OPTIMALITY CONDITIONS FOR LINEAR-CONVEX OPTIMAL CONTROL PROBLEMS WITH MIXED CONSTRAINTS

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Abstract. In this paper we provide sufficient optimality conditions for convex optimal control problems with mixed constraints. On one hand, the data delimiting the problem we consider is continuous and jointly convex on the state and control variables, but on the other hand, smoothness on the data of the problem, on the candidate to minimizer and/or on the multipliers are not needed. We also show that, under a suitable interior feasibility condition, the optimality conditions are necessary as well and can be written as a Maximum Principle in normal form. The novelty of this last part is that no additional regularity conditions on the mixed constraints, such as the Mangasarian-Fromovitz constraint qualification or the bounded slope condition, are required. A discussion about the regularity of the costate is also provided.

Keywords. Convex optimal control and Mixed constraints and Optimality conditions and Maximum Principle

1. Introduction

Optimality conditions for control problems with mixed constraints have been studied in a variety of settings. For example in [12], a Maximum Principle is presented for a very general control problem with smooth data and regular mixed constraints; see also [2, 4, 13, 5, 25, 1] for similar results. In [10, 8] nonsmooth data is considered and the regularity condition takes the form of a bounded slope condition; see also [17, 18] where the Mangasarian-Fromovitz condition is used instead. Recently, in [6] smooth data is considered but the regularity condition is dropped. There are also several works devoted to study second-order conditions; see for instance [23, 33, 20, 9, 24] and the references therein.

In this paper, we study Linear-Convex control problems, which means that the cost and constraints of the problem are jointly convex in the state and control variables and the dynamics is jointly linear in the state and control variables. In this context, optimality conditions can be derived using the duality theory developed by Rockafellar in [28] for Bolza problems posed over absolutely continuous arcs and later extended in [29] to cover arcs of bounded variation. According to this theory, if an *extremal* process of the Bolza problem has a *coextremal* process associated, then the extremal is a solution of the Bolza problem, the coextremal is a solution to a properly defined *dual Bolza problem* and there is no duality gap. Therefore, to derive optimality conditions for our control problem, we reformulate it as a convex Bolza problem in order to use the tools in [28] and [29]. In doing so, the coextremals play the role of the adjoint states (or costates), which are expected to have jumps at the boundary of the region defined by the mixed constraints. Since the costates are solutions to a dual problem as explained before, extending the admissible processes to cover arcs of bounded variation in the reformulated problem is a necessity in order to derive meaningful optimality conditions for the original problem.

It is important to note the advantages and disadvantages of employing this duality theory:

On one hand, we can prove sufficiency of the optimality conditions without imposing smoothness on the data of the problem, on the candidate to minimizer or/and on the multipliers. However, we need to restrict our attention to control problems with convex data. Let us also mention that, as far as we are aware, the study of (first order) sufficient optimality conditions for nonconvex problems with mixed constraints dates back from the 60's; see for instance [19] or the discussion in [14, Section 8]. In these works, in order to make the Maximum Principle also sufficient, besides smoothness on the data of the problem, additional regularity on the costate multiplier is required (piecewise smoothness at least). As pointed out above, in our setting no further regularity is demanded on the costate multiplier. Actually, in Theorem 2.1 arcs of bounded variation are allowed and the assumptions imposed are not related to the regularity of the multipliers, but to the structure of the constraints.

On the other hand, we can derive necessary optimality conditions for problems with mixed constraints without requiring any of the usual regularity conditions, such as the Mangasarian-Fromovitz constraint qualification or the bounded slope condition. Instead, we require a Slater type qualification condition and some coercivity assumption on the data of the problem. These assumptions also allow us to guarantee the *normality* of the multipliers in our necessary conditions; that is, the cost multiplier can be taken equal to 1. It is also noteworthy that these conditions can be written as a Maximum Principle.

The organization of this paper is as follows. In Section 2, we present the problem to be treated, the assumptions and the main theorems. In Section 3, we present the duality approach we use to prove our main theorems and provide some intermediate results required for proving our theorems. In Section 4, we provide the proof of the main results, first the sufficiency and then the necessity of the optimality conditions. Finally, in Section 5 we make a short discussion about the results we have presented in the paper, showing in particular an extension to problems with multiple mixed constraints and commenting on the regularity of the costate multiplier and its relation with the bounded slope condition. Also, an example is provided to demonstrate our findings.

