CONSTRAINED AND IMPULSIVE LINEAR QUADRATIC CONTROL PROBLEMS

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ABSTRACT. The aim of this paper is to study the value function of a time-continuous linear quadratic optimal control problem with input and state constraints. No coercive assumptions are made, which leads to optimal control problems whose trajectories are of bounded variation rather than merely absolutely continuous. Our approach is based on classical convex analysis, and we establish a Legendre-Fenchel type equality between the value function of the linear quadratic problem and its dual problem.

1. INTRODUCTION

We study Linear Quadratic (LQ) optimal control problems with input and state constraints. The dual problem thus lacks coercivity, and so it is natural to drop the coercivity in the primal problem and consider state trajectories that are arcs of bounded variation.

Coercive unconstrained LQ optimal control problems have been widely studied in the literature. In particular, a Hamilton-Jacobi theory has been well established by mean of the Riccati equation; see for example Anderson and Moore (2007); Boltyanski and Poznyak (2011). However, little attention has been addressed to input and/or state constrained problems for continuous-time linear systems. We mention that there is considerable work regarding discrete-time problems; see for example Bemporad et al. (2002); Lewis et al. (2013).

In this paper we take a duality approach to study the value functions of the LQ problem and its dual. In particular, we show (Theorem 2.1) that the lower semicontinuous envelop of the value function are conjugate to each other. Furthermore, these lower semicontinuous envelops correspond to value function of some extended problem to arcs of bounded variation. Moreover, we obtain under suitable conditions (Corollary 2.1), that the value functions are dual to each other on the relative interior of their domains.

The techniques we exhibit in this work are part of an abstract approach we are currently investigating to construct a Hamilton-Jacobi theory and develop a characteristic method for Fully convex optimal control problem with state constraints; see Hermosilla and Wolenski (2016). These results aim at generalizing previous work for coercive unconstrained Fully Convex Control problems obtained in Rockafellar and Wolenski (2000).

The paper is organized as follows: in Section 2 we present the LQ problem, its corresponding dual problem, and their extensions to impulsive systems. We also state the main results. In Section 3 we use a standard technique to implicitly include the constraints; the result is a reformulation resembling a calculus of variations problem. Section 4 contains the proofs of the main results stated in Section 2. Finally, in Section 5 we discuss our results and outline future work.

1.1. **Notation, basic definitions, and preliminaries.** Suppose $f : E \to \mathbb{R} \cup \{\pm\infty\}$ is a function with E being a topological vector space. The effective domain of f is the set $\mathrm{dom}(f) := \{x \in E \mid f(x) < +\infty\}$. Then f is called (i) *proper* if $\mathrm{dom}(f) \neq \emptyset$ and $f(x) > -\infty$ for all $x \in E$; (ii) *convex* if $\mathrm{epi}(f) := \{(x,r) \in E \times \mathbb{R} : f(x) \leq r\}$ is a convex set, and (iii) *lower semicontinuous* if $\mathrm{epi}(f)$ is a closed set. The set of functions satisfying (i)-(iii) is denoted by $\mathcal{F}(E)$.

When $E = \mathbb{R}^k$, $|\cdot|$ denotes the Euclidean norm and $\langle \cdot, \cdot \rangle$ is the Euclidean inner product on $\mathbb{R}^k$. The (Legendre-Fenchel) conjugate of $f : \mathbb{R}^k \to \mathbb{R} \cup \{\pm\infty\}$ is

$$f^*(y) := \sup\{\langle x, y \rangle - f(x) \mid x \in \mathrm{dom}(f)\}.$$

It belongs to $\mathcal{F}(\mathbb{R}^k)$ if f is proper and convex, and satisfies $(f^*)^* = f$ whenever $f \in \mathcal{F}(\mathbb{R}^k)$. If $f : \mathbb{R}^k \to \mathbb{R} \cup \{+\infty\}$ is proper and convex, then f^{**} is the lower semicontinuous envelop of f. The recession function f_∞ of $f \in \mathcal{F}(\mathbb{R}^k)$ is given by

$$f_\infty(d) := \sup\{f(x+d) - f(x) \mid x \in \mathrm{dom}(f)\}$$

for a direction $d \in \mathbb{R}^k$, and it also belongs to $\mathcal{F}(\mathbb{R}^k)$ with the additional property of being positively homogeneous. This is because f_∞ is the support function of $\mathrm{dom}(f^*)$, which means that $f_\infty(d) = \sigma_{\mathrm{dom}(f^*)}(d)$, where $\sigma_{\mathbf{Z}}(d) := \sup_{z \in \mathbf{Z}} \langle d, z \rangle$. The subdifferential is the set

$$\partial f(x) := \{y \in \mathbb{R}^k \mid f(z) \geq f(x) + \langle y, z - x \rangle, \ \forall z \in \mathbb{R}^k\}.$$

The indicator of $\mathbf{Z} \subseteq \mathbb{R}^k$ is denoted by $\delta_{\mathbf{Z}}$ and equals 0 on $\mathbf{Z}$ and $+\infty$ off of $\mathbf{Z}$. Also, $\mathbf{Z}^\circ$ and $\mathbf{Z}_\infty$ are the polar and recession cones of $\mathbf{Z}$, respectively:

$$\mathbf{Z}^\circ = \{d \in \mathbb{R}^k \mid \langle d, z \rangle \leq 0, \quad \forall z \in \mathbf{Z}\}$$
$$\mathbf{Z}_\infty = \{d \in \mathbb{R}^k \mid \exists z \in \mathbf{Z}, \ z + td \in \mathbf{Z}, \ \forall t > 0\}$$

The relative interior of a convex set $\mathbf{Z}$ is denoted by $\mathrm{ri}(\mathbf{Z})$.

Given a matrix M of dimension k, we define

$$f_M(z) = \frac{1}{2}\langle Mz, z\rangle, \quad \forall z \in \mathbb{R}^k.$$

Also, given $\mathbf{Z} \subseteq \mathbb{R}^k$ nonempty, we define

$$g_{M,\mathbf{Z}}(p) = (f_M + \delta_{\mathbf{Z}})^*(p) = \sup_{z\in\mathbf{Z}}\{\langle z, p\rangle - f_M(z)\}, \quad \forall p \in \mathbb{R}^k$$

We also consider a matrix N of dimension $k \times l$ and define

$$h_{M,\mathbf{Z},N}(q) = \inf\{(f_M + \delta_{\mathbf{Z}})(z) \mid Nz = q\}, \quad \forall q \in \mathbb{R}^l$$

The following statement summarizes several facts that can be deduced from classical convex analysis; see for instance (Auslender and Teboulle, 2003, Theorem 2.5.4 and Proposition 2.6.1) (Rockafellar, 1970, Theorem 9.2).

