eta\|\,\|y-x\|^2, \qquad \forall x,y \in C,\ \forall \eta \in \mathcal{N}_C(x).$$

The graph of a set-valued map $\mathbf{C} : \mathbb{R} \rightrightarrows \mathbb{R}^N$, denoted by $\operatorname{gr}(\mathbf{C})$, is the collection of all $(\tau, x) \in \mathbb{R} \times \mathbb{R}^N$ satisfying $x \in \mathbf{C}(\tau)$.

2.1 Assumptions

Along this work, we assume that the data of the optimal control problem at hand satisfy the following conditions, which are going to be referred in the sequel as *Standing Assumptions*:

(H1) A is a nonempty and compact subset of $\mathbb{R}^m$.

(H2) the time-dependent state-constraint sets satisfies

- **(H2.i)** $\mathbf{C}(t)$ is a nonempty, closed and ρ uniformly prox-regular set of $\mathbb{R}^N$ for any $t \in \mathbb{R}$.
- **(H2.ii)** there exists $\kappa_{\mathbf{C}} > 0$ such that

$$\sup_{x\in\mathbb{R}^N} |\operatorname{dist}_{\mathbf{C}(t)}(x) - \operatorname{dist}_{\mathbf{C}(s)}(x)| \le \kappa_{\mathbf{C}}|t-s|, \qquad \forall t,s \in \mathbb{R}.$$

(H3) The dynamics satisfies

(H3.i) $f(x,\cdot)$ is continuous on A for any fixed $x \in \mathbb{R}^N$.

(H3.ii) there exists $\kappa_f > 0$ such that

$$\sup_{a\in A} \|f(x_1,a) - f(x_2,a)\| \le \kappa_f \|x_1 - x_2\|, \qquad \forall x_1, x_2 \in \mathbb{R}^N.$$

We observe that under **(H3.i)** and **(H3.ii)**, the following assertion holds:

(H3.iii) there is $\beta_f > 0$ such that

$$\sup_{a\in A} \|f(x,a)\| \le \beta_f(1 + \|x\|), \qquad \forall x \in \mathbb{R}^N.$$

(H4) ℓ is continuous on $\mathbb{R}^N \times A$ and satisfies in addition

(H4.i) $\exists \beta_\ell > 0$ such that $0 \le \ell(x,a) \le \beta_\ell$ for any $t \in \mathbb{R}$, $x \in \mathbf{C}(t)$ and $a \in A$.

(H4.ii) there exists $\kappa_\ell > 0$ such that

$$\sup_{a\in A} |\ell(x_1,a) - \ell(x_2,a)| \le \kappa_\ell \|x_1 - x_2\|, \qquad \forall x_1, x_2 \in \mathbb{R}^N.$$

The Standing Assumptions considered above are enough (but not sharp) to obtain the regularity of the value function we have claimed in the introduction. This regularity result, key for the analysis we do in the sequel, heavily relies on Thibault's works on perturbed sweeping processes (see, e.g., [14, 8]), which we summarize below.

Lemma 1. *For any $\alpha \in \mathcal{A}$, $\tau \in \mathbb{R}$ and $x \in \mathbf{C}(\tau)$ fixed, the dynamical system (2) has a unique absolutely continuous solution, which is denoted by $y^\alpha_{\tau,x}(\cdot)$. Moreover,*

(i) we have

$$\|y^\alpha_{\tau,x}(t) - x\| \le \left(e^{2\beta_f(t-\tau)} - 1\right)\left(1 + \frac{\kappa_{\mathbf{C}}}{2\beta_f} + \|x\|\right), \qquad \forall t \ge \tau,$$
$$\|\dot y^\alpha_{\tau,x}(t)\| \le \kappa_{\mathbf{C}} + 2\beta_f\left(1 + \frac{\kappa_{\mathbf{C}}}{2\beta_f} + \|x\|\right) e^{2\beta_f(t-\tau)}, \qquad \textit{for a.e. } t \ge \tau$$

(ii) for any $r > 0$ and $T > \tau$ there is $K^r_T \ge 1$, such that for any $s \in [\tau, T]$ and $\alpha \in \mathcal{A}$ we have

$$\max_{t\in[s,T]} \|y^\alpha_{s,x_1}(t) - y^\alpha_{s,x_2}(t)\| \le K^r_T \|x_1 - x_2\|, \qquad \forall x_1, x_2 \in \mathbf{C}(s) \cap \mathbb{B}(0,r).$$

(iii) A mapping $y(\cdot)$ is a solution of (2) if and only if it is a solution of the unconstrained differential inclusion

$$\begin{cases} \dot y(t) \in f(y(t),\alpha(t)) - m(y(t))\partial_P \operatorname{dist}_{\mathbf{C}(t)}(y(t)), & \textit{for a.e. } t > \tau \\ y(\tau) = x, \quad \textit{with } (\tau,x) \in \operatorname{gr}(\mathbf{C}), \end{cases} \tag{5}$$

where $m(x) := \kappa_{\mathbf{C}} + \beta_f(1 + \|x\|)$ for any $x \in \mathbb{R}^N$.

Proof. Let $\alpha \in \mathcal{A}$, $\tau \in \mathbb{R}$ and $x \in \mathbf{C}(\tau)$ be fixed. From **(H2)** and **(H3)**, it is clear that the moving set $t \mapsto \mathbf{C}(t)$ and the drift $(t, x) \mapsto f(x, \alpha(t))$ satisfy the assumptions of [14, Theorem 1] on $I = [\tau, T]$. Moreover, from [14, Theorem 1], we can deduce that (2) has a unique absolutely continuous solution.
(i): By virtue of **(H2.ii)** and [14, Proposition 1], we obtain that

$$\|\dot{y}^{\alpha}_{\tau,x}(t) - f(y^{\alpha}_{\tau,x}(t), \alpha(t))\| \leq \kappa_{\mathbf{C}} + \|f(y^{\alpha}_{\tau,x}(t), \alpha(t))\|, \qquad \text{for a.e. } t \geq \tau.$$

Consequently, by **(H3.iii)**, it follows that

$$\|\dot{y}^{\alpha}_{\tau,x}(t)\| \leq \kappa_{\mathbf{C}} + 2\beta_f(1 + \|y^{\alpha}_{\tau,x}(t)\|), \qquad \text{for a.e. } t \geq \tau.$$

Therefore, the result follows from Gronwall's Lemma (see [10, Theorem 4.1.4]).
(ii): It is a direct consequence of [14, Proposition 2]. It is enough to note that the Lipschitz constant in the proof of [14, Proposition 2] can be taken uniform with respect to the control α. Indeed, it has an explicit expression that depends only on τ, T, r, ρ, κ_f, β_f and $\kappa_{\mathbf{C}}$.
(iii): It follows from formula (4) and [23, Theorem 2.1]. □

Also, in order to avoid convexity assumptions on the dynamics and cost, we require a Relaxation Theorem for trajectories of the controlled sweeping processes.
Lemma 2. *Let $\tau \in \mathbb{R}$, $x \in \mathbf{C}(\tau)$ and $T > \tau$ be fixed, and let $y(\cdot)$ be a solution of*

$$\begin{cases} \dot{y}(t) \in \operatorname{co} f(y(t), A) - \mathcal{N}_{\mathbf{C}(t)}(y(t)), & \text{for a.e. } t \in [\tau, T], \\ y(\tau) = x, \end{cases}$$

then, for any $\epsilon > 0$ there is a solution $y_\varepsilon(\cdot)$ of

$$\begin{cases} \dot{y}_\varepsilon(t) \in f(y_\varepsilon(t), A) - \mathcal{N}_{\mathbf{C}(t)}(y_\varepsilon(t)), & \text{for a.e. } t \in [\tau, T], \\ y_\varepsilon(\tau) = x, \end{cases}$$

such that $\max_{t\in[\tau,T]} \|y(t) - y_\varepsilon(t)\| < \epsilon$.

Proof. We observe that the result follows from an application of [9, Theorem 3.3], which is stated in terms of measure-valued relaxed control. In particular, it is shown a version of Lemma 2 in the case in which the state solution $y(\cdot)$ is solution of

$$\begin{cases} \dot{y}(t) \in \int_A f(y(t), r)\mu(t)(dr) - \mathcal{N}_{\mathbf{C}(t)}(y(t)), & \mu \in \mathcal{S}, \quad \text{for a.e. } t \in [\tau, T], \\ y(\tau) = x, \qquad \text{with } x \in \mathbf{C}(\tau), \end{cases}$$

where $\mathcal{S}$ is the set of the Lebesgue measurable maps from $[\tau, T]$ to the set r.p.m.(A) of the Radon probability measures over A, that is

$$\mathcal{S} := \{\mu : [\tau, T] \to \text{r.p.m.}(A) \quad \text{Lebesgue measurable}\}.$$

In view of the assumptions **(H1)** and **(H3.i)**, the set $\operatorname{co}\{f(x,a): \ a \in A\}$ is compact. Hence, it follows from the arguments in [24, Theorem VI.3.2] that

$$\operatorname{co}\{f(x,a): \ a \in A\} = \left\{ \int_A f(x,r)\mu(dr): \ \mu \in \text{r.p.m.}(A) \right\},$$

implying the equivalence between [9, Theorem 3.3] and Lemma 2. □

Remark 2.1. *Notice that the preceding lemma can also be applied for an augmented dynamics, by replacing the mapping* $(x,a) \mapsto f(x,a)$ *with* $(t,x,a) \mapsto (t, f(x,a), e^{-\lambda t}\ell(x,a))$ *and the moving set* $t \mapsto \mathbf{C}(t)$ *with* $t \mapsto \mathbb{R} \times \mathbf{C}(t) \times \mathbb{R}$.

3 Main Results

In the light of Lemma 1, the value function given in (1) is a well-defined finite function on $\operatorname{gr}(\mathbf{C})$; note that ℓ is bounded along feasible arcs by **(H4.i)**. In particular, we also have that, as in standard optimal problems with state constraints, the value function satisfies a dynamic programming principle.

Lemma 3. *The mapping* $\vartheta\colon \operatorname{gr}(\mathbf{C}) \subset \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ *defined in* (1) *satisfies for any* $\tau, h \geq 0$ *and any* $x \in \mathbf{C}(\tau)$ *that:*

$$\vartheta(\tau,x) := \inf_{\alpha \in \mathcal{A}} \left\{ \int_\tau^{\tau+h} e^{\lambda(\tau-t)}\ell(y^\alpha_{\tau,x}(t),\alpha(t))dt + e^{-\lambda h}\vartheta\left(\tau+h, y^\alpha_{\tau,x}(\tau+h)\right) \right\}.$$

Proof. It follows from similar arguments as those used in [2, Remark III.3.10]. □

3.1 Continuity of the value function

We can now prove one of our main results, which concerns the regularity of the value function. We show that the value function of an optimal control problem of sweeping processes over an infinite horizon is a bounded continuous function on $\operatorname{gr}(\mathbf{C})$.

Theorem 3.1. *The mapping* $\vartheta\colon \operatorname{gr}(\mathbf{C}) \subset \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ *defined in* (1) *is bounded and continuous.*

Proof. Let $(\bar\tau, \bar x) \in \operatorname{gr}(\mathbf{C})$, that is, $\bar x \in \mathbf{C}(\bar\tau)$. Thanks to **(H4.i)**, it is easy to see that $|\vartheta(\bar\tau,\bar x)| \leq \frac{\beta_\ell}{\lambda}$, which implies that $\vartheta(\cdot)$ is bounded on $\operatorname{gr}(\mathbf{C})$.

For the sake of the exposition, let us divide the proof of the continuity into several steps:

Step 1: For $(\bar\tau, \bar x) \in \operatorname{gr}(\mathbf{C})$ and $\varepsilon > 0$, there exists $h \in (0,1)$ such that $6\beta_\ell\left(1 - e^{-\lambda h}\right) \leq \lambda\varepsilon$ and

$$\mathbb{B}(\bar x, \kappa_{\mathbf{C}} h) \cap \mathbf{C}(\tau_0) \neq \emptyset, \qquad \forall \tau_0 \in [\bar\tau - h, \bar\tau + h].$$

Proof of Step 1: It follows from the continuity of $t \mapsto \mathbf{C}(t)$ (see **(H2.ii)**), since we have $\operatorname{dist}_{\mathbf{C}(t)}(\bar x) \leq \kappa_{\mathbf{C}}|t - \bar\tau|$ for any $t \in \mathbb{R}$. □

Step 2: For $(\bar\tau, \bar x) \in \operatorname{gr}(\mathbf{C})$ and $\varepsilon > 0$, there is $\delta > 0$ such that for any $\tau_0 \in [\bar\tau - h, \bar\tau + h]$ and $x_0 \in \mathbb{B}(\bar x, \kappa_{\mathbf{C}} h) \cap \mathbf{C}(\tau_0)$ we have

$$|\vartheta(\tau_0, x) - \vartheta(\tau_0, x_0)| \leq \varepsilon, \qquad \forall x \in \mathbf{C}(\tau_0) \cap \mathbb{B}(x_0, \delta). \tag{6}$$

Here $h \in (0,1)$ is as in Step 1.
Proof of Step 2: Take $T > 1 + \bar{\tau}$ such that $4\beta_\ell \leq \varepsilon\lambda e^{\lambda(T-\bar{\tau}-1)}$ and define

$$\bar{r} := \sup_{\alpha\in\mathcal{A}} \sup_{\tau\in[\bar{\tau}-1,\bar{\tau}+1]} \sup_{x\in\mathbb{B}(\bar{x},\kappa_{\mathbf{C}})\cap\mathbf{C}(\tau)} \max_{t\in[\tau,T]} \|y^\alpha_{\tau,x}(t)\|,$$

which is finite by Lemma 1. Let $K_T^{\bar{r}} > 0$ be the constant given by Lemma 1 for $r = \bar{r}$ and $\tau = \bar{\tau} - 1$. Let $\delta \in (0, (1-h)\kappa_{\mathbf{C}})$ small enough such that

$$\frac{\kappa_\ell K_T^{\bar{r}}}{\lambda}\delta\left(1 - e^{-\lambda(T-\bar{\tau}+1)}\right) \leq \frac{\varepsilon}{4}. \tag{7}$$

Take $\alpha_0 \in \mathcal{A}$ be an $\varepsilon/4$ optimal solution associated with $\vartheta(\tau_0, x_0)$, that is,

$$\int_{\tau_0}^\infty e^{\lambda(\tau_0-t)}\ell(y^{\alpha_0}_{\tau_0,x_0}(t), \alpha_0(t))dt - \frac{\varepsilon}{4} \leq \vartheta(\tau_0, x_0).$$

Take any $x \in \mathbf{C}(\tau_0) \cap \mathbb{B}(x_0, \delta)$ arbitrary. Since $\delta < (1-h)\kappa_{\mathbf{C}}$ and $h < 1$, we have that $x \in \mathbb{B}(\bar{x}, \kappa_{\mathbf{C}})$ and it follows that for any $\alpha \in \mathcal{A}$ we have

$$\max_{t\in[\tau_0,T]} \|y^\alpha_{\tau_0,x}(t)\| \leq \bar{r}.$$

Furthermore, in the light of **(H4.ii)**, we have that for any $\alpha \in \mathcal{A}$ and a.e. $t \in [\tau_0, T]$

$$\left|\ell(y^\alpha_{\tau_0,x_0}(t), \alpha(t)) - \ell(y^\alpha_{\tau_0,x}(t), \alpha(t))\right| \leq \kappa_\ell \left\|y^\alpha_{\tau_0,x_0}(t) - y^\alpha_{\tau_0,x}(t)\right\|.$$

Consequently, combining Lemma 1 and the preceding estimate, we obtain that for any $\alpha \in \mathcal{A}$,

$$\int_{\tau_0}^T e^{\lambda(\tau_0-t)}\left|\ell(y^\alpha_{\tau_0,x_0}(t), \alpha(t)) - \ell(y^\alpha_{\tau_0,x}(t), \alpha(t))\right| dt \leq \frac{\kappa_\ell K_T^{\bar{r}}}{\lambda}\|x_0 - x\|\left(1 - e^{-\lambda(T-\tau_0)}\right) \leq \frac{\varepsilon}{4},$$

where we have used (7). Hence, the definition of α_0 leads then to

$$\vartheta(\tau_0, x) - \vartheta(\tau_0, x_0) \leq \int_{\tau_0}^\infty e^{\lambda(\tau_0-t)}\left(\ell(y^{\alpha_0}_{\tau_0,x}(t), \alpha_0(t)) - \ell(y^{\alpha_0}_{\tau_0,x_0}(t), \alpha_0(t))\right) dt + \frac{\varepsilon}{4}.$$