2. Statement of the Problem and Main Result

The optimal control problem we deal with in this paper is the following:

$$
(\mathrm{P}_{\chi_0}) \qquad \begin{cases}
\text{Minimize} & J(x,u) := \displaystyle\int_0^1 \ell(t,x(t),u(t))dt, \\
\text{over all} & (x,u) \in AC([0,1];\mathbb{R}^n) \times L^\infty([0,1];\mathbb{R}^m), \\
\text{such that} & \dot{x}(t) = A(t)x(t) + B(t)u(t) + \varphi(t) \quad \text{a.e. on } [0,1], \\
& f(t,x(t),u(t)) \leq 0 \quad \text{a.e. on } [0,1], \\
& \zeta(t) = C(t)x(t) + D(t)u(t) \quad \text{a.e. on } [0,1], \\
& x(t) \in K(t) \quad \text{on } [0,1], \\
& u(t) \in U(t) \quad \text{a.e. on } [0,1], \\
& x(0) = \chi_0.
\end{cases}
$$

Here, $\ell, f : [0,1] \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ are given real-valued functions, $\varphi : [0,1] \to \mathbb{R}^n$ is a given *drift*, $\chi_0 \in \mathbb{R}^n$ is a given initial condition, $A(t), B(t), C(t)$ and $D(t)$ are $n \times n, n \times m, l \times n$ and $l \times m$ matrices for each $t \in [0,1]$ fixed, respectively, and $\zeta : [0,1] \to \mathbb{R}^l$ is a given *arc*. Also $K : [0,1] \rightrightarrows \mathbb{R}^n$ and $U : [0,1] \rightrightarrows \mathbb{R}^m$ are a set-valued maps, representing the (pure) state constraints and the (pure) control constraints, respectively.

Any pair $(x,u) \in AC([0,1];\mathbb{R}^n) \times L^\infty([0,1];\mathbb{R}^m)$ will be called a *process*, and a process satisfying all the constraints defining (P_{χ_0}) will be called an *admissible process*. An *optimal solution* for (P_{χ_0}) is an admissible process (x^*, u^*) such that $J(x^*, u^*) \leq J(x,u)$ for all admissible processes (x,u).

2.1. **Standing assumptions.** Our task in this paper is to provide sufficient conditions for an admissible process to be an optimal solution for (P_{χ_0}). For this purpose, the emphasis of this work is on Linear-Convex optimal control problems, in which the running cost is a real-valued function, jointly convex in the state and control variables, the dynamics is governed by an affine system, the mix constraint corresponds to the sublevel set of some convex function and the (pure) state and (pure) control constraints are convex sets at each instant of time. It is worth pointing out that these problems cover for instance the so-called *Linear Quadratic* case.

To be more precise, the basic assumptions regarding the data of the problem of concern are the following:

- (A1) The functions ℓ, f, φ and ζ are all continuous on $[0,1] \times \mathbb{R}^n \times \mathbb{R}^m$.
- (A2) The matrix-valued maps $t \mapsto A(t)$, $t \mapsto B(t)$, $t \mapsto C(t)$ and $t \mapsto D(t)$ are continuous on $[0,1]$.
- (A3) For all $t \in [0,1]$ fixed, the sets $K(t)$ and $U(t)$ are closed, convex and nonempty, with the multifunctions K and U having closed graph.
- (A4) For all $t \in [0,1]$ fixed, the functions $\ell(t,\cdot)$ and $f(t,\cdot)$ are convex.

Notation. In the future, when a function or a mapping depends on the variable t, we sometimes denote this dependence with a subindex; for example, the function ℓ_t represents the function $(x,u) \mapsto \ell(t,x,u)$ and K_t the set-valued map $t \mapsto K(t)$. Also, for a given convex function $h : \mathbb{R}^k \to \mathbb{R} \cup \{+\infty\}$, as usual, $\partial h(x)$ stands for its (convex) subdifferential at $x \in \mathrm{dom}\,(h)$ and for a given convex set $S \subseteq \mathbb{R}^k$, we write $N_S(x)$ for the (convex) *normal cone* of S at the point $x \in S$ and $\mathrm{ri}\,(S)$ for its relative interior. If $F : \mathbb{R}^k \rightrightarrows \mathbb{R}^i$ is a multifunction, then $\mathrm{gr}\,(F)$ stands for its graph.