Lemma 1.1. *Given a symmetric positive semi-definite matrix M of dimension k, a nonempty convex closed subset $\mathbf{Z} \subseteq \mathbb{R}^k$ and a matrix N of dimension $k \times l$ we have that:*

- $f_M + \delta_{\mathbf{Z}} \in \mathcal{F}(\mathbb{R}^k)$ *and* $(f_M + \delta_{\mathbf{Z}})_\infty = \delta_{\ker(M)} + \delta_{\mathbf{Z}_\infty}$
- $g_{M,\mathbf{Z}} \in \mathcal{F}(\mathbb{R}^k)$ *with* $\overline{\mathrm{dom}(g_{M,\mathbf{Z}})} = (\ker(M) \cap \mathbf{Z}_\infty)^\circ$
- $h_{M,\mathbf{Z},N} \in \mathcal{F}(\mathbb{R}^l)$, *the infimum in the definition of* $h_{M,\mathbf{Z},N}$ *is attained whenever* $q \in \mathrm{dom}(h_{M,\mathbf{Z},N})$ *and*

$$(h_{M,\mathbf{Z},N})_\infty(d) = \inf\{\delta_{\ker(M)}(z) + \delta_{\mathbf{Z}_\infty}(z) \mid Nz = d\}$$

We suppose $T > 0$ is fixed. In our setting an arc is just a function $x : [0,T] \to \mathbb{R}^n$. The space of absolutely continuous and bounded variation arcs are denoted by **AC** and **BV**, respectively. If $x \in \mathbf{BV}$ then $x(t^-)$ and $x(t^+)$ stand for the left and right limits of x at t. Given a measure μ on $[0,T]$ we denote by $\mathbf{L}^1_n(d\mu)$ the (equivalence class of) $d\mu$ integrable arcs. The Lebesgue measure is dt.

2. LQ AND DUAL PROBLEMS

In this section we introduce the LQ problem and the value function associated with this mapping. Furthermore, we exhibit the dual problem and its corresponding value function. Due to the state constraints and a lack of coercivity, we show how to extend both problems where the state trajectories are arcs of bounded variation.

Throughout the paper, $\mathbf{X} \subseteq \mathbb{R}^n$ and $\mathbf{U} \subseteq \mathbb{R}^m$ are given sets, A, P, Q are $n \times n$ matrices, B is a $n \times m$ matrix and R is a $m \times m$ matrix. The following are the basic assumptions we consider:

Hypothesis 1. *$\mathbf{X}$ and $\mathbf{U}$ are convex closed nonempty sets, P and Q are symmetric positive definite matrices and R is a symmetric positive semi-definite matrix.*

Remark 2.1. *Note that Hypothesis 1, in particular the fact that P is positive definite implies that $g_{P,\mathbf{X}}$ is finite everywhere with* $\mathrm{dom}(g_{P,\mathbf{X}}) = \mathbb{R}^n$. *Note that the same holds if P is only positive semi-definite but $\mathbf{X}$ is compact; this is due to the fact either way* $\ker(P) \cap \mathbf{X}_\infty = \{0\}$.

Besides the basic assumptions (Hypothesis 1), we also require a type of constraint qualification, which will imply there are feasible arcs for both primal and dual problems.

Hypothesis 2. $\mathrm{int}(\mathbf{X}) \neq \emptyset$ *and* $\mathrm{int}(\mathbf{Y}) \neq \emptyset$, *where*

$$\mathbf{Y} = \{y \in \mathbb{R}^n \mid g_{R,\mathbf{U}}(B^* y) < +\infty\}$$

Moreover, there are $x, y \in \mathbf{AC}$ such that $x(t) \in \mathrm{int}(\mathbf{X})$ and $y(t) \in \mathrm{int}(\mathbf{Y})$ for any $t \in [0,T]$, and

$$\dot{x}(t) \in Ax(t) + B\mathbf{U}, \quad \textit{for a.e. } t \in [0,T].$$

Remark 2.2. *Note that Hypothesis 2 is trivially satisfied in some recognizable cases: if there are $\bar{x} \in \mathrm{int}(\mathbf{X})$ and $\bar{u} \in \mathbf{U}$ such that $0 = A\bar{x} + B\bar{u}$, then the arc defined via $x(t) := \bar{x}$ for any $t \in [0,T]$ satisfies the condition for* $\mathrm{int}(\mathbf{X})$. *Also, if R is positive definite or $\mathbf{U}$ is compact, then $\mathbf{Y} = \mathbb{R}^n$ and any $y \in \mathbf{AC}$ provides the condition for* $\mathrm{int}(\mathbf{Y})$.

2.1. **LQ problem.** We aim at studying the following optimization problem.

Problem 2.1 (LQ problem)**.** *Given $\tau \in [0,T]$ and $\xi \in \mathbf{X}$, minimize over all $x \in \mathbf{AC}$ and $u \in \mathbf{L}^1_m(dt)$*

$$\int_\tau^T [f_P(x(t)) + f_R(u(t))]dt + f_Q(x(T)) \tag{1}$$

subject to $x(\tau) = \xi$, the dynamical constraint

$$\dot{x}(t) = Ax(t) + Bu(t), \quad \textit{for a.e. } t \in [\tau,T] \tag{2}$$

the input constraint

$$u(t) \in \mathbf{U}, \textit{ for a.e. } t \in [\tau,T] \tag{3}$$

and the state constraint

$$x(t) \in \mathbf{X}, \ \forall t \in [\tau,T]. \tag{4}$$

The value function associated with this optimal control problem is denoted by $\vartheta_\tau(\xi)$ and equals the smallest value of the objective function (1) while satisfying the restrictions (2), (3) and (4). The parameters (τ,ξ) appear in the initial condition $x(\tau) = \xi$. Let us point out that, a priori, the value function is defined only on $\mathbf{X}$, and that in general, its effective domain is only a subset of $\mathbf{X}$. By convention, thus, we have $\vartheta_\tau(\xi) = +\infty$

whenever $\xi \notin \mathbf{X}$ and $\tau \in [0,T]$. Furthermore, it is worthwhile to notice that $\vartheta_T(\xi) = f_Q(\xi)$ for any $\xi \in \mathbf{X}$.