Moreover, thanks to the way $T > \tau_0$ has been taken, we also get

$$\vartheta(\tau_0, x) - \vartheta(\tau_0, x_0) \leq \int_{\tau_0}^T e^{\lambda(\tau_0-t)}\left|\ell(y^{\alpha_0}_{\tau_0,x_0}(t), \alpha_0(t)) - \ell(y^{\alpha_0}_{\tau_0,x}(t), \alpha_0(t))\right| dt + \frac{3\varepsilon}{4} \leq \varepsilon.$$

Finally, let $\alpha \in \mathcal{A}$ be an $\varepsilon/4$ optimal solution for the problem associated to $\vartheta(\tau_0, x)$, that is,

$$\int_{\tau_0}^{\infty} e^{\lambda(\tau_0 - t)} \ell(y^{\alpha}_{\tau_0,x}(t), \alpha(t))dt - \frac{\varepsilon}{4} \leq \vartheta(\tau_0, x).$$

Since $\alpha \in \mathcal{A}$ is also feasible for the problem associated to $\vartheta(\tau_0, x_0)$, it follows that

$$\vartheta(\tau_0, x_0) - \vartheta(\tau_0, x) \leq \int_{\tau_0}^{\infty} e^{\lambda(\tau_0 - t)} \left(\ell(y^{\alpha}_{\tau_0,x_0}(t), \alpha(t)) - \ell(y^{\alpha}_{\tau_0,x}(t), \alpha(t))\right) dt + \frac{\varepsilon}{4}.$$

Thus, repeating the same arguments as before, we get

$$\vartheta(\tau_0, x_0) - \vartheta(\tau_0, x) \leq \varepsilon,$$

which implies (6). The proof of Step 2 is then completed. □

Step 3: Fix $(\tau, x) \in \mathrm{gr}\,(\mathbf{C})$. Then, for any $\varepsilon, h > 0$ there is $\bar{\alpha} \in \mathcal{A}$ such that

$$|\vartheta(\tau, x) - \vartheta(\tau + h, y^{\bar{\alpha}}_{\tau,x}(\tau + h))| \leq \frac{2\beta_\ell}{\lambda} \left(1 - e^{-\lambda h}\right) + \frac{\varepsilon}{3},$$

where $\bar{\alpha} \in \mathcal{A}$ is such that

$$\int_{\tau}^{\tau+h} e^{\lambda(\tau - t)} \ell(y^{\bar{\alpha}}_{\tau,x}(t), \alpha(t))dt + e^{-\lambda h} \vartheta\left(\tau + h, y^{\bar{\alpha}}_{\tau,x}(\tau + h)\right) \leq \vartheta(\tau, x) + \frac{\varepsilon}{3}.$$

Proof of Step 3: By the Dynamic Programming Principle (Lemma 3), for any $\alpha \in \mathcal{A}$ we have

$$\vartheta(\tau, x) \leq \int_{\tau}^{\tau+h} e^{\lambda(\tau - t)} \ell(y^{\alpha}_{\tau,x}(t), \alpha(t))dt + e^{-\lambda h} \vartheta\left(\tau + h, y^{\alpha}_{\tau,x}(\tau + h)\right).$$

The claim follows from appropriate bounds on integrals and on the value function. □

Step 4: Fix $(\bar{\tau}, \bar{x}) \in \mathrm{gr}\,(\mathbf{C})$. Then, for all $\varepsilon > 0$, there exists $\rho > 0$ such that if $\tau \in (\bar{\tau} - \rho, \bar{\tau} + \rho)$ and $\|x - \bar{x}\| \leq \kappa_{\mathbf{C}}\rho$, then $|\vartheta(\tau, x) - \vartheta(\bar{\tau}, \bar{x})| \leq \varepsilon$.

Proof of Step 4: Let $\varepsilon > 0$ and take $h \in (0, 1)$ as in Step 1. Then, by virtue of Step 3, for any $\tau \in (\bar{\tau} - h, \bar{\tau}]$ there is $\alpha \in \mathcal{A}$ such that

$$|\vartheta(\tau, x) - \vartheta(\bar{\tau}, \bar{x})| \leq \frac{2\varepsilon}{3} + |\vartheta(\bar{\tau}, y^{\alpha}_{\tau,x}(\bar{\tau})) - \vartheta(\bar{\tau}, \bar{x})|.$$

Similarly, for any $\tau \in [\bar{\tau}, \bar{\tau} + h)$ there is $\bar{\alpha} \in \mathcal{A}$ such that

$$|\vartheta(\tau, x) - \vartheta(\bar{\tau}, \bar{x})| \leq |\vartheta(\tau, x) - \vartheta(\tau, y^{\bar{\alpha}}_{\bar{\tau},\bar{x}}(\tau))| + \frac{2\varepsilon}{3}.$$

Now, by Lemma 1, for any $\delta > 0$ we can take $\rho \in (0, h)$ such that if

$$\tau \in (\bar{\tau} - \rho, \bar{\tau} + \rho) \quad \text{and} \quad \|x - \bar{x}\| \leq \kappa_{\mathbf{C}}\rho,$$

then

- if $\tau \leq \bar{\tau}$, we get $\|y^{\alpha}_{\tau,x}(\bar{\tau}) - \bar{x}\| \leq \delta$,
- if $\tau \geq \bar{\tau}$, we get $\|y^{\bar{\alpha}}_{\bar{\tau},\bar{x}}(\tau) - x\| \leq \delta$.

Consequently, by the way $h \in (0,1)$ has been taken, and letting $\delta > 0$ be such that (6) holds with $\frac{\varepsilon}{3}$ instead of ε, we obtain the following:

- if $\tau \in (\bar{\tau} - \rho, \bar{\tau}]$ and $\|x - \bar{x}\| \leq \kappa_{\mathbf{C}}\rho$, then $|\vartheta(\bar{\tau}, y^{\alpha}_{\tau,x}(\bar{\tau})) - \vartheta(\bar{\tau}, \bar{x})| \leq \frac{\varepsilon}{3}$,
- if $\tau \in [\bar{\tau}, \bar{\tau} + \rho)$ and $\|x - \bar{x}\| \leq \kappa_{\mathbf{C}}\rho$, then $|\vartheta(\tau, x) - \vartheta(\tau, y^{\bar{\alpha}}_{\bar{\tau},\bar{x}}(\tau))| \leq \frac{\varepsilon}{3}$,

which proves the continuity of the value function. □

3.2 Existence of optimal solutions

The following result establishes the existence of optimal controls for the problem (1). To prove this result, we must impose a convexity assumption, which is standard in optimal control theory. We emphasize that this convexity assumption will only be required in for this results, and no longer required afterwards.

(H5) For all $(\tau, x) \in \mathrm{gr}\,(\mathbf{C})$, the sets $f(x, A)$ and $\ell(x, A)$ are closed and convex.

Theorem 3.2 (Existence of optimal solutions)**.** *In addition to the Standing Assumptions, suppose that $\lambda > 2\beta_f$ and **(H5)** holds. Then, for any $(\tau, x) \in \mathrm{gr}\,(\mathbf{C})$ there is $\alpha \in \mathcal{A}$ such that*

$$\vartheta(\tau, x) = \int_{\tau}^{+\infty} e^{\lambda(\tau - t)} \ell(y^{\alpha}_{\tau,x}(t), \alpha(t))dt.$$

Proof. We adapt the ideas from [16, Proposition 3.2].

Fix $(\tau, x) \in \mathrm{gr}\,(\mathbf{C})$. According to Theorem 3.1, the value function is bounded. Hence, for all $n \in \mathbb{N}$, there exists $\alpha_n \in \mathcal{A}$ such that

$$\lim_{n \to +\infty} \int_{\tau}^{+\infty} e^{\lambda(\tau - t)} \ell(y^{\alpha_n}_{\tau,x}(t), \alpha_n(t))dt = \vartheta(\tau, x).$$

By virtue of Lemma 1, we have the following bounds,

$$\begin{aligned}
&\|y^{\alpha_n}_{\tau,x}(t)\| \leq \|x\| + \left(e^{2\beta_f(t-\tau)} - 1\right)\left(1 + \frac{\kappa_{\mathbf{C}}}{2\beta_f} + \|x\|\right) && \text{for } t \geq \tau;\\
&\|\dot{y}^{\alpha_n}_{\tau,x}(t)\| \leq \kappa_{\mathbf{C}} + 2\beta_f\left(1 + \frac{\kappa_{\mathbf{C}}}{2\beta_f} + \|x\|\right)e^{2\beta_f(t-\tau)} && \text{a.e. } t \geq \tau;\\
&|\ell(y^{\alpha_n}_{\tau,x}(t), \alpha_n(t)| \leq \beta_\ell && \text{for } t \geq \tau,
\end{aligned}$$

Consider the measure $d\mu = e^{\lambda(\tau - t)}dt$ and let $L^1_\mu := L^1([\tau, +\infty); d\mu)$ be the space of integrable functions on $[\tau, +\infty)$ for the measure $d\mu$. Consequently, we denote by $W^{1,1}_\mu$ the Sobolev space of functions in L^1_μ which have their weak derivative also in L^1_μ. Since $\lambda > 2\beta_f$, it follows that $y^{\alpha_n}_{\tau,x}$ are uniformly bounded in $W^{1,1}_\mu$. Hence, by virtue of [1, Theorem 0.3.4] and the Dunford-Pettis Theorem, there exist a function $y \in W^{1,1}_\mu$ and

$z \in L^1_\mu$ and subsequences (without relabeling) such that

$y^{\alpha_n}_{\tau,x}$ converges uniformly to y on compact subsets of $[0,+\infty)$;
$\dot y^{\alpha_n}_{\tau,x}$ converges weakly to $\dot y$ in L^1_μ;
$z_n := \ell(y^{\alpha_n}_{\tau,x}(\cdot),\alpha_n(\cdot))$ converges weakly to z in L^1_μ.

Then, by assertion (iii) of Lemma 1, hypothesis **(H5)** and the Convergence Theorem (see [1, Theorem 1.4.1]), we obtain that

$$\begin{cases} \dot y(t) \in f(y(t),A) - m(y(t))\partial_P \operatorname{dist}_{\mathbf{C}(t)}(y(t)) \text{ a.e. } t \geq \tau. \\ y(\tau) = x, \text{ with } x \in \mathbf{C}(t). \end{cases}$$

Besides, the sets $\mathbf{C}(t)$ are closed, which implies that $y(t) \in \mathbf{C}(t)$ for all $t \geq \tau$. Moreover, by virtue of **(H3.ii)**, we obtain that

$$z_n(t) \in \ell(y(t),A) + \kappa_\ell \|y(t) - y^{\alpha_n}_{\tau,x}(t)\|\,\mathbb{B} \quad \text{ for a.e. } t > \tau.$$

Since z_n converges weakly to z in L^1_μ, we obtain that $z(t) \in \operatorname{co}(\ell(y(t),A))$ for a.e. $t > \tau$ (see, e.g., [15, Proposition 2.3.31]). Then, by **(H5)**, we obtain that

$$z(t) \in \ell(y(t),A) \text{ for a.e. } t > \tau.$$

Hence, by the formula (4) and the Measurable Selection Theorem, there exists a measurable function $\alpha \in \mathcal{A}$ such that

$$\begin{aligned} &\dot y(t) \in f(y(t),\alpha(t)) - \mathcal{N}_{\mathbf{C}(t)}(y(t)) && \text{a.e. } t \geq \tau; \\ &z(t) = \ell(y(t),\alpha(t)) && \text{a.e. } t \geq \tau. \end{aligned}$$

Therefore, $y \equiv y^\alpha_{\tau,x}$. Finally, by weak convergence in L^1_μ of z_n to z, we obtain

$$\int_\tau^{+\infty} e^{\lambda(\tau-t)}\ell(y^\alpha_{\tau,x}(t),\alpha(t))dt = \lim_{n\to+\infty}\int_\tau^{+\infty} e^{\lambda(\tau-t)}\ell(y^{\alpha_n}_{\tau,x}(t),\alpha_n(t))dt = \vartheta(\tau,x),$$

which proves that α is a minimizer of the problem. □

3.3 Characterization of the value function

The main result of this paper concerns a characterization of the value function $\vartheta(\cdot)$ as the unique viscosity solution of the HJB equation

$$\lambda\vartheta(\tau,x) - \partial_\tau\vartheta(\tau,x) + H(x,\nabla_x\vartheta(\tau,x)) = 0, \qquad \tau \in \mathbb{R},\ x \in \operatorname{int}(\mathbf{C}(\tau)), \tag{8}$$

where

$$H(x,\zeta) := \max\{-\langle\zeta, f(x,a)\rangle - \ell(x,a) \colon a \in A\},$$

that in addition satisfies some extra conditions on the boundary of the moving set; the fact that additional information is actually mandatory for characterizing the value function is discussed in A. To be more precise, our main result about this characterization reads as follows.

Theorem 3.3. *The mapping* $\vartheta\colon \operatorname{gr}(\mathbf{C}) \subset \mathbb{R}\times\mathbb{R}^N \to \mathbb{R}$ *defined in* (1) *is the unique bounded and continuous function that solves* (8) *in the viscosity sense, which in addition is a:*

- *viscosity supersolution in* $\operatorname{gr}(\mathbf{C})$ *of*

$$\lambda\vartheta(\tau,x) - \partial_\tau\vartheta(\tau,x) + H(x,\nabla_x\vartheta(\tau,x)) + \mathfrak{S}_{\mathbf{C}}(\tau,x,\nabla_x\vartheta(\tau,x)) = 0, \tag{9}$$

- *viscosity subsolution in* $\operatorname{gr}(\mathbf{C})$ *of*

$$\lambda\vartheta(\tau,x) - \partial_\tau\vartheta(\tau,x) + H(x,\nabla_x\vartheta(\tau,x)) - \mathfrak{S}_{\mathbf{C}}(\tau,x,-\nabla_x\vartheta(\tau,x)) = 0, \tag{10}$$

where

$$\mathfrak{S}_{\mathbf{C}}(\tau,x,\zeta) := \sup\left\{\langle\zeta,v\rangle\colon v\in m(x)\partial_P \operatorname{dist}_{\mathbf{C}(\tau)}(x)\right\}. \tag{11}$$

and $m(\cdot)$ *is the function defined in Lemma 1.*

Proof. This is a direct consequence of Theorem 5.1, Corollary 5.1 and Proposition 4.1, which are stated and proven in the next sections. □

Remark 3.1. *In Theorem 3.3, the notion of viscosity supersolution in the closed set* $\operatorname{gr}(\mathbf{C})$ *is the same one as used in problems with state constraints; see for instance [2, Definition IV.5.6]. The notion of viscosity subsolution in the closed set* $\operatorname{gr}(\mathbf{C})$ *is the corresponding modification, that is,*

$$\lambda\vartheta(\tau,x) - \partial_\tau g(\tau,x) + H(x,\nabla_x g(\tau,x)) - \mathfrak{S}_{\mathbf{C}}(\tau,x,-\nabla_x g(\tau,x)) \le 0,$$

for any continuously differentiable function $g:\mathbb{R}\times\mathbb{R}^N\to\mathbb{R}$ *so that*

$$\vartheta - g \text{ attains a local maximum, relative to } \operatorname{gr}(\mathbf{C}),\text{ at } (\tau,x).$$