Given $k \in \mathbb{N}$, we denote the Euclidean norm of a vector $x \in \mathbb{R}^k$ by $|x|$ and the inner dot of two vectors $x, y \in \mathbb{R}^k$ by $\langle x, y \rangle$. In our setting, an *arc* is a function $\gamma : [0,1] \to \mathbb{R}^k$. If M is a $k \times l$ matrix, then M^* is the corresponding transpose matrix. We denote by $AC_k = AC([0,1];\mathbb{R}^k)$ the space of absolutely continuous arcs, $L^\infty_k = L^\infty([0,1];\mathbb{R}^k)$ the space of essentially bounded arcs and $BV_k = BV([0,1];\mathbb{R}^k)$ the space of arcs of bounded variation; each of these spaces is equipped with their usual norms. For $n, m \in \mathbb{N}$ given, we write $\mathbb{R}^{n+m}$ for the product space $\mathbb{R}^n \times \mathbb{R}^m$ endowed with the Euclidean norm. For $z \in BV_k$, we write $z(0)$ and $z(1)$ for the left and right limits of $z(t)$ at $t = 0$ and $t = 1$, respectively.

2.2. **Main results.** Before presenting our main result, let us introduce some definitions. Consider first for any $t \in [0,1]$ the set

$$\Omega(t) := \{(x,u) \colon\ f_t(x,u) \leq 0,\ \zeta_t = C_t x + D_t u,\ x \in K_t,\ u \in U_t\}.$$

Consider as well for any $t \in [0,1]$ the set

$$\mathbf{X}(t) := \{x\colon \exists u \in U_t \text{ such that } (x,u) \in \Omega(t)\},$$

which represents the *actual* state constraints present in (P_{χ_0}) and the *Lagrangian* $L : [0,1] \times \mathbb{R}^{n+m} \times \mathbb{R}^{n+m} \to \mathbb{R} \cup \{+\infty\}$ given by

$$L(t,x,y,v,u) := \begin{cases} \ell_t(x,u) & v = A_t x + B_t u + \varphi_t,\ (x,u) \in \Omega(t), \\ +\infty & \text{otherwise.} \end{cases} \tag{1}$$

Under the assumptions we have done so far, it is not difficult to see that for any $t \in [0,1]$ fixed, $\Omega(t)$ is a closed convex set and L_t is a convex and lower semicontinuous function. Moreover, for any $t \in [0,1]$ fixed it holds that

$$\mathbf{X}(t) \times \mathbb{R}^m = \{(x,y)\colon \exists (v,u) \text{ such that } L_t(x,y,v,u) < \infty\}.$$

Furthermore, we can also see that L is lower semicontinuous function jointly in all its variable, therefore as pointed out in [30, Example 14.30], L is a *Lebesgue-normal integrand* in the sense of [30, Definition 14.27].

In addition to the basic assumptions we have done up to now, we will require the following technical conditions for ensuring that $\mathbf{X}$ is a set-valued map with closed graph and also that the Lagrangian is bounded below.

(A5) There is an upper semicontinuous function $\psi : [0,1] \times \mathbb{R}^n \to \mathbb{R}$ such that

$$\sup_{u \in U_t} \{|u|\colon f_t(x,u) \le 0\} \le \psi(t,x), \qquad \forall (t,x) \in \mathrm{gr}\,(K)\,.$$

(A6) There is $\kappa_0 \in \mathbb{R}$ and a function $\rho : [0,1] \times \mathbb{R}^m \to \mathbb{R}$ such that $\rho(\cdot, 0)$ is upper semicontinuous and

$$\sup_{(x,u)\in\Omega(t)} \{|x|\colon \ell_t(x,u) - \langle z, u\rangle \le \kappa_0\} \le \rho(t,z), \qquad \forall z \in \mathbb{R}^m,\ t \in [0,1].$$

Remark 2.1. *Notice that (A5) and (A6) hold in several situations of interest. For example, they are straightforward if $K_t \times U_t \subseteq S$, for any $t \in [0,1]$ with S being a compact subset of $\mathbb{R}^{n+m}$.*

A noteworthy case when (A5) holds is when

$$f_t(x,u) = r(t,x) + s(t,u)$$

with s satisfying a coercivity condition of the form $s(t,u) \ge c(t)(1 + |u|^k)$ for some continuous positive function $c : [0,1] \to]0,+\infty[$ and some $k \in \mathbb{N}$. Notice too that (A6) holds if for example

$$\ell_t(x,u) \ge h(t,x,u) + |x|^\alpha + \gamma |u|^\beta$$

for some continuous bounded below function $h : [0,1] \to \mathbb{R}$, $\alpha \ge 1$, $\beta > 1$ and $\gamma > 0$, that is, if ℓ is coercive in u and has (at least) linear growth in x.