It is not difficult to check the following claims about ϑ_τ.

Lemma 2.1. *Under Hypothesis 1 and 2 the value function ϑ_τ is proper nonnegative and convex for any $\tau \in [0,T]$.*

2.2. **Dual problem.** The convexity in Problem 2.1 allows to developed a duality theory inspired by the theory of conjugate functions of convex analysis. We define the dual to Problem 2.1 next.

Problem 2.2 (Dual problem)**.** *Given $\tau \in [0,T]$ and $\eta \in \mathbf{Y}$, minimize over all $y \in \mathbf{AC}$ the functional*

$$\int_\tau^T [g_{P,\mathbf{X}}(\dot{y}(t) + A^* y(t)) + g_{R,\mathbf{U}}(B^* y(t))]dt + f_{Q^{-1}}(y(T))$$

subject to $y(\tau) = -\eta$ and to the state constraint

$$y(t) \in \overline{\mathbf{Y}}, \quad \forall t \in [\tau, T] \tag{5}$$

We denote by $\omega_\tau(\eta)$ the value function associated with Problem 2.2, that is, the least value of the objective function in Problem 2.2 over arcs satisfying the initial condition $y(\tau) = -\eta$ and the state constraint (5). It is again worthwhile to notice that ω_τ is defined on $\mathbb{R}^n$ and that $\omega_T(\eta) = f_{Q^{-1}}(\eta)$ for any $\eta \in \overline{\mathbf{Y}}$ (because $f_{Q^{-1}}$ is even).

By Lemma 1.1, we readily see that ω_τ is also convex.

Lemma 2.2. *Under Hypothesis 1, ω_τ is convex for $\tau \in [0,T]$.*

Let us point out that the value functions ϑ_τ and ω_τ satisfy a weak duality relation.

Proposition 2.1. *Under Hypothesis 1 , for any $\tau \in [0,T]$*

$$\vartheta_\tau(\xi) + \omega_\tau(\eta) \geq \langle \xi, \eta \rangle, \quad \forall \xi, \eta \in \mathbb{R}^n$$

If Hypothesis 2 holds, then ω_τ is also a proper function.

2.3. **Extended LQ problems and their duals.** When state constraints are involved, it is expected that the adjoint arc will have jumps whenever the constraint is active, and this naturally leads to the dual problem minimizing over **BV** rather than **AC**. The philosophy of convex analysis is that symmetry between primal and dual problems should be adhered to, and hence the primal problem should be extended to minimizing over arcs of bounded variation as well. For these reasons, and following the ideas introduced in Rockafellar (1976) we extend Problem 2.1 and Problem 2.2 to minimization problems over **BV** in the following manner. Any $z \in \mathbf{BV}$ induces a Borel measure dz and has the Lebesgue decomposition of form

$$dz(t) = \dot{z}(t)dt + \pi_z(t)d\mu(t),$$

where $\dot{z}(t)dt$ is the absolutely continuous part and $\pi_z(t)$ is the singular part with respect to Lebesgue measure.

The extended LQ problem is defined as follows.

Problem 2.3 (Extended LQ problem)**.** *Given* $\tau \in [0,T]$ *and* $\xi \in \mathbb{R}^n$*, minimize over all* $x \in \mathbf{BV}$*,* $u \in \mathbf{L}^1_m(dt)$ *and* $\theta \in \mathbf{L}^1_m(d\mu)$ *the functional*

$$\int_\tau^T [f_P(x(t)) + f_R(u(t))]dt + f_Q(x(T^+)) \tag{6}$$

subject to $x(\tau^-) = \xi$*, the dynamical constraint* (2)*, the input constraint* (3)*, the state constraint* (4)*, the impulsive input constraint*

$$\theta(t) \in \ker(R) \cap \mathbf{U}_\infty, \quad d\mu\text{-a.e. } t \in [0,T]. \tag{7}$$

and the impulsive dynamical constraint

$$\pi_x(t) = B\theta(t), \quad d\mu\text{-a.e. } t \in [0,T]. \tag{8}$$

Let us emphasize that the state constraint (4) can be interpreted in several different ways because of possible jumps at the initial and/or terminal times. For this reason we consider two value functions associated with Problem 2.3, one whose domain is contained in **X** (denoted by $\vartheta_\tau^{\text{ext}}(\xi)$) and another whose domain may be larger than **X** (denoted by $\mathbf{v}_\tau^{\text{ext}}(\xi)$). Both value functions will be defined as the infimum value of the objective (6) in Problem 2.3 over **BV** arcs satisfying the initial condition $x(\tau^-) = \xi$ and rest of the constraints. The constraint (4) is interpreted as follows:

- The value function $\vartheta_\tau^{\text{ext}}(\xi)$ will be associated with the state constraints:

  $$\xi,\ x(T^+),\ x(t) \in \mathbf{X}, \quad \forall t \in (\tau, T)$$

  Moreover, we set $\vartheta_\tau^{\text{ext}}(\xi) = +\infty$ if $\xi \notin \mathbf{X}$.
- The value function $\mathbf{v}_\tau^{\text{ext}}(\xi)$ will be associated with the state constraints:

  $$x(t) \in \mathbf{X}, \quad \forall t \in (\tau, T)$$

Note that for any $\tau \in [0,T]$, we have

$$\mathbf{v}_\tau^{\text{ext}}(\xi) \le \vartheta_\tau^{\text{ext}}(\xi) \le \vartheta_\tau(\xi), \quad \forall \xi \in \mathbb{R}^n.$$

Also, as claimed for ϑ_τ, it is not difficult to see that the extended value functions are also convex and proper.

Lemma 2.3. *Under Hypotheses 1 and 2, ϑ_τ^{ext} and $\mathbf{v}_\tau^{ext}$ are proper convex functions for any $\tau \in [0,T]$.*

The extended dual problem is as follows.