Remark 3.2. *Let us point out that these notions of viscosity super and sub solution on the closed set* $\operatorname{gr}(\mathbf{C})$ *coincide, in our case, with the definitions introduced by Ishii in [17] for the boundary value problem of the Dirichlet type. Indeed, let us set* $\mathsf{x} = (\tau,x)$ *and* $\mathsf{p} = (\theta,\zeta)$*, and consider the functions*

$$F(\mathsf{x},u,\mathsf{p}) = -\lambda u - \theta + H(x,\zeta),$$

$$B_*(\mathsf{x},u,\mathsf{p}) = F(\mathsf{x},u,\mathsf{p}) - \mathfrak{S}_{\mathbf{C}}(\tau,x,-\zeta) \quad \text{and} \quad B^*(\mathsf{x},u,\mathsf{p}) = F(\mathsf{x},u,\mathsf{p}) + \mathfrak{S}_{\mathbf{C}}(\tau,x,\zeta).$$

From Lemma 4, it will follow that B_* *and* B^* *are lower and upper semicontinuous functions, respectively. Let us set* $\Omega = \mathit{int}(\operatorname{gr}(\mathbf{C}))$*, and suppose* $\Omega \neq \emptyset$ *and that* $\operatorname{gr}(\mathbf{C}) = \overline{\Omega}$*; this is not necessarily always the case as for example if* $\mathit{int}(\operatorname{gr}(\mathbf{C})) = \emptyset$*. Under these circumstances, we have that being a viscosity supersolution in* $\operatorname{gr}(\mathbf{C})$ *of*

(9) *is the same as being a viscosity supersolution in the Ishii sense (see [17, page 107]) of the boundary value problem*

$$\begin{cases} F(\mathbf{x}, u, \nabla u) = 0, & \textit{in } \Omega, \\ B^*(\mathbf{x}, u, \nabla u) = 0 \quad \textit{or} \quad F(\mathbf{x}, u, \nabla u) = 0, & \textit{on } \partial\Omega. \end{cases}$$

Similarly, the same happens for a viscosity subsolution in $\operatorname{gr}(\mathbf{C})$ *of* (10)*, however in this case it corresponds to the boundary value problem*

$$\begin{cases} F(\mathbf{x}, u, \nabla u) = 0, & \textit{in } \Omega, \\ B_*(\mathbf{x}, u, \nabla u) = 0 \quad \textit{or} \quad F(\mathbf{x}, u, \nabla u) = 0, & \textit{on } \partial\Omega. \end{cases}$$

Notice too that if for some function B, we have that B_* and B^* are its lower and upper semicontinuous envelopes, respectively, then Theorem 3.3 can be stated as follows:

The mapping $\vartheta\colon \operatorname{gr}(\mathbf{C}) \subset \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ is the unique bounded and continuous viscosity solution in the Ishii (see e.g., [17]) of the boundary value problem

$$\begin{cases} F(\mathbf{x}, u, \nabla u) = 0, & \text{in } \Omega, \\ B(\mathbf{x}, u, \nabla u) = 0 \quad \text{or} \quad F(\mathbf{x}, u, \nabla u) = 0, & \text{on } \partial\Omega. \end{cases}$$

Apparently, the existence of such function B can be justified, although it could be a highly discontinuous function. For example, one can take

$$B(\mathsf{x}, u, p) = \begin{cases} B_*(\mathsf{x}, u, p) & \textit{if } p \in \mathbb{Q}^N \\ B^*(\mathsf{x}, u, p) & \textit{otherwise.} \end{cases}$$

The function $\mathfrak{S}_{\mathbf{C}}$ plays a fundamental role in this work, and it is what provides the information required for characterizing the value function as unique solution to an HJB equation. As it may seem apparent, this function is seldom continuous. Nonetheless, the continuity of this function is not mandatory for the technique we use to prove Theorem 3.3, and we only require upper semicontinuous in the first two arguments, which under the Standing Assumptions holds true as proved below.

Lemma 4. *The mapping $\mathfrak{S}_{\mathbf{C}} : \mathbb{R} \times \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ is upper semicontinuous.*

Proof. From [8, Proposition 4.1] it follows that the mapping

$$(\tau, x) \mapsto \Upsilon(\tau, x) := m(x) \partial_P \operatorname{dist}_{\mathbf{C}(\tau)}(x),$$

has closed graph on $\operatorname{gr}(\mathbf{C})$. Since $\partial_P \operatorname{dist}_{\mathbf{C}(\tau)}(x)$ is a closed subset of $\mathbb{B}$ (thanks to (4)), it follows that the mapping Υ is upper semicontinuous; see for instance [1, Theorem 1.1.1]. Therefore, from the Maximum Theorem (see e.g., [1, Theorem 1.2.5]) it follows that for any given $\zeta \in \mathbb{R}^N$, the mapping $(\tau, x) \mapsto \mathfrak{S}_{\mathbf{C}}(\tau, x, \zeta)$ is upper semicontinuous. The conclusion follows then from noticing that for any $x, \zeta, \bar{\zeta} \in \mathbb{R}^N$ and $\tau \in \mathbb{R}$ we have.

$$\mathfrak{S}_{\mathbf{C}}(\tau, x, \zeta) \leq \mathfrak{S}_{\mathbf{C}}(\tau, x, \bar{\zeta}) + m(x)\|\zeta - \bar{\zeta}\|.$$

□

In order to prove our main result, we require several intermediate steps, many of them based on non-smooth analysis techniques, in particular, on invariance principles as in [16], which we develop in the next sections.

4 Monotonicity along trajectories and invariance

Let us start by studying monotonicity properties of the value function along trajectories and their relation with invariance principles. This is a key step for later providing a characterization of the value function in terms of an HJB equation as mentioned before.

Definition 4.1. *We say that* $\varphi\colon \operatorname{gr}(\mathbf{C}) \subset \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ *is*

1. nearly weakly decreasing for the optimal control problem of sweeping process *if for any* $(\tau, x) \in \operatorname{gr}(\mathbf{C})$*,* $T \geq \tau$ *and* $\varepsilon > 0$ *there is* $\bar{\alpha} \in \mathcal{A}$ *such that*

$$\varphi(\tau,x)+\varepsilon \geq \int_\tau^T e^{\lambda(\tau-t)}\ell(y_{\tau,x}^{\bar{\alpha}}(t),\bar{\alpha}(t))dt + e^{\lambda(\tau-T)}\varphi\left(T, y_{\tau,x}^{\bar{\alpha}}(T)\right).$$

2. strongly increasing for the optimal control problem of sweeping process *if for any* $(\tau, x) \in \operatorname{gr}(\mathbf{C})$*,* $T \geq \tau$ *and any* $\alpha \in \mathcal{A}$ *such that*

$$\varphi(\tau,x) \leq \int_\tau^T e^{\lambda(\tau-t)}\ell(y_{\tau,x}^{\alpha}(t),\alpha(t))dt + e^{\lambda(\tau-T)}\varphi\left(T, y_{\tau,x}^{\alpha}(T)\right).$$

In the light of the Dynamic Programming Principle (Lemma 3), it follows that the value function defined in (1) is both, nearly weakly decreasing and strongly increasing for the optimal control problem of sweeping processes. The following lemma shows that any function $\varphi\colon \operatorname{gr}(\mathbf{C}) \subset \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ that is nearly weakly decreasing or strongly increasing in the sense of Definition 4.1 can be compared with respect to the value function.

Lemma 5. *Let* $\varphi\colon \operatorname{gr}(\mathbf{C}) \subset \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ *be a bounded function.*

1. *If* φ *is nearly weakly decreasing for the optimal control problem of sweeping processes, then* $\vartheta(\tau, x) \leq \varphi(\tau, x)$ *for all* $(\tau, x) \in \operatorname{gr}(\mathbf{C})$*.*
2. *If* φ *is strongly increasing for the optimal control problem of sweeping processes, then* $\vartheta(\tau, x) \geq \varphi(\tau, x)$ *for all* $(\tau, x) \in \operatorname{gr}(\mathbf{C})$*.*

Proof. First of all, note that for any $(\tau, x) \in \operatorname{gr}(\mathbf{C})$, $\alpha \in \mathcal{A}$ and $T \geq \tau$ we have that $\left(T, y_{\tau,x}^{\alpha}(T)\right) \in \operatorname{gr}(\mathbf{C})$ and thus, since φ is bounded on $\operatorname{gr}(\mathbf{C})$, it follows that

$$\lim_{T\to+\infty} e^{\lambda(\tau-T)}\varphi\left(T, y_{\tau,x}^{\alpha}(T)\right) = 0. \tag{12}$$

On the one hand, suppose that φ is nearly weakly decreasing for the optimal control problem of sweeping processes. Take $(\tau, x) \in \operatorname{gr}(\mathbf{C})$ and $\varepsilon > 0$ arbitrary, and let $T \geq \tau$

be such that $\beta_\ell e^{\lambda(\tau-T)} \leq \varepsilon\lambda$. Take $\bar{\alpha} \in \mathcal{A}$ given by Definition 4.1. Then

$$\varphi(\tau,x)+\varepsilon \geq \int_\tau^T e^{\lambda(\tau-t)}\ell(y_{\tau,x}^{\bar{\alpha}}(t),\bar{\alpha}(t))dt + e^{\lambda(\tau-T)}\varphi\left(T, y_{\tau,x}^{\bar{\alpha}}(T)\right).$$

Notice that

$$\varepsilon \geq \frac{\beta_\ell}{\lambda}e^{\lambda(\tau-T)} \geq \int_T^\infty e^{\lambda(\tau-t)}\ell(y_{\tau,x}^{\bar{\alpha}}(t),\bar{\alpha}(t))dt,$$

which implies that

$$\varphi(\tau,x)+2\varepsilon \geq \int_\tau^{+\infty} e^{\lambda(\tau-t)}\ell(y_{\tau,x}^{\bar{\alpha}}(t),\bar{\alpha}(t))dt + e^{\lambda(\tau-T)}\varphi\left(T, y_{\tau,x}^{\bar{\alpha}}(T)\right).$$

Therefore, by the definition of the value function we obtain the inequality

$$e^{\lambda(\tau-T)}\varphi\left(T, y_{\tau,x}^{\bar{\alpha}}(T)\right) + \vartheta(\tau,x) \leq \varphi(\tau,x) + 2\varepsilon.$$

From (12), letting $T \to +\infty$ we get $\vartheta(\tau,x) \leq \varphi(\tau,x)+2\varepsilon$, and since $\varepsilon > 0$ is arbitrary, we conclude.

On the other hand, suppose now that φ is strongly increasing for the optimal control problem of sweeping processes. Take $(\tau,x) \in \mathrm{gr}\,(\mathbf{C})$ and $\bar{\alpha} \in \mathcal{A}$ arbitrary. By definition, it follows that

$$\varphi(\tau,x) \leq \int_\tau^T e^{\lambda(\tau-t)}\ell(y_{\tau,x}^{\bar{\alpha}}(t),\bar{\alpha}(t)) + e^{\lambda(\tau-T)}\varphi\left(T, y_{\tau,x}^{\bar{\alpha}}(T)\right) \text{ for all } T \geq \tau.$$

Then by (12), letting $T \to +\infty$ and using the definition of the value function, we conclude the proof. □

In view of the previous results, we can state an intermediate characterization of the value function in terms of Definition 4.1.

Proposition 4.1. *The value function ϑ given by* (1) *is the unique bounded function defined on* $\mathrm{gr}\,(\mathbf{C})$ *that is nearly weakly decreasing and strongly increasing for the optimal control problem of sweeping process at the same time.*

4.1 Invariance with an unbounded dynamics

We now study some invariance principles and show how these are related to monotonicity properties of value functions. Let us begin by considering first the case with an unbounded dynamics. Let us set $\mathbf{P} = \mathbb{R} \times \mathbb{R} \times \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}$ and denote the corresponding variable $\mathbf{p} = (\sigma,\tau,x,z,\gamma)$. For a given function $\varphi\colon \mathrm{gr}\,(\mathbf{C}) \subset \mathbb{R}\times\mathbb{R}^N \to \mathbb{R}$ we define

$$\mathbf{Ep}(\varphi) := \left\{\mathbf{p} \in \mathbf{P}\colon e^{-\lambda\sigma}\varphi(\tau,x) + z \leq \gamma\right\}.$$

Let us consider the differential inclusion

$$\dot{\mathbf{p}}(t) \in \mathcal{F}(\mathbf{p}(t)), \qquad \text{a.e. on } [0,+\infty), \qquad \mathbf{p}(0) = \mathbf{p}_0, \tag{13}$$

where $\mathcal{F}\colon \mathbf{P} \rightrightarrows \mathbf{P}$ is the set-valued map given by

$$\mathcal{F}(\mathbf{p}) = \{(1,1)\} \times \left(\mathrm{co}\left\{\left(f(x,a), e^{-\lambda\sigma}\ell(x,a)\right) :\ a \in A\right\} - \mathcal{N}_{\mathbf{C}(\tau)}(x)\right) \times \{0\}\times\{0\}.$$

Note that

$$\mathrm{dom}(\mathcal{F}) = \left\{(\sigma,\tau,x,z,\gamma) \in \mathbf{P}\colon\ \mathcal{N}_{\mathbf{C}(\tau)}(x) \neq \emptyset\right\}.$$

But, $\mathcal{N}_{\mathbf{C}(\tau)}(x) \neq \emptyset$ if and only if $(\tau,x) \in \mathrm{gr}(\mathbf{C})$. Therefore, any absolutely continuous trajectory $t \mapsto \mathbf{p}(t) = (\sigma(t),\tau(t),y(t),z(t),\gamma(t))$ solution of (13) must satisfy as well $y(t) \in \mathbf{C}(\tau(t))$ for any $t \geq 0$.

We can now provide a link between the weak decreasing property for the optimal control problem of sweeping processes and a weak invariance principle for $(\mathbf{Ep}(\varphi),\mathcal{F})$. In our setting $(\mathbf{Ep}(\varphi),\mathcal{F})$ is said to be *weakly invariant* if for any $\mathbf{p}_0 \in \mathbf{Ep}(\varphi)$, there is an absolutely continuous trajectory $t \mapsto \mathbf{p}(t)$, solution of (13) such that $\mathbf{p}(t) \in \mathbf{Ep}(\varphi)$ for any $t \geq 0$

Proposition 4.2. *Let $\varphi : \mathrm{gr}(\mathbf{C}) \subset \mathbb{R}\times\mathbb{R}^N \to \mathbb{R}$ be a given continuous function and suppose that $(\mathbf{Ep}(\varphi),\mathcal{F})$ is weakly invariant. Then, φ is nearly weakly decreasing for the optimal control problem of sweeping processes.*

Proof. Let $(\tau_0,x_0) \in \mathrm{gr}(\mathbf{C})$, $\varepsilon > 0$ and $T \geq \tau_0$. Set $h = T - \tau_0$, and define $\mathbf{p}_0 = (0,\tau_0,x_0,0,\varphi(\tau_0,x_0))$. It is clear that $\mathbf{p}_0 \in \mathbf{Ep}(\varphi)$. Thus, since $(\mathbf{Ep}(\varphi),\mathcal{F})$ is weakly invariant, we can find an absolutely continuous trajectory $t \mapsto \mathbf{p}(t) = (\sigma(t),\tau(t),y(t),z(t),\gamma(t))$ solution of (13) such that

$$e^{-\lambda\sigma(t)}\varphi(\tau(t),y(t)) + z(t) \leq \gamma(t), \qquad \forall t \geq 0.$$

By the definition of $\mathcal{F}$ we get that $\sigma(t) = t$, $\tau(t) = t + \tau_0$ and $\gamma(t) = \varphi(\tau_0,x_0)$ for any $t \geq 0$. Moreover, it follows that for a.e. $t \in [0,+\infty)$ one has

$$\begin{pmatrix}\dot{y}(t)\\ \dot{z}(t)\end{pmatrix} \in \mathrm{co}\left\{\begin{pmatrix}f(y(t),a)\\ e^{-\lambda t}\ell(y(t),a)\end{pmatrix}\ :\ a \in A\right\} - \left(\mathcal{N}_{\mathbf{C}(t+\tau_0)}(y(t)) \times \{0\}\right).$$