Remark 2.2. *From the assumptions we have done so far it also follows that*

$$\inf_{t\in[0,1]} \inf_{(x,u)\in\Omega(t)} \ell(t,x,u) > -\infty$$

If it were not the case, for any $k \in \mathbb{N}$ there would be some $t_k \in [0,1]$ and $(x_k,u_k) \in \Omega(t_k)$ such that $\ell(t_k,x_k,u_k) \to -\infty$. Passing into a subsequence if necessary, let us assume that $t_k \to t$ for some $t \in [0,1]$. Thus, for $k \in \mathbb{N}$ large enough by (A6) we would have that $|x_k| \le \rho(t_k, 0)$ and by (A5) we would have $|u_k| \le \psi(t_k,x_k)$. Therefore, x_k and u_k are uniformly bounded, and without loss of generality we may assume there are $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$ such that $x_k \to x$ and $u_k \to u$. By continuity *of the data (assumptions (A1) to (A3)), it follows that* $\ell(t_k, x_k, u_k) \to \ell(t, x, u)$ *as well as* $(x, u) \in \Omega(t)$*, which leads to a contradiction. In particular, it follows that for any measurable arcs* $x, v : [0, 1] \to \mathbb{R}^n$ *and* $y, u : [0, 1] \to \mathbb{R}^m$*, we have*

$$\int_0^1 L(t, x(t), y(t), v(t), u(t))dt > -\infty.$$

Recall that any arc of bounded variation $z \in BV_k$ induces a Borel measure $dz(t)$ which has a Lebesgue decomposition of the form

$$dz(t) = \dot{z}(t)dt + \xi_z(t)d\theta(t),$$

where $\dot{z}(t)$ and $\xi_z(t)$ are the densities associated with the absolutely continuous and singular part of the measure $dz(t)$ (with respect to the Lebesgue measure). Here θ is a given a singular (regular) measure with respect to the Lebesgue measure on $[0, 1]$, which remains fixed from this point onward.

Now we are in a position to present the first results of this paper which provides sufficient conditions for a process to be an optimal solution of (P_{χ_0}).

Theorem 2.1. *Suppose that*

$$\text{(H)} \qquad \mathrm{ri}\,(K_t) \times \mathrm{ri}\,(U_t) \bigcap \{(x, u) \colon f_t(x, u) < 0\} \neq \emptyset, \qquad \textit{for a.e. } t \in [0, 1].$$

Let (x, u) *be an admissible process for* (P_{χ_0}) *for which there exist an arc* $p \in BV_n$*, a nonnegative measurable function* $\mu : [0, 1] \to [0, +\infty[$ *and a measurable arc* $\lambda : [0, 1] \to \mathbb{R}^l$ *satisfying the following conditions:*

(i) $p(1) = 0$*;*
(ii) $\mu(t)f(t, x(t), u(t)) = 0$ *for a.e.* $t \in [0, 1]$*;*
(iii) for a.e. $t \in [0, 1]$

$$\begin{pmatrix} \dot{p}(t) + A_t^* p(t) - C_t^* \lambda(t) \\ B_t^* p(t) - D_t^* \lambda(t) \end{pmatrix} \in \partial \ell_t(x(t), u(t)) + \mu(t)\partial f_t(x(t), u(t)) + \begin{pmatrix} N_{K_t}(x(t)) \\ N_{U_t}(u(t)) \end{pmatrix}.$$

(iv) $\xi_p(t) \in N_{\mathbf{X}(t)}(x(t))$ $d\theta$*-a.e.*

Then, (x, u) *is a solution for* (P_{χ_0})*.*

Remark 2.3. *It is noteworthy that if* ℓ_t *and* f_t *are both smooth function and there are no state constraints, then the conditions in Theorem 2.1 correspond to the smooth Maximum Principle for problems with mixed constraints. It is enough to see that (iii) in Theorem 2.1 is equivalent to costate equation*

$$-\dot{p}(t) = \nabla_x \mathcal{H}(t, x(t), u(t), p(t), \lambda(t), \mu(t)) \tag{2}$$

and first order optimality condition

$$0 \in -\nabla_u \mathcal{H}(t, x(t), u(t), p(t), \lambda(t), \mu(t)) + N_{U_t}(u(t)), \tag{3}$$

where

$$\mathcal{H}(t, x, u, p, \lambda, \mu) := \langle A_t x + B_t u + \varphi_t, p\rangle - \ell_t(x, u) - \langle C_t x + D_t u - \zeta_t, \lambda\rangle - \mu f_t(x, u).$$