Problem 2.4 (Extended dual problem)**.** *Given $\tau \in [0,T]$ and $\eta \in \mathbb{R}^n$, minimize over all $y \in \mathbf{BV}$ the functional*

$$\int_\tau^T [g_{P,\mathbf{X}}(\dot{y}(t) + A^* y(t)) + g_{R,\mathbf{U}}(B^* y(t))]dt \\ + \int_\tau^T \sigma_{\mathbf{X}}(\pi_y(t))d\mu(t) + f_{Q^{-1}}(y(T^+))$$

subject to $y(\tau^-) = -\eta$ and to the state constraints (5).

In this case we also consider two value functions, $\omega_\tau^{\text{ext}}(\eta)$ and $\mathbf{w}_\tau^{\text{ext}}(\eta)$ derived from two different interpretations of the state constraint (5). Both are defined as the least possible value that the cost of Problem 2.4 can take while satisfying the initial condition $y(\tau^-) = -\eta$, but (5) is interpreted as follows:

- The value function $\omega_\tau^{\text{ext}}(\eta)$ will be associated with the state constraints:
$$\eta,\ y(T^+),\ y(t) \in \overline{\mathbf{Y}}, \quad \forall t \in (\tau, T)$$
Furthermore, we set $\omega_\tau^{\text{ext}}(\eta) = +\infty$ if $-\eta \notin \overline{\mathbf{Y}}$.
- The value function $\mathbf{w}_\tau^{\text{ext}}(\eta)$ will be associated with the state constraints:
$$y(t) \in \overline{\mathbf{Y}}, \quad \forall t \in (\tau, T)$$

Note as well that for any $\tau \in [0,T]$

$$\mathbf{w}_\tau^{\text{ext}}(\xi) \le \omega_\tau^{\text{ext}}(\xi) \le \omega_\tau(\xi), \quad \forall \eta \in \mathbb{R}^n.$$

Similarly as for the value function ω_τ, Lemma 1.1 and Proposition 2.1 yield to the following statement.

Lemma 2.4. *Under Hypotheses 1 and 2, ω_τ^{ext} and $\mathbf{w}_\tau^{ext}$ are proper convex functions for any $\tau \in [0,T]$.*

Notice that the formulations of Problem 2.3 and Problem 2.4 are independent of the singular measure μ. On the one hand, it is clear that if $\mathbf{U}$ is bounded or R is invertible, then Problem 2.3 recovers the formulation of Problem 2.1 because (7) implies that $\theta(t) = 0$ for $d\mu$-a.e. $t \in [0,T]$; actually this is true whenever $\ker(R) \cap \mathbf{U}_\infty = \{0\}$. On the other hand, only if $\mathbf{X} = \mathbb{R}^n$, that is no state constraints are considered on the LQ problem, Problem 2.4 recovers the formulation of Problem 2.2. Let us emphasize then that if $\mathbf{Y} = \mathbf{X} = \mathbb{R}^n$, the three primal and dual value functions have exactly the same value, and hence, the problems can be studied in the light of the unconstrained theory developed in Rockafellar and Wolenski (2000).

2.4. **Main results.** The main theorem presented in this paper is a stronger version of Proposition 2.1 and it reads as follows.

Theorem 2.1. *Under Hypotheses 1 and 2, for any* $\tau \in [0,T]$*, the lower semicontinuous envelop of* ϑ_τ *and* ω_τ *agree respectively with* ϑ_τ^{ext} *and* ω_τ^{ext}*. Moreover, we have that*

$$(\mathbf{w}_\tau^{ext})^*(\xi) = \vartheta_\tau(\xi), \quad \forall \xi \in \mathrm{ri}(\mathrm{dom}(\vartheta_\tau))$$
$$(\mathbf{v}_\tau^{ext})^*(\eta) = \omega_\tau(\eta), \quad \forall \eta \in \mathrm{ri}(\mathrm{dom}(\omega_\tau)).$$

Theorem 2.1 was proved in (Rockafellar and Wolenski, 2000, Theorem 5.1) when no state constraints are involved ($\mathbf{X} = \mathbb{R}^n$) and the cost is coercive in the control (R is positive definite). However, the presence of state constraints and possibly recession directions on the cost demands further developments that go beyond the exposition and proof presented in that paper. The technique we use to prove the main result requires to embed the original problem into a larger space, namely, the space of arcs of bounded variation. For this purpose, we first formulate the LQ problem at hand as a calculus of variation problem by means of a standard infinite penalization technique.

Theorem 2.1 will be a direct consequence of the following intermediate result, which determines the behavior of the value functions of the extended problems.

Theorem 2.2. *Under Hypotheses 1 and 2, for any* $\tau \in [0,T]$ *we have that* ϑ_τ^{ext}, $\mathbf{v}_\tau^{ext}$, ω_τ^{ext}, $\mathbf{w}_\tau^{ext} \in \mathcal{F}(\mathbb{R}^n)$ *and the pairs* $(\vartheta_\tau^{ext}, \mathbf{w}_\tau^{ext})$ *and* $(\mathbf{v}_\tau^{ext}, \omega_\tau^{ext})$ *are conjugate to each other.*

Let us mention that in the case the cost of Problem 2.1 is coercive or more generally if $\ker(R) \cap \mathbf{U}_\infty = \{0\}$, Theorem 2.2 provides a finer version of Theorem 2.1.

Corollary 2.1. *Suppose Hypothesis 1 and 2 hold. Assume further that* $\ker(R) \cap \mathbf{U}_\infty = \{0\}$*. Then, for any* $\tau \in [0,T]$ *we have that* ϑ_τ *and* ω_τ^{ext} *are conjugate to each other, and moreover* $(\omega_\tau)^* = \vartheta_\tau$*.*

3. Formulation as a calculus of variation problem

In this section, we exhibit a way to describe the LQ problem as a fully convex Bolza problem and then, we present the corresponding extended problem over arcs of bounded variation. This section explains as well how the dual problems have been obtained. For this purpose let us introduce $L : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ via

$$L(x,v) = (f_P + \delta_{\mathbf{X}})(x) + h_{R,\mathbf{U},B}(v - Ax)$$

It is not difficult to check that, under Hypothesis 1, the Lagrangian L is a proper nonnegative and convex function. On the other hand, by Lemma 1.1, the Lagrangian L is also lower semicontinuous and the infimum in the definition of $h_{R,\mathbf{U},B}$ is attained whenever $L(x,v)$ is finite. In other words, there is $u \in \mathbf{U}$ such that $v = Ax + Bu$ and

$$h_{R,\mathbf{U},B}(v - Ax) = f_R(u)$$

It is also easy to check that

$$\mathbf{X} = \{x \in \mathbb{R}^n \mid \exists v \in \mathbb{R}^n \text{ such that } L(x,v) < +\infty\}$$