By the Relaxation Theorem (see Lemma 2 and Remark 2.1), there is a sequence $\{(y_n,z_n)\}$ that converges uniformly on $[0,h]$ to (y,z), with $(y_n(0),z_n(0)) = (x_0,0)$ and such that for a.e. $t \in [0,h]$

$$\begin{pmatrix}\dot{y}_n(t)\\ \dot{z}_n(t)\end{pmatrix} \in \left\{\begin{pmatrix}f(y_n(t),a)\\ e^{-\lambda t}\ell(y_n(t),a)\end{pmatrix}\ :\ a \in A\right\} - \left(\mathcal{N}_{\mathbf{C}(t+\tau_0)}(y_n(t)) \times \{0\}\right).$$

By [20, Theorem 14.26], we have that $t \mapsto \mathcal{N}_{\mathbf{C}(t+\tau_0)}(y_n(t))$ is measurable, and then by [20, Theorem 14.16] we can find some measurable map $t \mapsto (\alpha_n(t),\eta_n(t))$ with $\alpha_n(t) \in A$ and $\eta_n(t) \in \mathcal{N}_{\mathbf{C}(t+\tau_0)}(y_n(t))$ defined a.e. on $[0,h]$ such that

$$\dot{y}_n(t) = f(y_n(t),\alpha_n(t)) - \eta_n(t) \text{ and } \dot{z}_n(t) = e^{-\lambda t}\ell(y_n(t),\alpha_n(t)), \qquad \text{a.e. on } [0,h].$$

Consider $\bar{y}_n\colon [\tau_0,+\infty) \to \mathbb{R}^N$ and $\bar{\alpha}_n : [\tau_0,+\infty) \to A$, given by

- $\bar{y}_n(t+\tau_0) = y_n(t)$ and $\bar{\alpha}_n(t+\tau_0) = \alpha_n(t)$ if $t \in [0,h]$;

- $\bar{y}_n(t+\tau_0) = y^{\bar{\alpha}}_{T,y_n(T)}(t)$ and $\bar{\alpha}_n(t+\tau_0) = \bar{\alpha}(t)$ if $t > h$,

where $\bar{\alpha}(t) \equiv a$ for some $a \in \mathcal{A}$ arbitrary. It is clear that $\bar{y}_n$ is a solution of (2) with initial condition $\bar{y}_n(\tau_0) = x_0$ and related to the control $t \mapsto \bar{\alpha}_n(t)$. However, this controlled sweeping process has a unique solution (Lemma 1). Therefore, $\bar{y}_n = y^{\bar{\alpha}_n}_{\tau_0,x_0}$ and since $z_n(0) = 0$, we get

$$z_n(h) = \int_{\tau_0}^{T} e^{\lambda(\tau_0 - s)} \ell(y^{\bar{\alpha}_n}_{\tau_0,x_0}(s), \bar{\alpha}_n(s)) ds.$$

Take now $n \in \mathbb{N}$ large enough such that

$$e^{\lambda(\tau_0 - T)} \varphi(T, y^{\bar{\alpha}_n}_{\tau_0,x_0}(T)) \leq e^{\lambda(\tau_0 - T)} \varphi(T, y(h)) + \frac{\varepsilon}{2} \quad \text{and} \quad z_n(h) \leq z(h) + \frac{\varepsilon}{2}.$$

This can be done since $y^{\bar{\alpha}_n}_{\tau_0,x_0}(T) = y_n(h) \to y(h)$ and we are also assuming that φ is continuous function. It follows then

$$e^{\lambda(\tau_0 - T)} \varphi(T, y^{\bar{\alpha}_n}_{\tau_0,x_0}(T)) + \int_{\tau_0}^{T} e^{\lambda(\tau_0 - s)} \ell(y^{\bar{\alpha}_n}_{\tau_0,x_0}(s), \bar{\alpha}_n(s)) ds \leq \varphi(\tau_0, x_0) + \varepsilon.$$

This completes the proof. □

In a similar way, we can establish a link between strong increasingness for the optimal control problem of sweeping processes and a strong invariance principle. Given a function $\varphi : \mathrm{gr}\,(\mathbf{C}) \subset \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ we define

$$\mathbf{Hy}(\varphi) := \left\{ \mathbf{p} \in \mathbf{P} \colon e^{-\lambda \sigma} \varphi(\tau, x) + z \geq \gamma \right\}.$$

We say that $(\mathbf{Hy}(\varphi), \mathcal{F})$ is *strongly invariant* if for any $\mathbf{p}_0 \in \mathbf{Hy}(\varphi)$ and any absolutely continuous trajectory $t \mapsto \mathbf{p}(t)$ solution of (13) we have $\mathbf{p}(t) \in \mathbf{Hy}(\varphi)$ for any $t \geq 0$.

Proposition 4.3. *Let $\varphi : \mathrm{gr}\,(\mathbf{C}) \subset \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ be a given function and suppose that $(\mathbf{Hy}(\varphi), \mathcal{F})$ is strongly invariant. Then, φ is strongly increasing for the optimal control problem of sweeping processes.*

Proof. Let $(\tau_0, x_0) \in \mathrm{gr}\,(\mathbf{C})$ and $\alpha \in \mathcal{A}$. Consider the absolutely continuous curve defined for $t \geq 0$ given by

$$\mathbf{p}_\alpha(t) := \left(t, t + \tau_0, y^{\alpha}_{\tau_0,x_0}(t + \tau_0), \int_{\tau_0}^{t+\tau_0} e^{\lambda(\tau_0 - s)} \ell(y^{\alpha}_{\tau_0,x_0}(s), \alpha(s)) ds, \varphi(\tau_0, x_0) \right).$$

Clearly, $t \mapsto \mathbf{p}_\alpha(t)$ is a solution of (13) with $\mathbf{p}_0 = (0, \tau_0, x_0, 0, \varphi(\tau_0, x_0))$. Also, $\mathbf{p}_0 \in \mathbf{Hy}(\varphi)$ and thus, since $(\mathbf{Hy}(\varphi), \mathcal{F})$ is strongly invariant, we must have that the trajectory $\mathbf{p}_\alpha(t) \in \mathbf{Hy}(\varphi)$ for any $t \geq 0$. In other words, for any $t \geq 0$ we have

$$e^{\lambda(\tau_0 - t - \tau_0)}\varphi(t + \tau_0, y^\alpha_{\tau_0,x_0}(t + \tau_0)) + \int_{\tau_0}^{t+\tau_0} e^{\lambda(\tau_0 - s)}\ell(y^\alpha_{\tau_0,x_0}(s), \alpha(s))ds \geq \varphi(\tau_0, x_0).$$

But, since $\alpha \in \mathcal{A}$ is arbitrary, this last inequality is equivalent to say that φ is strongly increasing for the optimal control problem of sweeping processes, so the conclusion follows. □

4.2 Invariance with a bounded dynamics

Now, in order to characterize invariance principles in terms of variational inequalities, and later on in terms of HJB inequalities, it is more convenient to work with an unconstrained and bounded dynamics. Indeed, from Lemma 1, a mapping $y(\cdot)$ solves (2) if and only if it solves the unconstrained and bounded dynamics (5). Consequently, we can get the following variant of Proposition 4.2 and Proposition 4.3 for the set-valued map $\Gamma : \mathbf{P} \rightrightarrows \mathbf{P}$ given by

$$\Gamma(\mathbf{p}) = \{(1,1)\}\times\big(\mathrm{co}\,\big\{\big(f(x,a), e^{-\lambda\sigma}\ell(x,a)\big) :\ a \in A\big\} - m(x)\partial_P\, \mathrm{dist}_{\mathbf{C}(\tau)}(x)\big)\times\{0\}\times\{0\}$$

and the differential inclusion

$$\dot{\mathbf{p}}(t) \in \Gamma(\mathbf{p}(t)), \qquad \text{a.e. on } [0, +\infty), \qquad \mathbf{p}(0) = \mathbf{p}. \tag{14}$$

Proposition 4.4. *Let $\varphi : \mathrm{gr}\,(\mathbf{C}) \subset \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ be a given function.*

1. *Suppose that φ is continuous and $(\mathbf{Ep}(\varphi), \Gamma)$ is weakly invariant, that is, for any $\mathbf{p} \in \mathbf{Ep}(\varphi)$, there is an absolutely continuous trajectory $t \mapsto \mathbf{p}(t)$, solution of (14) such that $\mathbf{p}(t) \in \mathbf{Ep}(\varphi)$ for any $t \geq 0$. Then, φ is nearly weakly decreasing for the optimal control problem of the sweeping process.*
2. *Suppose that $(\mathbf{Hy}(\varphi), \Gamma)$ is strongly invariant, that is, for any $\mathbf{p} \in \mathbf{Hy}(\varphi)$ and any absolutely continuous trajectory $t \mapsto \mathbf{p}(t)$ solution of (14) we have that $\mathbf{p}(t) \in \mathbf{Hy}(\varphi)$ for any $t \geq 0$. Then, φ is strongly increasing for the optimal control problem of the sweeping process.*

Proof. It follows from Proposition 4.2 and Proposition 4.3, combined with the fact that (13) is equivalent, as a dynamical system, to (14). The latter being a direct consequence of formula (4) and [23, Theorem 3.1]. □

4.3 Characterization of Invariance Principles

Let us now focus on some characterizations of the invariance principles discussed in the preceding part. For the sake of clarity, let us write the multifunction Γ as follows

$$\Gamma(\mathbf{p}) = \Gamma_C(\mathbf{p}) - \Gamma_D(\mathbf{p}),$$

where

$$\Gamma_C(\mathbf{p}) := \text{co}\left\{\left(1, 1, f(x,a), e^{-\lambda\sigma}\ell(x,a), 0\right) : a \in A\right\}$$

and

$$\Gamma_D(\mathbf{p}) := \left\{(0, 0, m(x)\xi, 0, 0) : \xi \in \partial_P \operatorname{dist}_{\mathbf{C}(\tau)}(x)\right\}.$$

It is clear that under the standing assumptions, the set-valued map $\mathbf{p} \mapsto \Gamma_C(\mathbf{p})$ is continuous, with nonempty compact convex images. Moreover, as pointed out in the proof of Lemma 4, it also holds that the set-valued map $\mathbf{p} \mapsto \Gamma_D(\mathbf{p})$ is upper semi-continuous; see also [7, Theorem 5.8]. It is not difficult to see that Γ_D has nonempty compact convex images, and has linear growth with respect to the $\mathbf{p}$ variable.

Let us now show a Hamiltonian strong invariance characterization for the initial value problem (14).

Theorem 4.1. *Let $\mathcal{K} \subset \mathbb{R} \times \text{gr}\,(\mathbf{C}) \times \mathbb{R} \times \mathbb{R}$ be a nonempty closed set. Then $(\mathcal{K}, \Gamma)$ is strongly invariant for (14) if and only if for every $\mathbf{p} \in \mathcal{K}$, we have*

$$\min_{v \in -\Gamma_D(\mathbf{p})} \langle v, \eta \rangle + \max_{v \in \Gamma_C(\mathbf{p})} \langle v, \eta \rangle \leq 0, \qquad \forall\, \eta \in \mathcal{N}_{\mathcal{K}}(\mathbf{p}).$$

Proof. **(Necessity)**. Fix $\mathbf{p}_0 = (\sigma_0, \tau_0, x_0, z_0, \gamma_0) \in \mathcal{K}$, $\eta_0 \in \mathcal{N}_{\mathcal{K}}(\mathbf{p}_0)$. Take $v_0 \in \Gamma_C(\mathbf{p}_0)$ such that

$$\max_{v \in \Gamma_C(\mathbf{p}_0)} \langle v, \eta_0 \rangle = \langle v_0, \eta_0 \rangle.$$

Since $\max_{v \in S} \langle v, \eta_0 \rangle = \max_{v \in \text{co}(S)} \langle v, \eta_0 \rangle$, for any nonempty compact set $S \subset \mathbf{P}$ By definition of Γ_C, we may assume that there is $a_0 \in A$ such that

$$v_0 = \left(1, 1, f(x_0, a_0), e^{-\lambda\sigma_0}\ell(x_0, a_0), 0\right).$$

Let us consider the mapping $\phi : \mathbf{P} \to \mathbf{P}$ given by

$$\phi(\mathbf{p}) := \left(1, 1, f(x, a_0), e^{-\lambda\sigma}\ell(x, a_0), 0\right), \qquad \forall \mathbf{p} = (\sigma, \tau, x, z, \gamma).$$

Clearly, ϕ is a continuous selection of Γ_C such that $\phi(\mathbf{p}_0) = v_0$. Since, for any $\mathbf{p} = (\sigma, \tau, x, z, \gamma) \in \mathcal{K}$, the Cauchy problem

$$\dot{\mathbf{p}}(t) \in -\Gamma_D(\mathbf{p}(t)) + \phi(\mathbf{p}(t)), \qquad \text{a.e. on } [0+\infty), \qquad \mathbf{p}(0) = \mathbf{p}. \tag{15}$$

admits solutions, it follows from the strong invariance of $\mathcal{K}$ with respect to (14) that $\mathcal{K}$ is also weakly invariant with respect to (15). Thus, from the classical Weak Invariance Theorem (e.g., [10, Theorem 4.2.10]), we obtain that for all $\mathbf{p} = (\sigma, \tau, x, z, \gamma) \in \mathcal{K}$

$$\min_{v \in -\Gamma_D(\mathbf{p})} \langle v, \eta \rangle + \langle \phi(\mathbf{p}), \eta \rangle \leq 0, \qquad \forall\, \eta \in \mathcal{N}_{\mathcal{K}}(\mathbf{p}).$$

In particular, by taking $\mathbf{p} = \mathbf{p}_0$ and $\eta = \eta_0$ in the latter inequality, we obtain

$$\min_{v \in -\Gamma_D(\mathbf{p}_0)} \langle v, \eta \rangle + \max_{v \in \Gamma_C(\mathbf{p}_0)} \langle v, \eta_0 \rangle \leq 0.$$

Since $\mathbf{p}_0 = (\sigma_0, \tau_0, x_0, z_0, \gamma_0) \in \mathcal{K}$ and $\eta_0 \in \mathcal{N}_{\mathcal{K}}(\mathbf{p}_0)$ were arbitrarily chosen, the first part of the proof is complete.

(Sufficiency). Take $\mathbf{p}_0 = (\sigma_0, \tau_0, x_0, z_0, \gamma_0) \in \mathcal{K}$ and any $T > 0$ and a solution $\mathbf{p} : [0,T] \to \mathbf{P}$ of (14) with initial condition $\mathbf{p}(0) = \mathbf{p}_0$. We aim at proving that $\mathbf{p}(t) = (\sigma(t), \tau(t), y(t), z(t), \gamma(t)) \in \mathcal{K}$ for each $t \in [0,T]$.

To this aim, observe first that the structure of the right hand side of (14) implies that $\sigma(t) = \sigma_0 + t$, $\tau(t) = \tau_0 + t$ and $\gamma(t) = \gamma_0$. Moreover, through standard measurable selection theorems (see, e.g., [1, Proposition 1.4 and Theorem 1.1]), the existence of some measurable functions $g : [0,T] \to \mathbf{P}$ and $\xi : [0,T] \to \mathbb{R}^N$ such that

$$\xi(t) \in \partial_P \operatorname{dist}_{\mathbf{C}(\tau(t))}(y(t)) \quad \text{and} \quad g(t) \in \Gamma_C(\mathbf{p}(t)), \qquad \text{for a.e. } t \in [0,T],$$

and for which

$$\dot{\mathbf{p}}(t) = (0, 0, -m(y(t))\xi(t), 0, 0) + g(t).$$

In particular, since $(\tau_0, x_0) \in \operatorname{gr}(\mathbf{C})$, we have that $y(0) = x_0 \in \mathbf{C}(\tau_0)$. Besides,

$$\dot{y}(t) \in \operatorname{co}(f(y(t), A)) - m(y(t))\xi(t), \qquad \text{for a.e. } t \in [0,T],$$

it follows that $y(t) \in \mathbf{C}(\tau_0 + t)$ for any $t \in [0,T]$; see for instance [23].