By convexity, we get that (3) is equivalent to the maximality condition

$$\mathcal{H}(t, x(t), u(t), p(t), \lambda(t), \mu(t)) = \max_{u \in U_t} \mathcal{H}(t, x(t), u, p(t), \lambda(t), \mu(t)). \tag{4}$$

The strategy we use to prove Theorem 2.1 relies on Rockafellar's duality theory for convex problems of Bolza type; see for instance [28, 29]. In this approach, the Bolza problem is paired with a dual problem which has the same structure as the primal one. The underlying convex structure of the problem is what provides the sufficiency of the conditions in Theorem 2.1.

In the next section we develop the main core of the strategy described above. In particular, we introduce an auxiliary convex problem of Bolza type, which encloses all the information of the optimal control problem of concern.

Our second result establishes that the sufficient conditions in Theorem 2.1 are also necessary when an additional Slater type qualification condition holds.

Theorem 2.2. *Suppose that $C_t = 0$ for all $t \in [0,1]$ and that there exists a process $(\bar{x}, \bar{u})$ and $r_0 > e^{\|A\|_\infty}|\chi_0 - \bar{x}(0)|$ such that*

$$\text{(H')} \qquad \begin{cases} \dot{\bar{x}}(t) = A(t)\bar{x}(t) + B(t)\bar{u}(t) + \varphi(t) & a.e. \text{ on } [0,1], \\ \operatorname{ess-sup}_{t\in[0,1]} f_t(\bar{x}(t), \bar{u}(t)) < 0 & \\ \zeta(t) = D(t)\bar{u}(t) & a.e. \text{ on } [0,1], \\ \mathbb{B}_{\mathbb{R}^n}(\bar{x}(t), r_0) \subseteq K_t & on\ [0,1], \\ \bar{u}(t) \in U_t & a.e. \text{ on } [0,1]. \end{cases}$$

Asume in addition that there is $\kappa_\ell > 0$ such that

$$|\ell_t(x,u) - \ell_t(y,v)| \leq \kappa_\ell(|x-y| + |u-v|), \quad \forall x, y \in \mathbb{R}^n\, u, v \in \mathbb{R}^m,\ t \in [0,1].$$

Let (x^*, u^*) be a solution for (P_{χ_0}) and assume that (H) holds also. Then, there exist an arc $p \in BV_n$, a nonnegative measurable function $\mu : [0,1] \to [0, +\infty[$ and a measurable arc $\lambda : [0,1] \to \mathbb{R}^l$ satisfying the conditions of Theorem 2.1.

Remark 2.4. *It is pertinent to note that from Theorem 2.2 one can obtain a nonsmooth Maximum Principle in normal form. Indeed, notice first that*

$$\partial\ell_t(\hat{x}, \hat{u}) \subseteq \partial_x\ell_t(\hat{x}, \hat{u}) \times \partial_u\ell_t(\hat{x}, \hat{u}) \quad \text{and} \quad \partial f_t(\hat{x}, \hat{u}) \subseteq \partial_x f_t(\hat{x}, \hat{u}) \times \partial_u f_t(\hat{x}, \hat{u}),$$

where $\partial_x h(\hat{x}, \hat{u})$ and $\partial_u h(\hat{x}, \hat{u})$ stand for the subdifferential of the functions $h(\cdot, \hat{u})$ at $x = \hat{x}$ and $h(\hat{x}, \cdot)$ at $u = \hat{u}$, respectively.

Therefore, from condition (iii) in Theorem 2.1 we can obtain

$$(5) \quad \dot{p}(t) + A_t^* p(t) - C_t^*\lambda(t) \in \partial_x\ell_t(x(t), u(t)) + \mu(t)\partial_x f_t(x(t), u(t)) + N_{K_t}(x(t)),$$

$$(6) \qquad B_t^* p(t) - D_t^*\lambda(t) \in \partial_u\ell_t(x(t), u(t)) + \mu(t)\partial_u f_t(x(t), u(t)) + N_{U_t}(u(t)).$$