The value function associated with this Lagrangian and the terminal cost f_Q is given by

$$V_\tau(\xi) = \inf_{\substack{x \in \mathbf{AC} \\ x(\tau) = \xi}} \int_\tau^T L(x(t), \dot{x}(t))dt + f_Q(x(T))$$

Note that $V_\tau(\xi) \leq \vartheta_\tau(\xi)$ for any $\xi \in \mathbf{X}$ and $\tau \in [0,T]$. Indeed, if $x \in \mathbf{AC}$ and $u \in \mathbf{L}^1_m(dt)$ are a feasible arc and control for Problem 2.1, then $\dot{x}(t) = Ax(t) + Bu(t)$ and

$$L(x(t), \dot{x}(t)) \leq f_P(x(t)) + f_R(u(t)), \quad \text{for a.e. } t \in [\tau, T].$$

Therefore, the cost of Problem 2.1 is minored by the cost of Problem 2.3. We claim that we actually have

$$\vartheta_\tau(\xi) = V_\tau(\xi), \quad \forall \xi \in \mathbf{X}. \tag{9}$$

Recall that L is nonnegative and moreover, we have that by allowing the integral cost to take infinite values, we are handling implicitly the constraints over the input and state of the system. Indeed, note that $V_\tau(\xi) \in \mathbb{R}$ implies $x(t) \in \mathbf{X}$ for any $t \in [\tau, T]$ (because $x \in \mathbf{AC}$) and for almost every $t \in [\tau, T]$ we have

$$\phi(t) := \inf_{u \in \mathbf{U}} \{f_R(u) \mid \dot{x}(t) = Ax(t) + Bu\} < +\infty.$$

This in turn means that the mapping defined by

$$\mathbb{U}(t) := \{u \in \mathbf{U} \mid \phi(t) = f_R(u) \text{ and } \dot{x}(t) = Ax(t) + Bu\}$$

if $\dot{x}(t)$ exists and $\phi(t)$ is finite, and $\mathbb{U}(t) := \mathbf{U}$ otherwise, has nonempty closed images for any $t \in [\tau, T]$. Also, for any $u \in \mathbf{U}$, the mapping $t \mapsto f_R(u) + \delta_{\{0\}}(\dot{x}(t) - Ax(t) - Bu)$ is measurable on $[\tau, T]$ and so $t \mapsto \phi(t)$ is also measurable. Consequently, for any open subset of $O \subseteq \mathbb{R}^m$ we have that the set $\{t \in [\tau, T] \mid U(t) \cap O)\}$ is also a measurable set. Therefore, by the

(Rockafellar and Wets, 2009, Corollary 14.6), there is $u(t) \in \mathbf{U}$ such that the infimum in the definition of $\phi(t)$ is attained at $u(t)$ with the mapping $t \mapsto u(t)$ being measurable. This also provides the dynamical constraint $\dot{x}(t) = Ax(t) + Bu(t)$ for a.e. $t \in [0,T]$, which implies that there is no loss of generality in assuming that $u \in \mathbf{L}^1_m(dt)$ because $Bu \in \mathbf{L}^1_n(dt)$. Therefore, (9) folows.

On the other hand, following the definitions provided in Rockafellar and Wolenski (2000), we have that the dual value function associated with V is

$$W_\tau(\eta) = \inf_{\substack{y \in \mathbf{AC} \\ y(\tau^-) = -\eta}} \int_\tau^T K(y(t),\dot{y}(t))dt + f_{Q^{-1}}(y(T))$$

with $K : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ being given by

$$K(y,w) = L^*(w,y)$$

Using directly the definition of the conjugate we get

$$K(y,w) = g_{P,\mathbf{X}}(w + A^* y) + g_{R,\mathbf{U}}(B^* y)$$

It is also possible to check that Hypothesis 1 leads to

$$\mathbf{Y} = \{y \in \mathbb{R}^n \mid \exists w \in \mathbb{R}^n \text{ such that } K(y,w) < +\infty\}$$

from where we conclude that

$$\omega_\tau(\eta) = W_\tau(\eta), \quad \forall \eta \in \overline{\mathbf{Y}}. \tag{10}$$

3.1. **Formulation for the extended problems.** Following the theory developed in Rockafellar (1976), we now introduce the extended version of the calculus of variation formulation of the LQ problem to arcs of bounded variation. This problem can be stated in two ways, depending on whether or not we allow to jump at the initial and terminal times from outside $\mathbf{X}$ into the state constraint. We consider the following value functions

$$V_\tau^{\text{ext}}(\xi) = \inf_{\substack{\mathbf{x} \in \mathbf{BV} \\ x(\tau^-) = \xi}} \left\{ J_\tau(x) + (f_Q + \delta_{\mathbf{X}})(x(T^+)) + \delta_{\mathbf{X}}(\xi) \right\}$$

and $\mathbb{V}_\tau^{\text{ext}}(\xi) = \inf\limits_{\substack{\mathbf{x} \in \mathbf{BV} \\ x(\tau^-) = \xi}} \left\{ J_\tau(x) + f_Q(x(T^+)) \right\}$ where the functional $J_\tau : \mathbf{BV} \to \mathbb{R} \cup \{+\infty\}$ is given by

$$J_\tau(x) = \int_\tau^T L(x(t),\dot{x}(t))dt + \int_\tau^T L_\infty(0, \pi_x(t))d\mu(t)$$

Note that $\mathbb{V}_\tau^{\text{ext}}(\xi) \leq V_\tau^{\text{ext}}(\xi)$ and that $\mathbb{V}_\tau^{\text{ext}}(\xi)$ is the value function that allows to jump from outside the state constraint at the initial and terminal times.