Using the notation $\mathbf{p} = (\sigma, \tau, x, z, \gamma)$, let us define the set

$$S_\rho := \left\{ (t, \mathbf{p}) \in [0,T] \times \mathbf{P} \colon d_{\mathbf{C}(\tau(t))}(x) < \rho,\ d_{\mathbf{C}(\tau)}(y(t)) < \rho,\ d_{\mathbf{C}(\tau)}(x) < \frac{\rho}{2} \right\},$$

and the set-valued map $\tilde{G} \colon S_\rho \rightrightarrows \mathbf{P}$ given by

$$\tilde{G}(t, \mathbf{p}) := \{ v \in \Gamma(\mathbf{p}) \colon \langle \dot{\mathbf{p}}(t) - v, \mathbf{p}(t) - \mathbf{p} \rangle \leq C(t,x) \|\mathbf{p}(t) - \mathbf{p}\|^2 + 2m(x)|\tau(t) - \tau| \}, \quad (16)$$

where

$$C(t,x) := \kappa_f + e^{-\lambda\sigma_0}\kappa_\ell + \lambda\beta_\ell + m(y(t))\frac{8}{\rho} + m(x)\frac{16}{\rho} + \beta_f.$$

Notice that $(t, \mathbf{p}(t)) \in S_\rho$ for any $t \in [0,T]$. In particular, $S_\rho \neq \emptyset$. Finally, let us define the multifunction $\tilde{F} : [0,T] \times \mathbf{P} \rightrightarrows \mathbf{P}$, which is an extension of $\tilde{G}$ to the whole space $[0,T] \times \mathbf{P}$, as follows:

$$\tilde{F}(t, \mathbf{p}) = \begin{cases} \tilde{G}(t, \mathbf{p}) & \text{if } (t, \mathbf{p}) \in S_\rho, \\ \operatorname{co}\{\Gamma(\mathbf{p}), \{0\}\} & \text{otherwise.} \end{cases}$$

We claim that the set-valued map $\tilde{F}$ has the following properties:

(i) $\emptyset \neq \tilde{F}(t, \mathbf{p})$, for all $\mathbf{p} \in \mathbf{P}$ and a.e. $t \in [0,T]$;

(ii) $\min_{v \in \tilde{F}(t,\mathbf{p})} \langle v, \eta \rangle \leq 0$ for all $\mathbf{p} \in \mathcal{K}$, $\eta \in \mathcal{N}_{\mathcal{K}}(\mathbf{p})$ and a.e. $t \in [0,T]$;

(iii) $\tilde{F}$ has compact and convex values and is almost upper semicontinuous on $[0,T] \times \mathbb{R}^N$, i.e., for every closed interval $I \subset [0,T]$ and each $\varepsilon > 0$, there exists a closed set $\mathcal{N}_\varepsilon \subset I$ with Lebesgue measure $\mu(I \setminus \mathcal{N}_\varepsilon) < \varepsilon$ and so that the graph $\operatorname{gr}\left(\tilde{F}\right)$ is closed in $\mathcal{N}_\varepsilon \times \mathbb{R}^N \times \mathbb{R}^N$.

In order to see the nonemptyness of $\tilde{F}$, we establish first some inequalities. Fix $t \in [0,T]$ such that $\dot{\mathbf{p}}(t)$ exists and take $\mathbf{p} = (\sigma, \tau, x, z, \gamma) \in \mathbf{P}$ such that $(t, \mathbf{p}) \in S_\rho$. By virtue of [1, Proposition 1.1.6], the Lipschitz continuity of $x \mapsto f(x,a)$ and of the map $x \mapsto \ell(x,a)$, there exists $\bar{g}_t \in \Gamma_C(\mathbf{p})$ such that

$$\|g(t) - \bar{g}_t\| \leq \left(\kappa_f + e^{-\lambda\sigma_0}\kappa_\ell + \lambda\beta_\ell\right)\|\mathbf{p}(t) - \mathbf{p}\|. \tag{17}$$

Let us now observe that, in view of the prox-regularity assumption **(H2.i)**, the sets $\partial_P \operatorname{dist}_{\mathbf{C}(\tau(t))}(y(t))$ and $\partial_P \operatorname{dist}_{\mathbf{C}(\tau)}(x)$ are both non-empty. On the one hand, for any $\xi(t) \in \partial_P \operatorname{dist}_{\mathbf{C}(\tau(t))}(y(t))$, from [7, Theorem 2.14] we have that

$$\langle \xi(t), x - y(t)\rangle \leq \operatorname{dist}_{\mathbf{C}(\tau(t))}(x) + \frac{8}{\rho}\|y(t) - x\|^2, \tag{18}$$

and that for any $\xi \in \partial_P \operatorname{dist}_{\mathbf{C}(\tau)}(x)$, one has that

$$\langle \xi, y(t) - x\rangle \leq \operatorname{dist}_{\mathbf{C}(\tau)}(y(t)) - \operatorname{dist}_{\mathbf{C}(\tau)}(x) + \frac{16}{\rho}\|y(t) - x\|^2. \tag{19}$$

Define $\bar{v} = (0, 0, -m(x)\xi, 0, 0) + \bar{g}_t$, where ξ is an arbitrary element of the set $\partial_P \operatorname{dist}_{\mathbf{C}(\tau)}(x)$ (the latter set being non-empty). Then, by taking into account (17), (18) and (19), we obtain

$$\begin{aligned}
&\langle \dot{\mathbf{p}}(t) - \bar{v}_t, \mathbf{p}(t) - \mathbf{p}\rangle \\
&= \langle g(t) - \bar{g}_t, \mathbf{p}(t) - \mathbf{p}\rangle - m(y(t))\langle \xi(t), y(t) - x\rangle + m(x)\langle \xi, y(t) - x\rangle \\
&= \langle g(t) - \bar{g}, \mathbf{p}(t) - \mathbf{p}\rangle + m(y(t))\langle \xi(t), x - y(t)\rangle + m(x)\langle \xi, y(t) - x\rangle \\
&\leq (\kappa_f + e^{-\lambda\sigma_0}\kappa_\ell + \lambda\beta_\ell)\|\mathbf{p}(t) - \mathbf{p}\|^2 + m(y(t))\frac{8}{\rho}\|y(t) - x\|^2 + m(x)\frac{16}{\rho}\|y(t) - x\|^2 \\
&+ m(y(t))(\operatorname{dist}_{\mathbf{C}(\tau)}(x) - \operatorname{dist}_{\mathbf{C}(\tau(t))}(y(t))) + m(x)(\operatorname{dist}_{\mathbf{C}(\tau)}(y(t)) - \operatorname{dist}_{\mathbf{C}(\tau)}(x)) \\
&= (\kappa_f + e^{-\lambda\sigma_0}\kappa_\ell + \lambda\beta_\ell)\|\mathbf{p}(t) - \mathbf{p}\|^2 + m(y(t))\frac{8}{\rho}\|y(t) - x\|^2 + m(x)\frac{16}{\rho}\|y(t) - x\|^2 \\
&+ (m(y(t)) - m(x))(\operatorname{dist}_{\mathbf{C}(\tau)}(x) - \operatorname{dist}_{\mathbf{C}(\tau(t))}(y(t))) \\
&+ m(x)(\operatorname{dist}_{\mathbf{C}(\tau)}(y(t)) - \operatorname{dist}_{\mathbf{C}(\tau(t))}(y(t)) + \operatorname{dist}_{\mathbf{C}(\tau)}(x) - \operatorname{dist}_{\mathbf{C}(\tau)}(x)) \\
&\leq (\kappa_f + e^{-\lambda\sigma_0}\kappa_\ell + \lambda\beta_\ell)\|\mathbf{p}(t) - \mathbf{p}\|^2 + m(y(t))\frac{8}{\rho}\|y(t) - x\|^2 + m(x)\frac{16}{\rho}\|y(t) - x\|^2 \\
&+ \beta_f\|y(t) - x\|^2 + 2m(x)|\tau(t) - \tau| \\
&= C(t,x)\|\mathbf{p}(t) - \mathbf{p}\|^2 + 2m(x)|\tau(t) - \tau|.
\end{aligned}$$

Thus, $\bar{v} \in \tilde{F}(t, \mathbf{p})$, which shows (i).

We observe that in the above argument the only condition imposed on ξ was $\xi \in \partial_P \operatorname{dist}_{\mathbf{C}(\tau)}(x)$. Then, given any $\mathbf{p} \in \mathcal{K}$ and $\eta \in \mathcal{N}_{\mathcal{K}}(\mathbf{p})$, if $(t, \mathbf{p}) \notin S_\rho$, then $0 \in \tilde{F}(t, \mathbf{p})$ and so

$$\min_{v \in \tilde{F}(t,\mathbf{p})} \langle v, \eta \rangle \leq 0.$$

Otherwise, we can choose $\bar{\xi} \in \partial_P \operatorname{dist}_{\mathbf{C}(\tau)}(x)$ such that

$$\min_{v \in \Gamma_D(\mathbf{p})} \langle v, \eta \rangle = \langle (0, 0, -m(x)\bar{\xi}, 0, 0), \eta \rangle.$$

Therefore, by taking $\bar{v}_t = \big(0, 0, -m(x)\bar{\xi}, 0, 0\big) + \bar{g}_t \in \tilde{F}(t, \mathbf{p})$, we have

$$\min_{v \in \tilde{F}(t,\mathbf{p})} \langle v, \eta \rangle \leq \langle \bar{v}_t, \eta \rangle \leq \min_{v \in -\Gamma_D(\mathbf{p})} \langle v, \eta \rangle + \max_{v \in \Gamma_C(\mathbf{p})} \langle v, \eta \rangle \leq 0,$$

where the last inequality holds true by hypothesis. Thus, we have proved (ii). We point out that (iii) is easily checked, so we left the details to the reader.
Next, by a known weak flow invariance result (see, e.g., [13, Theorem 1]), the Cauchy problem

$$\begin{cases} \dot{\mathbf{p}}(t) & \in & \tilde{F}(t, \mathbf{p}(t)), \\ \mathbf{p}(0) & = & \mathbf{p}_0 \end{cases}$$

admits a solution $\tilde{\mathbf{p}} : [0, \tilde{T}] \to P$ such that $\tilde{\mathbf{p}}(t) \in \mathcal{K}$ for all $t \in [0, \tilde{T}]$. Set $T' = \min\{T, \tilde{T}\}$. We claim that $\mathbf{p}(t) = \tilde{\mathbf{p}}(t)$ for all $t \in [\tau_0, T']$. Indeed, we first observe that $\tau(t) = \tilde{\tau}(t)$ since $\tau(0) = \tilde{\tau}(0)$ and $\dot{\tau}(t) = \dot{\tilde{\tau}}(t) \equiv 1$ a.e. $t \in [0, T']$. Furthermore, since $\tilde{\mathbf{p}}$ is a $\tilde{F}$-trajectory, by taking $v = \dot{\tilde{\mathbf{p}}}(t)$ and $\mathbf{p} = \tilde{\mathbf{p}}(t)$ in (16), we obtain for a.e. $t \in [0, T']$,

$$\langle \dot{\mathbf{p}}(t) - \dot{\tilde{\mathbf{p}}}(t), \mathbf{p}(t) - \tilde{\mathbf{p}}(t) \rangle \leq C(t, \tilde{x}(t)) \|\mathbf{p}(t) - \tilde{\mathbf{p}}(t)\|^2.$$

Since $\mathbf{p}(0) = \tilde{\mathbf{p}}(0) = p_0$, it follows from Gronwall's Lemma that $\mathbf{p}(t) = \tilde{\mathbf{p}}(t)$ in $[0, T']$. If $T' < T$, we can repeat the same arguments starting from the point $\mathbf{p}(T')$ in place of $\mathbf{p}_0$. This proves that $\mathbf{p}(t) \in \mathcal{K}$ for all $t \in [0, T]$. Since the trajectory $\mathbf{p}(\cdot)$ was arbitrarily chosen, the proof is complete. ☐

In a similar note, we can prove a characterization for the weak invariance.
Theorem 4.2. *Let $\mathcal{K} \subset \mathbb{R} \times \operatorname{gr}(\mathbf{C}) \times \mathbb{R} \times \mathbb{R}$ be a nonempty closed set. Then, $(\mathcal{K}, \Gamma)$ is weakly invariant for (14) if and only if for every $\mathbf{p}_0 \in \mathcal{K}$, we have*

$$\min_{v \in -\Gamma_D(\mathbf{p}_0)} \langle v, \eta \rangle + \min_{v \in \Gamma_C(\mathbf{p}_0)} \langle v, \eta \rangle \leq 0, \qquad \forall \eta \in \mathcal{N}_{\mathcal{K}}(\mathbf{p}_0). \tag{20}$$

Proof. This is a straightforward consequence of [10, theorem 4.2.4] and the fact that

$$\min_{v \in -\Gamma_D(\mathbf{p}_0)} \langle v, \eta \rangle + \min_{v \in \Gamma_C(\mathbf{p}_0)} \langle v, \eta \rangle = \min_{v \in \Gamma(\mathbf{p}_0)} \langle v, \eta \rangle$$

☐

5 Hamiltonian characterization of monotonicity

This section aims at providing some characterizations in terms of HJB inequalities for a function to be nearly weakly decreasing and strongly increasing for the optimal control problem of sweeping processes. These characterizations will be proven by means of the proximal sub- and superdifferential. First we recall the notion of viscosity subgradient.

Definition 5.1. *Let $\psi : \Omega \subset \mathbb{R}^k \to \mathbb{R}$ be an l.s.c. function. A vector $\xi \in \mathbb{R}^k$ is called a viscosity subgradient of ψ at $x \in \Omega$ if and only if there exists a continuously differentiable function $g : \mathbb{R}^k \to \mathbb{R}$ so that*

$$\nabla g(x) = \xi \text{ and } \psi - g \text{ attains a local minimum at } x.$$

Remark 5.1. *The set of all viscosity subgradients of ψ at x is called the* viscosity subdifferential *and it is denoted by $D^-\varphi(x)$ or $\partial_D\varphi(x)$ (depending on the community); see for instance [2, Lemma II.1.7] or [10, Proposition 3.4.12].*

Observe that if for some $\sigma > 0$, we have

$$g(y) = \psi(x) + \langle \xi, y - x\rangle - \sigma|y - x|^2,$$

then ξ is actually a proximal subgradient of ψ at x; see [10, Theorem 1.2.5]. Recall that (see Section 2) the set of all proximal subgradients of ψ at x is called the proximal subdifferential, and is denoted by $\partial_P\varphi(x)$. Although the inclusion $\partial_P\varphi(x) \subset D^-\varphi(x)$ may be strict, it is possible to approximate any viscosity subgradient by a sequence of proximal subgradients; see, for instance, [10, Proposition 3.4.5].

As mentioned in Section 2, the proximal subdifferential and the proximal normal cone are related to each other via the relation

$$\xi \in \partial_P\psi(x) \quad \Longleftrightarrow \quad (\xi, -1) \in \mathcal{N}_{\mathrm{epi}(\psi)}\left(x, \psi(x)\right), \qquad \forall x \in \Omega.$$

The characterization for the nearly weakly decreasing case follows from rather standard arguments and its proof similar to what can be found in mainstream references in optimal control; see for instance [10, Theorem 4.5.7]. We provide the details of the proof for the sake of completeness. Recall that the usual (maximized) Hamiltonian is given by

$$H(x, \zeta) := \max_{a\in A} \{-\langle \zeta, f(x, a)\rangle - \ell(x, a)\}$$

and also that we have defined

$$\mathfrak{S}_{\mathbf{C}}(\tau, x, \zeta) := \sup\left\{\langle \zeta, v\rangle \colon v \in m(x)\partial_P \operatorname{dist}_{\mathbf{C}(\tau)}(x)\right\}.$$

Theorem 5.1. *Consider a given continuous function $\varphi\colon \operatorname{gr}(\mathbf{C}) \subset \mathbb{R}\times\mathbb{R}^N \to \mathbb{R}$. Then the following assertions are equivalent:*

(a) *φ is nearly weakly decreasing for the optimal control problem of sweeping processes.*

(b) *φ is a* proximal supersolution*, that is, for any $(\tau, x) \in \operatorname{gr}(\mathbf{C})$ we have*

$$-\theta + H(x, \zeta) + \mathfrak{S}_{\mathbf{C}}(\tau, x, \zeta) + \lambda\varphi(\tau, x) \geq 0, \qquad \forall(\theta, \zeta) \in \partial_P\varphi(\tau, x). \tag{21}$$

(c) *φ is a* viscosity supersolution*, that is, for any $(\tau, x) \in \mathrm{gr}\,(\mathbf{C})$ we have*

$$-\theta + H(x,\zeta) + \mathfrak{S}_{\mathbf{C}}(\tau,x,\zeta) + \lambda\varphi(\tau,x) \geq 0, \qquad \forall(\theta,\zeta) \in D^{-}\varphi(\tau,x). \tag{22}$$

Proof. The fact that **(c)** $\Rightarrow$ **(b)** is straightforward from the definition of the corresponding subdifferentials; see Remark 5.1. Consequently, the equivalence **(c)** $\Leftrightarrow$ **(b)** is actually a consequence of [10, Proposition 3.4.5] combined with Lemma 4.

Thus, to conclude it remains to prove that **(a)** $\Leftrightarrow$ **(b)**. Let us first prove the implication **(a)** $\Rightarrow$ **(b)**: Suppose that φ is nearly weakly decreasing for the optimal control problem of sweeping processes. Let $(\tau,x) \in \mathrm{gr}\,(\mathbf{C})$. If $\partial_P\varphi(\tau,x) = \emptyset$, then (22) trivially holds. Otherwise, assume that $\partial_P\varphi(\tau,x) \neq \emptyset$ and let $(\theta,\zeta) \in \partial_P\varphi(\tau,x)$. Since φ is nearly weakly decreasing, given $\varepsilon > 0$ and $T \geq \tau$, there exists $\bar{\alpha} \in \mathcal{A}$ such that for all

$$\varepsilon(T-\tau)+\varphi(\tau,x) \geq \int_\tau^T e^{\lambda(\tau-t)}\ell(y^{\bar{\alpha}}_{\tau,x}(t),\bar{\alpha}(t))dt + e^{\lambda(\tau-T)}\varphi(T,y^{\bar{\alpha}}_{\tau,x}(T)).$$

Moreover, by the definition of proximal subdifferential, there exist $\sigma,\delta > 0$ such that for all $T \in [\tau,\tau+\delta)$

$$\varphi(T,y^{\bar{\alpha}}_{\tau,x}(T)) \geq \varphi(\tau,x) + \theta(T-\tau) + \langle\zeta, y^{\bar{\alpha}}_{\tau,x}(T) - x\rangle - \eta(T,\tau),$$

where $\eta(T,\tau) := \sigma|T-\tau|^2 + \sigma\|y^{\bar{\alpha}}_{\tau,x}(T) - x\|^2$. Therefore, combining the latter inequalities, we obtain that for all $T \in [\tau,\tau+\delta)$

$$\begin{aligned}\varepsilon(T-\tau) \geq (e^{\lambda(\tau-T)}-1)\varphi(\tau,x) + \int_\tau^T e^{\lambda(\tau-t)}\ell(y^{\bar{\alpha}}_{\tau,x}(t),\bar{\alpha}(t))dt + e^{\lambda(\tau-T)}\theta(T-\tau)\\ + e^{\lambda(\tau-T)}\langle\zeta, y^{\bar{\alpha}}_{\tau,x}(T)-x\rangle - e^{\lambda(\tau-T)}\eta(T,\tau).\end{aligned}$$

Moreover, for $T \in [\tau,\tau+\delta)$

$$\begin{aligned}&\int_\tau^T e^{\lambda(\tau-t)}\ell(y^{\bar{\alpha}}_{\tau,x}(t),\bar{\alpha}(t))dt + e^{\lambda(\tau-T)}\langle\zeta, y^{\bar{\alpha}}_{\tau,x}(T)-x\rangle\\ &\qquad= \int_\tau^T \left[e^{\lambda(\tau-t)}\ell(y^{\bar{\alpha}}_{\tau,x}(t),\bar{\alpha}(t)) + e^{\lambda(\tau-T)}\langle\zeta,\dot{y}^{\bar{\alpha}}_{\tau,x}(t)\rangle\right]dt\\ &\qquad\geq \int_\tau^T \min_{a\in A}\left\{e^{\lambda(\tau-t)}\ell(y^{\bar{\alpha}}_{\tau,x}(t),a) + e^{\lambda(\tau-T)}\langle\zeta, f(y^{\bar{\alpha}}_{\tau,x}(t),a)\rangle\right\}dt\\ &\qquad- e^{\lambda(\tau-T)}\int_\tau^T \mathfrak{S}_{\mathbf{C}}(t,y^{\bar{\alpha}}_{\tau,x}(t),\zeta)dt.\end{aligned}$$

The last inequality comes from (5) and the definition of $\mathfrak{S}_{\mathbf{C}}$. Therefore, for $T \in [\tau, \tau+\delta)$

$$\begin{aligned}\varepsilon(T-\tau) \geq\ & (e^{\lambda(\tau-T)}-1)\varphi(\tau,x) + e^{\lambda(\tau-T)}\theta(T-\tau) - e^{\lambda(\tau-T)}\eta(T,\tau) \\ & + \int_\tau^T \min_{a\in A}\left\{e^{\lambda(\tau-t)}\ell(y^{\bar{\alpha}}_{\tau,x}(t),a) + e^{\lambda(\tau-T)}\langle \zeta, f(y^{\bar{\alpha}}_{\tau,x}(t),a)\rangle\right\} dt \\ & - e^{\lambda(\tau-T)}\int_\tau^T \mathfrak{S}_{\mathbf{C}}(t, y^{\bar{\alpha}}_{\tau,x}(t),\zeta)dt.\end{aligned}$$

Notice that by Lemma 1, it follows that

$$\|y^{\bar{\alpha}}_{\tau,x}(t) - x\| \leq \left(e^{2\beta_f(t-\tau)}-1\right)\left(1+\|x\|+\frac{\kappa_{\mathbf{C}}}{2\beta_f}\right), \qquad \forall t \in [\tau, T].$$

Also for any $a \in A$ and $t \in [\tau, T]$ fixed we have

$$e^{\lambda(\tau-t)}\ell(y^{\bar{\alpha}}_{\tau,x}(t),a) \geq e^{\lambda(\tau-T)}\ell(x,a) - \kappa_\ell\|y^{\bar{\alpha}}_{\tau,x}(t)-x\| - \beta_\ell\left(e^{\lambda(\tau-t)} - e^{\lambda(\tau-T)}\right)$$

and

$$\langle\zeta, f(y^{\bar{\alpha}}_{\tau,x}(t),a)\rangle \geq \langle \zeta, f(x,a)\rangle - \kappa_f\|\zeta\|\|y^{\bar{\alpha}}_{\tau,x}(t)-x\|$$

Consequently,

$$\begin{aligned}&\int_\tau^T \min_{a\in A}\left\{e^{\lambda(\tau-t)}\ell(y^{\bar{\alpha}}_{\tau,x}(t),a) + e^{\lambda(\tau-T)}\langle \zeta, f(y^{\bar{\alpha}}_{\tau,x}(t),a)\rangle\right\} dt \\ &\geq e^{\lambda(\tau-T)}(T-\tau)\min_{a\in A}\{\ell(x,a)+\langle\zeta,f(x,a)\rangle\} - \beta_\ell\int_\tau^T\left(e^{\lambda(\tau-t)}-e^{\lambda(\tau-T)}\right)dt \\ &- \left(1+\|x\|+\frac{\kappa_{\mathbf{C}}}{2\beta_f}\right)(\kappa_\ell+\kappa_f\|\zeta\|)\int_\tau^T\left(e^{2\beta_f(t-\tau)}-1\right)dt\end{aligned}$$

Notice that, since $(t,y)\mapsto \mathfrak{S}_{\mathbf{C}}(t,y,\zeta)$ is upper semicontinuous at (τ,x) and $t\mapsto y^a_{\tau,x}(t)$ is continuous at τ, it follows that $t\mapsto\mathfrak{S}_{\mathbf{C}}(t,y^a_{\tau,x}(t),\zeta)$ is upper semicontinuous at $t=\tau$. Therefore, we may assume that

$$\mathfrak{S}_{\mathbf{C}}(t,y^a_{\tau,x}(t),\zeta) \leq \mathfrak{S}_{\mathbf{C}}(\tau,x,\zeta)+\varepsilon, \qquad \forall t\in[\tau,\tau+\delta).$$

Consequently, we get

$$\frac{1}{(T-\tau)}\int_\tau^T \mathfrak{S}_{\mathbf{C}}(t,y^{\bar{\alpha}}_{\tau,x}(t),\zeta)dt \leq \mathfrak{S}_{\mathbf{C}}(\tau,x,\zeta)+\varepsilon.$$

Therefore, gathering the information provided by the preceding inequalities we get for $T \in [\tau, \tau + \delta)$

$$-\varepsilon \leq \frac{e^{\lambda(\tau-T)} - 1}{\tau - T}\varphi(\tau, x) - e^{\lambda(\tau-T)}\theta + \frac{e^{\lambda(\tau-T)}\eta(T,\tau)}{T-\tau}$$
$$+ e^{\lambda(\tau-T)}H(x,\zeta) + \frac{\beta_\ell}{T-\tau}\int_\tau^T \left(e^{\lambda(\tau-t)} - e^{\lambda(\tau-T)}\right) dt$$
$$+ \left(1 + \|x\| + \frac{\kappa_{\mathbf{C}}}{2\beta_f}\right)\frac{\kappa_\ell + \kappa_f\|\zeta\|}{T-\tau}\int_\tau^T \left(e^{2\beta_f(t-\tau)} - 1\right) dt$$
$$+ e^{\lambda(\tau-T)}\left(\mathfrak{S}_{\mathbf{C}}(\tau, x, \zeta) + \varepsilon\right).$$

Thus, letting $T \to \tau$ and $\varepsilon \to 0$, we obtain (22).

Now, we turn to the second part of the proof ((**a**) $\Leftarrow$ (**b**)). According to Proposition 4.4, it is enough to show that $(\mathbf{Ep}(\varphi), \Gamma)$ is weakly invariant. Thus, we aim to prove that (22) implies that

$$\min_{v \in -\Gamma_D(p)} \langle \eta, v\rangle + \min_{v\in\Gamma_C(p)} \langle \eta, v\rangle \leq 0, \qquad \forall \eta \in \mathcal{N}_{\mathbf{Ep}(\varphi)}(p).$$

The conclusion will then follow from Theorem 4.2.

Let $\eta \in \mathcal{N}_{\mathbf{Ep}(\varphi)}(\mathbf{p})$ with $\mathbf{p} = (\sigma, \tau, x, z, \gamma)$. Consider

$$\psi(\sigma, \tau, x, z) := e^{-\lambda\sigma}\varphi(\tau, x) + z.$$

Notice that $\mathbf{Ep}(\varphi) = \text{epi}(\psi)$, and so, $\eta = (\xi, -q)$ for some $q \geq 0$ and, if $q \neq 0$, we also have $\frac{1}{q}\xi \in \partial_P\psi(\sigma, \tau, x, z)$. Les us consider now two cases:
Case $q > 0$: under these circumstances, we have that $\gamma = \psi(\sigma, \tau, x, z)$ and

$$\frac{1}{q}\xi \in \partial_P\psi(\sigma, \tau, x, z) \subset \{-\lambda e^{-\lambda\sigma}\varphi(\tau, x)\} \times e^{-\lambda\sigma}\partial_P\varphi(\tau, x) \times \{1\}$$

Therefore, for some $(\theta, \zeta) \in \partial_P\varphi(\tau, x)$, $a \in A$ and $\mathbf{g} \in \partial_P \operatorname{dist}_{\mathbf{C}(t)}(x)$ we have

$$\min_{v\in\Gamma(p)} \langle \eta, v\rangle \leq qe^{-\lambda\sigma}\left(-\lambda\varphi(\tau, x) + \theta + \langle \zeta, f(x,a) - m(x)\mathbf{g}\rangle + \ell(x,a)\right),$$

which implies that

$$\min_{v\in\Gamma(p)} \langle \eta, v\rangle \leq -qe^{-\lambda\sigma}\left(\lambda\varphi(\tau, x) - \theta + H(x,\zeta) + \mathfrak{S}_{\mathbf{C}}(\tau, x, \zeta)\right).$$

Hence, by (22) we get $\min\{\langle \eta, v\rangle : v \in \Gamma(t,p)\} \leq 0$.
Case $q = 0$: We can use Rockafellar's horizonality theorem (see [19]) and proceed as in [16, Section 5.2]. Indeed, Suppose now that $q = 0$, there exist some sequences $\{(\sigma_n, \tau_n, x_n, z_n)\} \subset \text{dom}(\psi)$, $\{(\xi_n)\} \subset \mathbb{R}^{N+2}$ and $\{q_n\} \subset (0, \infty)$ such that

$$\mathbf{p}_n := (\sigma_n, \tau_n, x_n, z_n, \psi(\sigma_n, \tau_n, x_n, z_n)) \to (\sigma, \tau, x, z, \psi(\tau, x, z))$$

and

$$\eta_n := (\xi_n, q_n) \to (\xi, 0), \qquad \frac{1}{q_n}\xi_n \in \partial_P \psi(\sigma_n \tau_n, x_n, z_n).$$

Thus, using the same argument as above we can show

$$\min_{v\in\Gamma(\mathbf{p}_n)} \langle \eta_n, v\rangle \leq 0, \quad \forall n \in \mathbb{N}.$$

Hence, due to the fact that Γ has compact images, we can find some velocity $v_n \in \Gamma(\sigma_n, \tau_n, x_n, z_n, \psi(\sigma_n, \tau_n, x_n, z_n))$ where the minimum is attained. Since the images of Γ are locally bounded, the sequence $\{v_n\}$ can be taken uniformly bounded, and moreover, because the graph of Γ is closed, we can take the liminf in the last inequality, use that $\Gamma(\tau, x, z, \psi(\sigma, \tau, x, z)) = \Gamma(\sigma, \tau, x, z, \gamma)$ and we obtain (20).

□

Example 5.1. *To illustrate that the value function is a solution of* (22) *in Theorem 5.1, let us consider a simple example. Take* $f \equiv 0$, $\ell \equiv c > 0$, $\mathbf{C}(\tau) = [0,1]$ *and* $\lambda > 1$*. It is not difficult to see that* $\vartheta(\tau, x) = \frac{c}{\lambda}$ *for any* $\tau \in \mathbb{R}$ *and* $x \in [0,1]$*. Here* $\mathrm{gr}\,(\mathbf{C}) = \mathbb{R} \times [0,1]$.

Notice that the Standing Assumptions are satisfied for any $\kappa_{\mathbf{C}} > 0$ *and* $\beta_f > 0$*, so that* $m(x) = \kappa_{\mathbf{C}} + \beta_f(1 + |x|)$ *for any* $x \in \mathbb{R}$.