Lemma 1.1 combined with some calculus rules for recession functions lead to

$$L_\infty(0,d) = \inf\{\delta_{\ker(R)}(z) + \delta_{U_\infty}(z) \mid Bz = d\}.$$

This means that $V_\tau^{\text{ext}}(\xi)$ and $\mathbb{V}_\tau^{\text{ext}}(\xi)$ are nonnegative, and moreover, if $V^{\text{ext}}(\tau,\xi) < +\infty$, then for $d\mu$-a.e. on $[\tau,T]$ we must have $L_\infty(0,\pi_x(t)) = 0$ and

$$\pi_x(t) = B\theta(t) \quad \text{for some} \quad \theta(t) \in \ker(R) \cap \mathbf{U}_\infty$$

Using similar arguments to those to check (9) and (Rockafellar and Wets, 2009, Corollary 14.6), it is not difficult to see that $t \mapsto \theta(t)$ can be supposed measurable and that there is no lost in generality in assuming that $\theta \in \mathbf{L}^1_m(d\mu)$. Therefore, similarly as done for ϑ_τ and V_τ, we have that

$$\vartheta_\tau^{\text{ext}}(\xi) = V_\tau^{\text{ext}}(\xi) \quad \text{and} \quad \mathbf{v}_\tau^{\text{ext}}(\xi) = \mathbb{V}_\tau^{\text{ext}}(\xi) \tag{11}$$

The extended versions of the dual LQ problem to **BV** are given as follows.

$$W_\tau^{\text{ext}}(\eta) = \inf_{\substack{\mathbf{y}\in\mathbf{BV} \\ \mathbf{y}(\tau^-)=-\eta}} I_\tau(y) + (f_{Q^{-1}} + \delta_{\overline{\mathbf{Y}}})(y(T^+)) + \delta_{\overline{\mathbf{Y}}}(-\eta)$$

and $\mathbb{W}_\tau^{\text{ext}}(\eta) = \inf_{\substack{\mathbf{y}\in\mathbf{BV} \\ \mathbf{y}(\tau)=-\eta}} \left\{ I_\tau(y) + f_{Q^{-1}}(y(T^+)) \right\}$ where the functional $I_\tau : \mathbf{BV} \to \mathbb{R} \cup \{+\infty\}$ is given by

$$I_\tau(y) = \int_\tau^T K(y(t),\dot{y}(t))dt + \int_\tau^T K_\infty(0,\pi_y(t))d\mu(t)$$

Also, calculus rules for recession functions imply that

$$K_\infty(0,d) = \sigma_{\mathbf{X}}(d), \quad \forall d \in \mathbb{R}^n.$$

Hence, it is not difficult to see that

$$\omega_\tau^{\text{ext}}(\eta) = W_\tau^{\text{ext}}(\eta) \quad \text{and} \quad \mathbf{w}_\tau^{\text{ext}}(\eta) = \mathbb{W}_\tau^{\text{ext}}(\eta) \tag{12}$$

4. Proof of main results

In this section we provide the arguments that prove Theorem 2.1 and Theorem 2.2, we also prove Proposition 2.1. To begin with, we state an intermediate lemma and for sake of the exposition we introduce the following notation. For given $\Lambda : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$, $\tau \in [0,T]$ and $a,b \in \mathbb{R}^n$ we define the *fundamental kernel* via

$$E_\tau^\Lambda(a,b) := \inf_{z\in\mathbf{BV}} \left\{ \int_\tau^T \Lambda(z(t),\dot{z}(t))dt \,\middle|\, \begin{array}{l} z(\tau^-) = a, \\ z(T^+) = b \end{array} \right\}$$

Lemma 4.1. *[(Rockafellar, 1976, Theorem 3 and 3')] Assume that Hypothesis 1 and 2 hold.*

(1) *For any $a,b\in\mathbb{R}^n$ we have that*

$$E^L_\tau(a,b)=\sup\langle y(T^+),b\rangle-\langle y(\tau^-),a\rangle-I_\tau(y)$$
$$E^K_\tau(a,b)=\sup\langle x(T^+),b\rangle-\langle x(\tau^-),a\rangle-J_\tau(x)$$

where the supremum are taken over all $y\in\mathbf{BV}$ such that $y(\tau^-)$, $y(T^+)\in\overline{\mathbf{Y}}$ and over all $x\in\mathbf{BV}$ such that $x(\tau^-)$, $x(T^+)\in\mathbf{X}$, respectively.

(2) *For any $a,b\in\mathbf{X}$ we have that*

$$E^L_\tau(a,b)=\sup_{y\in\mathbf{BV}}\langle y(T^+),b\rangle-\langle y(\tau^-),a\rangle-I_\tau(y)$$

and if in addition $(a,b)\in\mathrm{ri}(\mathrm{dom}(E^L_\tau)\cap\mathbf{X}\times\mathbf{X})$

$$E^L_\tau(a,b)=\inf_{x\in\mathbf{AC}}\{J_\tau(x)\mid\ x(\tau)=a,\ x(T)=b\}$$

(3) *For any $a,b\in\overline{\mathbf{Y}}$ we have that*

$$E^K_\tau(a,b)=\sup_{x\in\mathbf{BV}}\langle x(T^+),b\rangle-\langle x(\tau^-),a\rangle-J_\tau(x)$$

and if in addition $(a,b)\in\mathrm{ri}(\mathrm{dom}(E^K_\tau)\cap\mathbf{Y}\times\mathbf{Y})$

$$E^K_\tau(a,b)=\inf_{y\in\mathbf{AC}}\{I_\tau(y)\mid\ y(\tau)=a,\ y(T)=b\}$$

4.1. **Proof of Proposition 2.1.** It is enough to check the inequality for $\xi\in\mathrm{dom}(\vartheta_\tau)$ and $\eta\in\mathbb{R}^n$ such that $\omega_\tau(\eta)<+\infty$, otherwise the conclusion is straightforward (considering the convention $+\infty-\infty=+\infty$). Note that we are not assuming a priori that $\omega_\tau(\eta)>-\infty$. Let $\varepsilon>0$ and $(x,u)\in\mathbf{AC}\times\mathbf{L}^1_m(dt)$ be an ε-suboptimal solution to Problem 2.1. Since $\omega_\tau(\eta)<+\infty$ there is $y\in\mathbf{AC}$ feasible for Problem 2.2, that is, $y(\tau)=-\eta$ and (5) holds. Note that, thanks to the Legendre-Fenchel inequality we have for a.e. on $[\tau,T]$

$$(f_P+\delta_{\mathbf{X}})(x)+g_{P,\mathbf{X}}(\dot{y}+A^*y)\geq\langle x,\dot{y}+A^*y\rangle$$
$$(f_R+\delta_{\mathbf{U}})(u)+g_{R,\mathbf{U}}(B^*y)\geq\langle u,B^*y\rangle$$

Let $\varphi(x,u,y)$ be the sum of costs in Problem 2.1 and Problem 2.2 associated with $x\in\mathbf{AC}$, $u\in\mathbf{L}^1_m(dt)$ and $y\in\mathbf{AC}$. By the dynamical constraint (2) we get that

$$\varphi(x,u,y)\geq\int_\tau^T\frac{d}{dt}\langle x(t),y(t)\rangle dt+f_Q(x(T))+f_{Q^{-1}}(y(T))$$

Since $(f_Q)^*=f_{Q^{-1}}$ and f_Q is even, the Legendre-Fenchel inequality leads to

$$f_Q(x(T))+f_{Q^{-1}}(y(T))\geq-\langle x(T),y(T)\rangle$$

Hence, since $y \in \mathbf{AC}$ is an arbitrary feasible trajectory for Problem 2.2, it follows that

$$\vartheta_\tau(\xi) + \omega_\tau(\eta) + \varepsilon \geq \langle x(\tau), -y(\tau) \rangle = \langle \xi, \eta \rangle$$

Letting $\varepsilon \to 0^+$ the inequality on the statement follows. Moreover, if Hypothesis 2 holds, then $\mathrm{dom}(\vartheta_\tau) \neq \emptyset$ and

$$\omega_\tau(\eta) \geq \langle \xi, \eta \rangle - \vartheta_\tau(\xi) > -\infty, \quad \forall \eta \in \mathbb{R}^n,\ \forall \xi \in \mathrm{dom}(\vartheta_\tau)$$

Finally, if $y \in \mathbf{AC}$ is the trajectory given by Hypothesis 2, then $\omega_\tau(y(\tau)) < +\infty$ for any $\tau \in [0,T]$. Consequently, ω_τ is proper for any $\tau \in [0,T]$ and the proof is complete.

4.2. **Proof of Theorem 2.1.** In the light of (9), (10), (11) and (12), the conclusion will follow from Theorem 2.2 combined with the facts that

$$V_\tau^{\mathrm{ext}}(\xi) = V_\tau(\xi), \quad \forall \xi \in \mathrm{ri}(\mathrm{dom}(V_\tau))$$
$$W_\tau^{\mathrm{ext}}(\eta) = W_\tau(\eta), \quad \forall \eta \in \mathrm{ri}(\mathrm{dom}(W_\tau))$$

We prove the first equation, the second one can be proved using symmetric arguments, and so we skip it.

Note that for any $\xi \in \mathrm{dom}(V_\tau)$ we have

$$V_\tau^{\mathrm{ext}}(\xi) = \inf_{\mathbf{x}_T \in \mathbf{X}} f_Q(\mathbf{x}_T) + E_\tau^L(\xi, \mathbf{x}_T)$$

Hence, if $\xi \in \mathrm{ri}(\mathrm{dom}(V_\tau))$ we have that

$$V_\tau^{\mathrm{ext}}(\xi) = \inf_{\substack{\mathbf{x}_T \in \mathbb{R}^n \\ (\xi, \mathbf{x}_T) \in \mathrm{ri}(\mathrm{dom}(E_\tau^L) \cap \mathbf{X} \times \mathbf{X})}} f_Q(\mathbf{x}_T) + E_\tau^L(\xi, \mathbf{x}_T)$$

By point (2) in Lemma 4.1 we get that

$$V_\tau^{\mathrm{ext}}(\xi) = \inf_{\substack{x \in \mathbf{AC},\ x(\tau) = \xi \\ (\xi, x(T)) \in \mathrm{ri}(\mathrm{dom}(E_\tau^L) \cap \mathbf{X} \times \mathbf{X})}} f_Q(x(T)) + J_\tau(x)$$

From where we get that $V_\tau^{\mathrm{ext}}(\xi) = V_\tau(\xi)$ and so the conclusion follows and the proof of the theorem is complete.

4.3. **Proof of Theorem 2.2.** Let $\tau \in [0,T]$ be given. Thanks to (11) and (12), in order to prove $\mathbf{w}_\tau^{\mathrm{ext}} = (\vartheta_\tau^{\mathrm{ext}})^*$ and $\mathbf{v}_\tau^{\mathrm{ext}} = (\omega_\tau^{\mathrm{ext}})^*$, it is enough to show that $\mathbb{W}_\tau^{\mathrm{ext}} = (V_\tau^{\mathrm{ext}})^*$ and $\mathbb{V}_\tau^{\mathrm{ext}} = (W_\tau^{\mathrm{ext}})^*$. Note that this also will prove that $\mathbf{v}_\tau^{\mathrm{ext}}$ and $\mathbf{w}_\tau^{\mathrm{ext}}$ are lower semicontinuous, since they can be written as the supremum of a family of affine functions.

Let us focus on $\mathbb{W}_\tau^{\mathrm{ext}} = (V_\tau^{\mathrm{ext}})^*$, the other one follows by symmetric arguments. We begin by pointing out that

$$\mathbb{W}_\tau^{\mathrm{ext}}(\eta) = \inf_{\mathbf{y}_T \in \mathbb{R}^n} f_{Q^{-1}}(\mathbf{y}_T) + E_\tau^K(-\eta, \mathbf{y}_T), \quad \forall \eta \in \mathbb{R}^n$$