Moreover, it is not difficult to see that for any $\tau \in \mathbb{R}$ *we have*

$$\partial_P \operatorname{dist}_{\mathbf{C}(\tau)}(x) = \begin{cases} \{-1\} & x < 0 \\ [-1,0] & x = 0 \\ \{0\} & x \in (0,1) \\ [0,1] & x = 1 \\ \{1\} & x > 1 \end{cases} \text{ and } \partial_P \vartheta(\tau, x) = \begin{cases} \{0\} \times (-\infty, 0] & x = 0 \\ \{(0,0)\} & x \in (0,1) \\ \{0\} \times [0, +\infty) & x = 1. \end{cases}$$

Consequently, since $H(x,\zeta) = -c$ *and*

$$\mathfrak{S}_{\mathbf{C}}(\tau, x, \zeta) = \begin{cases} (\kappa_{\mathbf{C}} + \beta_f) \max\{-\zeta, 0\} & x = 0 \\ 0 & x \in (0,1) \\ (\kappa_{\mathbf{C}} + 2\beta_f) \max\{\zeta, 0\} & x = 1 \end{cases}$$

we have for $x \in (0,1)$

$$-\theta + H(x,\zeta) + \mathfrak{S}_{\mathbf{C}}(\tau, x, \zeta) + \lambda\vartheta(\tau, x) = -0 - c - 0 + \lambda\frac{c}{\lambda} = 0$$

and at the boundary points, for $(\theta, \zeta) \in \partial_P \vartheta(\tau, 0)$ *we have*

$$-\theta + H(0,\zeta) + \mathfrak{S}_{\mathbf{C}}(\tau, 0, \zeta) + \lambda\vartheta(\tau, 0) = -0 - c - (\kappa_{\mathbf{C}} + \beta_f)\zeta + \lambda\frac{c}{\lambda} = -(\kappa_{\mathbf{C}} + \beta_f)\zeta \geq 0$$

and for $(\theta,\zeta)\in\partial_P\vartheta(\tau,1)$ *we have*

$$-\theta+H(1,\zeta)+\mathfrak{S}_{\mathbf{C}}(\tau,1,\zeta)+\lambda\vartheta(\tau,1)=-0-c+(\kappa_{\mathbf{C}}+2\beta_f)\zeta+\lambda\frac{c}{\lambda}=(\kappa_{\mathbf{C}}+2\beta_f)\zeta\geq 0.$$

Thus, ϑ *is a* proximal supersolution*.*

Let us now focus on the strongly increasing case. Similarly as in the preceding case, this characterization will be proven by means of of the *proximal superdifferential*, which corresponds, for any u.s.c. function $\psi:\Omega\subset\mathbb{R}^k\to\mathbb{R}$ to the set

$$\partial^P\psi(x)=-\partial_P(-\psi)(x).$$

In this case, the proximal superdifferential and the proximal normal cone are related to each other via the relation

$$\xi\in\partial^P\psi(x)\quad\Longleftrightarrow\quad(-\xi,1)\in\mathcal{N}_{\text{hypo}(\psi)}\left(x,\psi(x)\right),\qquad\forall x\in\Omega,$$

where hypo(ψ) is the hypograph of the function $\psi:\Omega\subset\mathbb{R}^k\to\mathbb{R}$.

Theorem 5.2. *Consider a given u.s.c. function* $\varphi\colon\ \mathrm{gr}\,(\mathbf{C})\subset\mathbb{R}\times\mathbb{R}^N\to\mathbb{R}$*. Then* φ *is strongly increasing for the optimal control problem of sweeping processes if and only if* φ *is a* proximal subsolution*, that is, for any* $(\tau,x)\in\mathrm{gr}\,(\mathbf{C})$ *we have*

$$-\theta+H(x,\zeta)-\mathfrak{S}_{\mathbf{C}}(\tau,x,-\zeta)+\lambda\varphi(\tau,x)\leq 0,\qquad\forall(\theta,\zeta)\in\partial^P\varphi(\tau,x).\tag{23}$$

Proof. We start by proving the implication $(\Rightarrow)$: Let $(\tau,x)\in\mathrm{gr}\,(\mathbf{C})$.

First of all note that $(\theta,\zeta)\in\partial^P\varphi(\tau,x)$ if and only if $\exists\sigma,\delta>0$ such that

$$\varphi(s,y)\leq\varphi(\tau,x)+\theta(s-\tau)+\langle\zeta,y-x\rangle+\sigma\left((s-\tau)^2+\|y-x\|^2\right)$$

for any $(s,y)\in\mathbb{B}((\tau,x),\delta)\cap\mathrm{gr}\,(\mathbf{C})$.

Take $a\in A$ and let $y^a_{\tau,x}$ be the trajectory associated with constant control $\alpha\equiv a$. Since φ is strongly increasing we have for any $h>0$

$$e^{\lambda h}\varphi(\tau,x)\leq\int_\tau^{\tau+h}e^{\lambda(\tau+h-t)}\ell(y^a_{\tau,x}(t),a)dt+\varphi(\tau+h,y^a_{\tau,x}(\tau+h)).$$

Therefore, for $h>0$ small enough we have

$$\begin{aligned}0\leq&\int_\tau^{\tau+h}\left(e^{\lambda(\tau+h-t)}\ell(y^a_{\tau,x}(t),a)+\langle\zeta,\dot y^a_{\tau,x}(t)\rangle\right)dt+\left(1-e^{\lambda h}\right)\varphi(\tau,x)+\theta h\\&+\sigma\left(h^2+|y^a_{\tau,x}(\tau+h)-x|^2\right).\end{aligned}$$

Notice that

$$\int_\tau^{\tau+h}e^{\lambda(\tau+h-t)}\ell(y^a_{\tau,x}(t),a)\leq\int_\tau^{\tau+h}\ell(y^a_{\tau,x}(t),a)+\beta_\ell h\left(\frac{e^{\lambda h}-1}{\lambda h}-1\right).$$

Thus, by Lemma 1 we get $\frac{1}{h}\|y^a_{\tau,x}(\tau+h)-x\|$ is uniformly bounded with respect to $h>0$ small enough, and therefore

$$0 \leq \frac{1}{h}\int_{\tau}^{\tau+h} \big(\ell(y^a_{\tau,x}(t),a)+\langle \zeta, \dot{y}^a_{\tau,x}(t)\rangle\big)\,dt - \lambda\varphi(\tau,x)+\theta+O(h),$$

with $O(h)\to 0$ if $h\to 0^+$. Since $y^a_{\tau,x}$ is also a solution of (5), we have

$$\langle \zeta, \dot{y}^a_{\tau,x}(t)\rangle \leq \langle \zeta, f(y^a_{\tau,x}(t),a)\rangle + \mathfrak{S}_{\mathbf{C}}(t, y^a_{\tau,x}(t),\zeta), \qquad \text{for a.e. } t\in[\tau,\tau+h].$$

Notice that

$$\ell(y^a_{\tau,x}(t),a)+\langle \zeta, f(y^a_{\tau,x}(t),a)\rangle \leq \ell(x,a)+\langle \zeta, f(x,a)\rangle + (\kappa_\ell+\kappa_f\|\zeta\|)\|y^a_{\tau,x}(t)-x\|.$$

Thus, in the light of Lemma 1, we get

$$\limsup_{h\to 0^+}\frac{1}{h}\int_{\tau}^{\tau+h}\big(\ell(y^a_{\tau,x}(t),a)+\langle \zeta, f(y^a_{\tau,x}(t),a)\rangle\big)\,dt \leq \ell(x,a)+\langle\zeta, f(x,a)\rangle,$$

so to conclude, since $a\in A$ is arbitrary, it remains to show that

$$\limsup_{h\to 0^+}\frac{1}{h}\int_{\tau}^{\tau+h}\mathfrak{S}_{\mathbf{C}}(t,y^a_{\tau,x}(t),\zeta)dt \leq \mathfrak{S}_{\mathbf{C}}(\tau,x,\zeta).$$

Let $\varepsilon>0$ and notice that, since $(t,y)\mapsto \mathfrak{S}_{\mathbf{C}}(t,y,\zeta)$ is upper semicontinuous at $(t,y)=(\tau,x)$ and $t\mapsto y^a_{\tau,x}(t)$ is continuous at $t=\tau$, $t\mapsto \mathfrak{S}_{\mathbf{C}}(t,y^a_{\tau,x}(t),\zeta)$ is upper semicontinuous at $t=\tau$. Therefore, there is $\delta>0$ such that

$$\mathfrak{S}_{\mathbf{C}}(t,y^a_{\tau,x}(t),\zeta)\leq \mathfrak{S}_{\mathbf{C}}(\tau,x,\zeta)+\varepsilon, \qquad \forall t\in[\tau,\tau+\delta].$$

Consequently, for any $h\in(0,\delta)$ we have

$$\frac{1}{h}\int_{\tau}^{\tau+h}\mathfrak{S}_{\mathbf{C}}(t,y^a_{\tau,x}(t),\zeta)dt \leq \mathfrak{S}_{\mathbf{C}}(\tau,x,\zeta)+\varepsilon.$$

Therefore, taking lim sup over $h\to 0^+$, and then letting $\varepsilon\to 0$ we get (22).

Let us now focus on the other implication ($\Leftarrow$). In the light of Proposition 4.4, it will be enough to show that $(\mathbf{Hy}(\varphi),\Gamma)$ is strongly invariant. Our vehicle to prove this will be Theorem 4.1.

Take $\mathbf{p}_0=(\sigma_0,\tau_0,x_0,z_0,\gamma_0)\in\mathbf{Hy}(\varphi)$. Recall that this means that

$$e^{-\lambda\sigma_0}\varphi(\tau_0,x_0)+z_0\geq\gamma_0.$$

We need to check that for all $t\in\mathbb{R}$

$$\min_{v\in-\Gamma_D(\mathbf{p}_0)}\langle v,\eta\rangle + \max_{v\in\Gamma_C(\mathbf{p}_0)}\langle v,\eta\rangle \leq 0 \qquad \forall\,\eta\in\mathcal{N}_{\mathbf{Hy}(\varphi)}(\mathbf{p}_0)\,. \tag{24}$$

Take $\eta_0 \in \mathcal{N}_{\mathbf{Hy}(\varphi)}(\mathbf{p}_0)$, then $\eta_0 = (\xi_0, q_0)$ for some $q_0 \geq 0$ and $-\frac{1}{q_0}\xi_0 \in \partial^P \psi(\sigma_0, \tau_0, x_0, z_0)$ whenever $q_0 > 0$, where ψ is as in the proof of Theorem 5.1, that is

$$\psi(\sigma, \tau, x, z) := e^{-\lambda\sigma}\varphi(\tau, x) + z$$

Let us analyze the case $q_0 > 0$ first. It is not difficult to see that we must have $\gamma_0 = \psi(\sigma_0, \tau_0, x_0, z_0)$ and also

$$\partial^P \psi(\sigma_0, \tau_0, x_0, z_0) \subset \{-\lambda e^{-\lambda\sigma_0}\varphi(\tau_0, x_0)\} \times e^{-\lambda\sigma_0}\partial^P \varphi(\tau_0, x_0) \times \{1\}$$

It follows that for some $(\theta_0, \zeta_0) \in \partial^P \varphi(\tau_0, x_0)$, we also have

$$-\frac{1}{q_0}\xi_0 = (-\lambda e^{-\lambda\sigma_0}\varphi(\tau_0, x_0), e^{-\lambda\sigma_0}\theta_0, e^{-\lambda\sigma_0}\zeta_0, 1)$$

and since any $v \in \Gamma_C(\mathbf{p}_0)$ can be written as convex combinations of vectors of the form $\tilde{v} = (1, 1, f(x_0, a), e^{-\lambda\sigma_0}\ell(x_0, a), 0)$ for some $a \in A$, we have

$$\langle v, \eta_0 \rangle \leq -q_0 e^{-\lambda\sigma_0}\left(-\lambda\varphi(\tau_0, x_0) + \theta_0 + H(x_0, \zeta_0)\right).$$

Take now, $v_0 \in -\Gamma_D(\mathbf{p}_0)$ such that

$$v_0 \cdot \eta_0 = \min_{v \in -\Gamma_D(\mathbf{p}_0)} \langle v, \eta_0 \rangle.$$

In particular, from the definition of $\Gamma_D(\mathbf{p}_0)$ it follows that

$$v_0 \cdot \eta_0 \leq -q_0 e^{-\lambda\sigma_0}\mathfrak{S}_{\mathbf{C}}(\tau_0, x_0, \zeta_0).$$

Therefore, for any $v \in \Gamma_C(\mathbf{p}_0)$ we have

$$\min_{v \in -\Gamma_D(\mathbf{p}_0)} \langle v, \eta_0 \rangle + \langle v, \eta_0 \rangle = -q_0 e^{-\lambda\sigma_0}\left(\theta_0 + H(x_0, \zeta_0) + \mathfrak{S}_{\mathbf{C}}(\tau_0, x_0, \zeta_0) - \lambda\varphi(\tau_0, x_0)\right)$$

It is then clear that (23) implies (24).

The case $q_0 = 0$ is also straightforward consequence of the Rockafellar's horizonality theorem. Here are the details. If $q_0 = 0$, there exist some sequences $\{(\sigma_n, \tau_n, x_n, z_n)\} \subset \mathrm{dom}(\psi)$, $\{(\xi_n)\} \subset \mathbb{R}^{N+2}$ and $\{q_n\} \subset (0, \infty)$ such that

$$\mathbf{p}_n := (\sigma_n, \tau_n, x_n, z_n, \psi(\sigma_n, \tau_n, x_n, z_n)) \to (\sigma, \tau, x, z, \psi(\sigma, \tau, x, z))$$

and

$$\eta_n := (\xi_n, q_n) \to (\xi, 0), \qquad \frac{1}{q_n}\xi_n \in \partial^P \psi(\sigma_n, \tau_n, x_n, z_n).$$

Thus, using the same argument as above, we can show

$$\min_{v\in -\Gamma_D(\mathbf{p}_n)} \langle v, \eta_n\rangle + \max_{v\in\Gamma_C(\mathbf{p}_n)} \langle v, \eta_n\rangle \leq 0, , \quad \forall n \in \mathbb{N}. \tag{25}$$

Finally, notice that, since Γ_C is in particular a lower semicontinuous multifunction, by the Maximum Theorem (e.g. [1, Theorem 1.2.4]), the mapping

$$(\mathbf{p}, \eta) \mapsto \max_{v\in\Gamma_C(\mathbf{p})} \langle v, \eta\rangle$$

is lower semicontinuous. Hence, due to the fact that Γ_D has compact images, we can find some velocity $v_n \in \Gamma(\sigma_n, \tau_n, x_n, z_n, \psi(\sigma_n, \tau_n, x_n, z_n))$ where the minimum is attained. Since the images of Γ_D are locally bounded, the sequence $\{v_n\}$ can be taken uniformly bounded, and moreover, because the graph of Γ_D is closed, we can take the liminf in the equation (25), use that $\Gamma_D(\tau, x, z, \psi(\sigma, \tau, x, z)) = \Gamma_D(\sigma, \tau, x, z, \gamma)$ and we obtain (20). □

Example 5.2. *As in Example 5.1, to illustrate that the value function is a solution of (23) in Theorem 5.2 we consider the following data:* $f \equiv 0$, $\ell \equiv c > 0$, $\mathbf{C}(\tau) = [0, 1]$ *and* $\lambda > 1$. *Recall that* $\vartheta(\tau, x) = \frac{c}{\lambda}$ *for any* $\tau \in \mathbb{R}$ *and* $x \in [0, 1]$.

In this case, it is not difficult to see that for any $\tau \in \mathbb{R}$ *we have*

$$\partial^P \vartheta(\tau, x) = \begin{cases} \{0\} \times [0, +\infty) & x = 0 \\ \{(0, 0)\} & x \in (0, 1) \\ \{0\} \times (-\infty, 0] & x = 1. \end{cases}$$

Consequently, since $H(x, \zeta) = -c$ *and*

$$\mathfrak{S}_{\mathbf{C}}(\tau, x, -\zeta) = \begin{cases} (\kappa_{\mathbf{C}} + \beta_f) \max\{\zeta, 0\} & x = 0 \\ 0 & x \in (0, 1) \\ (\kappa_{\mathbf{C}} + 2\beta_f) \max\{-\zeta, 0\} & x = 1 \end{cases}$$

we have for $x \in (0, 1)$

$$-\theta + H(x, \zeta) - \mathfrak{S}_{\mathbf{C}}(\tau, x, -\zeta) + \lambda\vartheta(\tau, x) = -0 - c - 0 + \lambda\frac{c}{\lambda} = 0$$

and at the boundary points, for $(\theta, \zeta) \in \partial^P \vartheta(\tau, 0)$ *we have*

$$-\theta + H(0, \zeta) - \mathfrak{S}_{\mathbf{C}}(\tau, 0, -\zeta) + \lambda\vartheta(\tau, 0) = -0 - c - (\kappa_{\mathbf{C}} + \beta_f)\zeta + \lambda\frac{c}{\lambda} = -(\kappa_{\mathbf{C}} + \beta_f)\zeta \leq 0$$

and for $(\theta, \zeta) \in \partial^P \vartheta(\tau, 1)$ *we have*

$$-\theta + H(1, \zeta) - \mathfrak{S}_{\mathbf{C}}(\tau, 1, -\zeta) + \lambda\vartheta(\tau, 1) = -0 - c + (\kappa_{\mathbf{C}} + 2\beta_f)\zeta + \lambda\frac{c}{\lambda} = (\kappa_{\mathbf{C}} + 2\beta_f)\zeta \leq 0.$$

Thus, ϑ is a proximal subsolution*.*

To conclude this section, we show that the notion of strongly increasing can also be characterized in the usual sense of viscosity supersolution.

Definition 5.2. *Let $\psi : \Omega \subset \mathbb{R}^k \to \mathbb{R}$ be an u.s.c. function. A vector $\xi \in \mathbb{R}^k$ is called a viscosity supergradient of ψ at $x \in \Omega$ if and only if there exists a continuously differentiable function $g : \mathbb{R}^k \to \mathbb{R}$ so that*

$$\nabla g(x) = \xi \text{ and } \psi - g \text{ attains a local maximum at } x.$$

The set of all viscosity supergradients of ψ at x is called the viscosity superdifferential and it is denoted by $D^+\varphi(x)$.

Corollary 5.1. *Consider a given u.s.c. function $\varphi\colon \operatorname{gr}(\mathbf{C}) \subset \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$. Then φ is strongly increasing for the optimal control problem of sweeping processes if and only if φ is a* viscosity subsolution*, that is, for any $(\tau, x) \in \operatorname{gr}(\mathbf{C})$ we have*

$$-\theta + H(x, \zeta) - \mathfrak{S}_{\mathbf{C}}(\tau, x, -\zeta) + \lambda\varphi(\tau, x) \le 0, \qquad \forall(\theta, \zeta) \in D^+\varphi(\tau, x). \tag{26}$$

Proof. It is enough to prove that (23) is equivalent to (26). As a matter of fact, it is not difficult to see that $\partial^P\varphi(\tau, x) \subset D^+\varphi(\tau, x)$; see for instance [10, Exercise 1.2.8]. Consequently, any solution of (26) is also a solution of (23). Moreover, it is also apparent that $D^+\varphi(\tau, x) = -D^-(-\varphi)(\tau, x)$. Therefore, using similar approximation arguments as in the first part of the proof of Theorem 5.1, we can prove that any solution of (23) is also a solution of (26), and so the proof is complete. □

6 Discussion

In this paper, we have proven a Uniqueness Theorem for the value function associated with an optimal control problem governed by a sweeping process with a controlled drift. The uniqueness has been accomplished under a very important feature provided by the sweeping process: the value function's continuity. As we may recall, this is not a common feature for standard optimal control problems with state constraints. Also, by means of an appropriate Relaxation Theorem, we have been able to prove our Uniqueness Theorem without requiring any convexity assumptions on the dynamics nor on the running cost, as, for instance, done in [16] for problems with standard optimal control problems with state constraints or in [11, 21] for optimal control problems of sweeping processes.

As pointed out in Remark 3.2, under appropriate conditions, Theorem 3.3 corresponds to a well-posedness result for a boundary value problem of the Dirichlet type for an HJB equation. A comparison principle was obtained in [17] for equations of this type, however it doesn't cover the equations derived from an optimal control problem governed by a sweeping process, since the boundary condition has a different structure; see [17, Theorem 2.1]. In a related note, many authors have studied problems with discontinuous Hamiltonians, either motivated by problems with unbounded data or with state constraints; see for instance [3, 18]. The discontinuity considered in these papers are of different nature than the one considered in this work. Indeed, in [3] the discontinuity comes from the fact that supremum in the definition of the Hamiltonian may no be finite and in [18] from considering a smaller set of admissible controls at the boundary than in the interior of the state constraints. Notice that in our case, we introduce a completely new term at the boundary (the functions $\mathfrak{S}_{\mathbf{C}}$), which is not related with the Hamiltonian in the interior of the state cosntraints.

As mentioned in the paper, the Standing Assumptions are not sharp for getting the continuity of the value function. For example, the running cost does not need to be globally Lipschitz continuous; Local Lipschitz continuity is enough. The Lipschitz continuity of the moving set **(H2.ii)** can be replaced with absolute continuity:

(H2.ii) there is $\nu : \mathbb{R} \to \mathbb{R}$ absolutely continuous such that for any fixed $x \in \mathbb{R}^N$ we have

$$|\operatorname{dist}_{\mathbf{C}(t)}(x) - \operatorname{dist}_{\mathbf{C}(s)}(x)| \leq |\nu(t) - \nu(s)|, \qquad \forall t, s \in \mathbb{R}.$$

Also, the dynamics can be non-autonomous with measurable dependence on the time variable, and **(H3)** can be replaced with

(H3.i) $f(\cdot, x, a)$ is measurable on $\mathbb{R}$ for any fixed $(x, a) \in \mathbb{R}^N \times A$.

(H3.ii) $f(t, x, \cdot)$ is continuous on A for any fixed $(t, x) \in \mathbb{R} \times \mathbb{R}^N$.

(H3.iii) $\exists \kappa_f \in L^1_{\text{loc}}$ nonnegative such that for any $x_1, x_2 \in \mathbb{R}^N$ and for a.e. $t \in \mathbb{R}$ we have

$$\sup_{a \in A} \|f(t, x_1, a) - f(t, x_2, a)\| \leq \kappa_f(t)\|x_1 - x_2\|.$$

(H3.iv) there is $\beta_f \in L^1_{\text{loc}}$ nonnegative such that

$$\sup_{a \in A} \|f(t, x, a)\| \leq \beta_f(t)(1 + \|x\|), \qquad \forall x \in \mathbb{R}^N, \text{ for a.e. } s \in \mathbb{R}.$$

The only consequence these changes have is that the Gronwall estimates in Lemma 1 are slightly different, but the proof of Theorem 3.1 remains almost the same.

However, the Standing Assumptions are needed in their actual form to prove the Uniqueness Theorem. Several technical difficulties, particularly with the strong invariance principle, arise when considering non-autonomous dynamics or an unbounded running cost, which require further analysis. This is beyond the scope of this paper.

Appendix A An example

To conclude this paper, we make a discussion about the fact that the notion of constrained viscosity solution of the classical HJB equation is not suitable for characterizing the value function associated with an optimal control problem governed by a sweeping process with a controlled drift. Hence, a new Hamiltonian is required for such a purpose. To do this, let us consider the following example, which is an adaptation of [2, Example IV.5.3] to a sweeping process context.

Consider the following data:

$$\mathbf{C} = \left\{x \in \mathbb{R}^2 \colon x_2 \leq x_1^2\right\}, \qquad A = [-1, 1], \qquad \lambda = 1, \qquad f(x, a) = (a, 0)$$

and

$$\ell(x,a) = \begin{cases} 1 & \text{if } x_1 < 0, \\ 1 - x_1 & \text{if } 0 \le x_1 \le 1, \\ 0 & \text{if } x_1 > 1. \end{cases}$$

Notice that here we are in the autonomous case.

One can see that optimal solutions of this problem tend to move to the right with maximal speed (control $\alpha(t) = 1$) until reaching the set $\{x \in \mathbf{C} \colon x_1 \ge 1\}$; from that point, any control $\alpha(t) \in [0,1]$ provides an optimal trajectory. Moreover, the influence of the normal cone in (2) can be disregarded for any optimal trajectory that starts in $\{x \in \mathbf{C} \colon x_1 \ge 0 \ \vee \ x_2 \le 0\}$ (gray area in Figure A1) because every arc starting from $\{x \in \mathbf{C} : x_1 \ge 1, x_2 \ge 1\}$ is optimal and may touch the boundary of $\mathbf{C}$ several times; for instance, at $t = 0$ if $x_1 = \sqrt{x_2}$ and never if $x_2 < 0$. Consequently, it is not difficult to see that the value function of this problem is the same as given in [2, Example IV.5.3] except in $\{x \in \mathbf{C} \colon x_1 < 0 < x_2\}$ (red area in Figure A1). In particular

$$\vartheta(x) = \begin{cases} 1 + e^{x_1 - 1} - e^{x_1} & \text{if } x_1, x_2 \le 0, \\ e^{x_1 - 1} - x_1 & \text{if } 0 \le x_1 \le 1, \\ 0 & \text{if } 1 \le x_1, \end{cases} \qquad x \in \mathbf{C}.$$

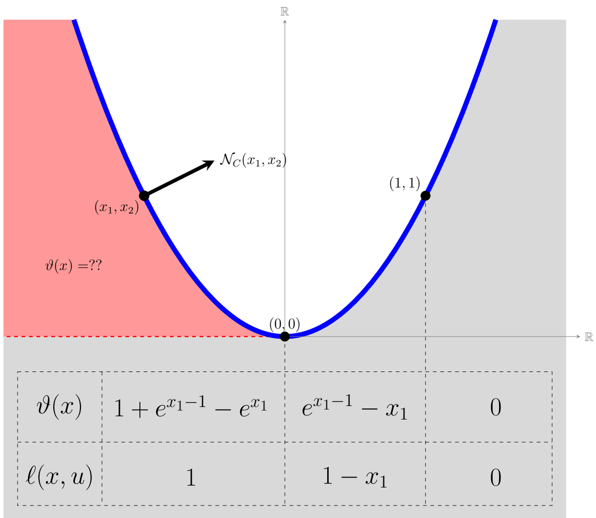

Fig. A1 Sketch of the example

An optimal trajectory that starts in $\{x \in \mathbf{C} \colon x_1 < 0 < x_2\}$, once it reaches the boundary of $\mathbf{C}$, it slides along that curve downward until getting at $(0,0)$; this is due to the influence of the normal cone in the dynamical system (2). From that point onwards it moves horizontally to the right as described earlier. To be more precise, let $y_x^* : [0, +\infty) \to \mathbf{C}$ be an optimal solution that starts at $y_x^*(0) = x \in \mathbf{C}$ such that

$x_1 < 0 < x_2$. Let $T(x) > 0$ be the time this optimal trajectory requires to reach $(0,0)$. Then, the optimal value of the problem, in this case, is given by

$$\vartheta(x) = \int_0^{T(x)} e^{-t} dt + e^{-T(x)}\vartheta(0,0) = 1 - e^{-T(x)} + e^{-T(x)-1}.$$

- Let $\tau_0(x) := \min\left\{t \in [0,+\infty) \colon y_y^*(t) \in \partial\mathbf{C}\right\} < T(x)$. Since the dynamics on $\operatorname{int}(\mathbf{C})$ is determined only by f because $\mathcal{N}_{\mathbf{C}}(\cdot) = \{(0,0)\}$ on $\operatorname{int}(\mathbf{C})$, it follows that
$$\tau_0(x) = -\sqrt{x_2} - x_1 \quad \text{and} \quad y_x^*(\tau_0(x)) = (-\sqrt{x_2}, x_2)\,.$$
- On the interval $[\tau_0(x), T(x)]$ the optimal trajectory slides along the boundary of $\mathbf{C}$ downward until getting at $(0,0)$, and so $y_x^*(T(x)) = (0,0)$. Because $\mathcal{N}_{\mathbf{C}}(x) = \{(-2ux_1, u) \colon u \geq 0\}$ for $x \in \partial\mathbf{C}$, the dynamics of the optimal trajectory on the time interval $[\tau_0(x), T(x)]$ is given by

  $$\dot{y}_1(t) = 1 + 2u(t)y_1(t), \qquad \dot{y}_2(t) = -u(t) \quad \text{and} \quad u(t) \geq 0.$$

  Moreover, the optimal trajectory is subject to the path constraint

  $$y_1^2(t) = y_2(t), \qquad \text{for any } t \in [\tau_0(x), T(x)].$$

  Thus, derivating this state constraint with respect to time, we can get an explicit formula for the parameter $u(t)$. Indeed,

  $$u(t) = -\frac{2y_1(t)}{4y_1^2(t)+1}, \qquad \text{for a.e. } t \in [\tau_0(x), T(x)].$$

  Note that $u(t) \in \left[0, \frac{1}{2}\right]$ for any $t \in [\tau_0(x), T(x)]$. Hence, the dynamical system that governs the behavior of the optimal trajectory on $[\tau_0(x), T(x)]$ can be reduced to an ODE system:

  $$\dot{y}_1(t) = \frac{1}{4y_1^2(t)+1} \quad \text{and} \quad \dot{y}_2(t) = \frac{2y_1(t)}{4y_1^2(t)+1}, \qquad \text{a.e. on } [\tau_0(x), T(x)].$$

  It follows that

  $$\frac{4}{3}y_1^3(t) + y_1(t) = -\frac{4}{3}\sqrt{x_2^3} - \sqrt{x_2} + (t - \tau_0(x)), \qquad \forall t \in [\tau_0(x), T(x)].$$

  From here, we get that

  $$T(x) = \frac{4}{3}\sqrt{x_2^3} + \sqrt{x_2} + \tau_0(x) = \frac{4}{3}\sqrt{x_2^3} - x_1,$$

  and consequently

  $$\vartheta(x) = 1 - \exp\left(x_1 - \frac{4}{3}\sqrt{x_2^3}\right) + \exp\left(x_1 - \frac{4}{3}\sqrt{x_2^3} - 1\right)$$

Note that ϑ turns out to be continuously differentiable on $\text{int}(\mathbf{C})$ with

$$\nabla\vartheta(x) = \begin{cases} \left(e^{-T(x)} - e^{-T(x)-1}\right)\begin{pmatrix} -1 \\ 2\sqrt{x_2} \end{pmatrix} & \text{if } x_1 < 0 < x_2 \\ \begin{pmatrix} e^{x_1-1} - e^{x_1} \\ 0 \end{pmatrix} & \text{if } x_1, x_2 < 0, \\ \begin{pmatrix} e^{x_1-1} - 1 \\ 0 \end{pmatrix} & \text{if } 0 < x_1 < 1, \\ \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \text{if } 1 \leq x_1, \end{cases} \qquad x \in \text{int}(\mathbf{C}),$$

where

$$\nabla T(x) = (-1, 2\sqrt{x_2})$$

The Hamiltonian of this problem is $H(x,p) = |p_1| - \ell(x,1)$, and it is not difficult to check that the value function is a classical solution of the HJB equation on $\text{int}(\mathbf{C})$.

A.1 Is it a supersolution on C?

Recall that if ϑ is a viscosity supersolution on $\mathbf{C}$ of the HJB equation (in the constrained viscosity sense), then for any given $\bar{x} \in \mathbf{C}$ and any $\varphi \in \mathcal{C}^1(\mathbb{R}^2)$ such that $\vartheta - \varphi$ has a local minimum relative to $\mathbf{C}$ at $\bar{x} \in \mathbf{C}$ one must have

$$\vartheta(\bar{x}) + H(x, \nabla\varphi(\bar{x})) \geq 0.$$

In particular, for $\bar{x} \in \partial\mathbf{C}$ with $\bar{x}_1 < 0$ we must have that any test function satisfies

$$\vartheta(\bar{x}) + |\partial_{x_1}\varphi(\bar{x})| - 1 \geq 0$$

In particular, if $\varphi(\bar{x}) = \vartheta(\bar{x})$ then the conditions reduces to

$$|\partial_{x_1}\varphi(\bar{x})| \geq e^{-T(\bar{x})} - e^{-T(\bar{x})-1}. \tag{A1}$$

Let $\varphi \in \mathcal{C}^1(\mathbb{R}^2)$ be given by

$$\varphi(x) := 1 - \exp\left(-\sqrt{x_2} - \frac{4}{3}\sqrt{x_2^3}\right) + \exp\left(-\sqrt{x_2} - \frac{4}{3}\sqrt{x_2^3} - 1\right), \qquad \forall x \in \mathbb{R}^2.$$

Note that $\varphi(\bar{x}) = \vartheta(\bar{x})$ and $|\partial_{x_1}\varphi(\bar{x})| = 0$. Moreover, since in $\mathbf{C}$ we have that $x_1 \leq -\sqrt{x_2}$ and $z \mapsto -e^z + e^{z-1}$ is a decreasing function, we have that $\varphi(x) \leq \vartheta(x)$ for any $x \in \mathbf{C}$ with $x_1 < 1$. Thus, φ is a suitable test function however since $|\partial_{x_1}\varphi(\bar{x})| = 0$ we get a contradict with (A1), and so ϑ is not a supersolution of the HJB equation on $\mathbf{C}$.

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Declarations

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