Thus, thanks to point (1) in Lemma 4.1 we have

$$\mathbb{W}_\tau^{\mathrm{ext}}(\eta) = \inf_{\mathbf{y}_T\in\mathbb{R}^n}\sup_{x\in\mathbf{BV}} -\Psi(\mathbf{y}_T,x)+f_{Q^{-1}}(\mathbf{y}_T)$$

where $\Psi:\mathbb{R}^n\times\mathbf{BV}\to\mathbb{R}\cup\{+\infty\}$ is defined via

$$\Psi(\mathbf{y}_T,x) := J_\tau(x)-\langle x(\tau^-),\eta\rangle-\langle x(T^+),\mathbf{y}_T\rangle$$

if $x(\tau^-),x(T^+)\in\mathbf{X}$ and $\Psi(\mathbf{y}_T,x):=+\infty$ otherwise. Note that $\mathbf{y}_T\mapsto\Psi(\mathbf{y}_T,x)-f_{Q^{-1}}(\mathbf{y}_T)$ is concave and finite, and also its levels sets are compact subsets of $\mathbb{R}^n$ for any $x\in\mathrm{dom}(J_\tau)$ with $x(\tau^-),x(T^+)\in\mathbf{X}$. Furthermore, $x\mapsto\Psi(\mathbf{y}_T,x)$ is convex for any $\mathbf{y}_T\in\mathbb{R}^n$. Note that the subset of $\mathrm{dom}(J_\tau)$ that satisfies the additional condition $x(\tau^-),x(T^+)\in\mathbf{X}$ is convex and nonempty thanks to Hypotheses 1 and 2. Hence, in the light of the minimax theorem (Zalinescu, 2002, Theorem 2.10.2) we have that

$$\mathbb{W}_\tau^{\mathrm{ext}}(\eta) = -\inf_{\mathbf{x}\in\mathbf{BV}}\sup_{\mathbf{y}_T\in\mathbb{R}^n}\Psi(\mathbf{y}_T,\mathbf{x})-f_{Q^{-1}}(\mathbf{y}_T)$$

But, given $x\in\mathbf{BV}$, we have that (since $f_{Q^{-1}}$ is even)

$$\sup_{\mathbf{y}_T\in\mathbb{R}^n}\langle x(T^+),-\mathbf{y}_T\rangle-f_{Q^{-1}}(\mathbf{y}_T)=f_Q(x(T^+))$$

and so $\mathbb{W}_\tau^{\mathrm{ext}}(\eta)$

$$\begin{aligned}
&= -\inf_{\substack{x\in\mathbf{BV}\\ x(\tau^-),\ x(T^+)\in\mathbf{X}}} J_\tau(x)-\langle x(\tau^-),\eta\rangle+f_Q(x(T^+))\\
&= -\inf_{\xi\in\mathbf{X}}-\langle\xi,\eta\rangle+\inf_{\substack{x\in\mathbf{BV}\\ x(\tau^-)=\xi}} J_\tau(x)+(f_Q+\delta_{\mathbf{X}})(x(T^+))\\
&= \sup_{\xi\in\mathbf{X}}\langle\xi,\eta\rangle-V_\tau^{\mathrm{ext}}(\xi)=(V_\tau^{\mathrm{ext}})^*(\eta)
\end{aligned}$$

Using similar arguments and point (3) in Lemma 4.1, we can show that $W_\tau^{\mathrm{ext}}(\eta)=\sup_{\xi\in\mathbb{R}^n}\langle\xi,\eta\rangle-\psi_\tau(\xi)$ where $\psi_\tau(\xi)=\inf_{\substack{x\in\mathbf{BV}\\ x(\tau^-)=\xi}} J_\tau(x)+g_{Q^{-1},\overline{\mathbf{Y}}}(-x(T^+))$. Therefore, W_τ^{ext} is lower semicontinuous, and since we know that it is convex and proper (Lemma 2.4 combined with (12)), we get $W_\tau^{\mathrm{ext}}(\eta)=(W_\tau^{\mathrm{ext}})^{**}(\eta)=(\mathbb{V}_\tau^{\mathrm{ext}})^*(\eta)$.

Symmetric arguments (but using point (2) in Lemma 4.1) allow to show that V_τ^{ext} is lower semicontinuous, and since it is also convex and proper (Lemma 2.3 combined with (11)), we obtain $V_\tau^{\mathrm{ext}}(\xi)=(V_\tau^{\mathrm{ext}})^{**}(\xi)=(\mathbb{W}_\tau^{\mathrm{ext}})^*(\xi)$.

Finally, using (11) and (12), the conclusion follows.

4.4. **Proof of Corollary 2.1.** Since $\ker(R) \cap \mathbf{U}_\infty = \{0\}$, by Lemma 1.1 we get that $\mathbf{Y} = \mathbb{R}^n$, which in turns implies that $\mathbf{v}_\tau^{\text{ext}} = \vartheta_\tau^{\text{ext}} = \vartheta_\tau$ and $\mathbf{w}_\tau^{\text{ext}} = \omega_\tau^{\text{ext}}$. In particular ϑ_τ is lower semicontinuous. Moreover, since $\omega_\tau^{\text{ext}} = (\omega_\tau)^{**}$ we get that $(\mathbf{w}_\tau^{\text{ext}})^* = (\omega_\tau)^*$, and so the conclusion follows.

5. CONCLUSIONS AND FUTURE WORKS

We have shown that the value function of a LQ optimal control problem can be understood as the conjugate of the value function of a suitable dual problem. We have exhibited that to describe the value function of a constrained LQ problem, it's required to extend such problems to **BV** arcs. This in turn has shown that allowing jumps at the initial and final times is essential in these problems.

The main purpose of establishing the results in this paper is to understand the generalized characteristic methods proposed in Rockafellar and Wolenski (2000) in the case of impulsive systems. Let us point out that, for $\xi \in \text{ri}(\text{dom}(\vartheta_\tau))$, Theorem 2.1 leads to

$$\eta \in \partial\vartheta_\tau(\xi) \quad \Longleftrightarrow \quad \vartheta_\tau(\xi) + \mathbf{w}_\tau^{\text{ext}}(\eta) = \langle \xi, \eta \rangle$$

We expect value functions to be constant along optimal trajectories. If $x \in \mathbf{AC}$ and $y \in \mathbf{BV}$ are optimal for $\vartheta_\tau(\xi)$ and $\mathbf{w}_\tau^{\text{ext}}(\eta)$, respectively, the question is whether or not $y(t) \in \partial\vartheta_\tau(x(t))$ and if (x,y) is a Hamiltonian trajectory with final condition $y(T^+) = Qx(T^+)$.

Finally, let us mention that, the results stated in this paper can be adapted to time-dependent data, because the Lemma 4.1 holds as well. We have chosen a time-independent presentation only for the sake of simplicity.

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