Similarly as in Remark 2.3, here (5) is the costate equation and (6) leads to the maximality condition. Notice that, if we set $\mathcal{H}$ is as in Remark 2.3, then (5) and (6) can be written as

$$-\dot{p}(t) \in \partial^x\mathcal{H}(t, x(t), u(t), p(t), \lambda(t), \mu(t)) - N_{K_t}(x(t))$$

and

$$0 \in -\partial^u\mathcal{H}(t, x(t), u(t), p(t), \lambda(t), \mu(t)) + N_{U_t}(u(t)),$$

respectively, where $\partial^x\mathcal{H}(t, \hat{x}, \hat{u}, \hat{p}, \hat{\lambda}, \hat{\mu})$ and $\partial^u\mathcal{H}(t, \hat{x}, \hat{u}, \hat{p}, \hat{\lambda}, \hat{\mu})$ are the negative of the subdifferential of the convex functions $-\mathcal{H}(t, \cdot, \hat{u}, \hat{p}, \hat{\lambda}, \hat{\mu})$ at $x = \hat{x}$ and $-\mathcal{H}(t, \hat{x}, \cdot, \hat{p}, \hat{\lambda}, \hat{\mu})$ at $u = \hat{u}$. Since $\mathcal{H}$ is a concave function on the variable u, the last inclusion is equivalent to the maximality condition (4).

Remark 2.5. *Notice the if $\bar{u}$ in (H') also satisfied that $\bar{u}(t) \in \mathrm{ri}\,(U_t)$ a.e. on $[0,1]$, then (H) holds immediately. However, in general we do not enforce that. It should also be pointed out that if no pure state constraints are considered (i.e. $K_t \equiv \mathbb{R}^n$), then $\bar{x}(0)$ in (H') is free and does not need to be related with χ_0. Also, since $r_0 > |\chi_0 - \bar{x}(0)|$, it follows that (H') implies that $\chi_0 \in \mathrm{int}\,(K_0)$.*

Remark 2.6. *It is well known that for problem with (pure) state constraints, the so-called* inward pointing condition *(IPC), allows to ensure normality of the Maximum Principle; see for instance [3] and the reference therein. As pointed out in Remark 2.4, Theorem 2.2 leads as well to a Maximum Principle in normal form, however by considering a less restrictive condition than the IPC; conditions such as the IPC implies (H') as has been shown in* [7, Theorem 1]. *As a matter of fact, the optimal control problem considered in Example 5.1 at the end of the paper satisfies (H') but not the IPC; see also Remark 5.1.*

3. The Auxiliary Problem and its Dual

As pointed out in [16], when state constraints are present in convex optimal control problems, these problems can be formulated as convex problems of Bolza type over arcs of bounded variation rather than over absolutely continuous ones. Considering this, we introduce for any given pair $(x, y) \in BV_{n+m}$ the cost functional

$$J^L(x,y) := \int_0^1 L_t(x(t), y(t), \dot{x}(t), \dot{y}(t))dt + \int_0^1 R^L(t, \xi_x(t), \xi_y(t))d\theta(t),$$

where L is the Lagrangian given by (1) and $R^L : [0,1] \times \mathbb{R}^{n+m} \to \mathbb{R} \cup \{+\infty\}$ is the *(partial)* recession function (see for instance [27, Theorem 8.5]) given by

$$R^L(t, d_x, d_y) := \sup_{(x,y,v,u)\in \mathrm{dom}(L_t)} L_t(x, y, v + d_x, u + d_y) - L_t(x, y, v, u).$$

The auxiliary problem we need for proving our main result is the following

$$\text{(P)} \qquad \min J^L(x,y), \qquad \text{over all the pair } (x,y) \in BV_{n+m}.$$

The underlying idea is the following; if (x, u) is an admissible process for (P_{χ_0}) satisfying the conditions of Theorem 2.1, then, to this admissible process, one can associate an optimal solution for (P). A posteriori, one can use problem (P) as a pivot for testing the optimality of the admissible process (x, u).

Following [29], in order to write (sufficient and necessary) optimality conditions for (P), we need to construct a suitable dual problem to (P). Thus, we introduce a *dual* Lagrangian $M : [0,1] \times \mathbb{R}^{n+m} \times \mathbb{R}^{n+m} \to \mathbb{R} \cup \{+\infty\}$ given by

$$M(t, p, q, r, s) := \sup_{x,y,v,u} \{\langle p, v\rangle + \langle q, u\rangle + \langle r, x\rangle + \langle s, y\rangle - L_t(x, y, v, u)\}.$$

Notice that $M_t(p, q, r, s) = +\infty$ if $s \neq 0$ and otherwise

$$\text{(7)} \qquad M_t(p, q, r, 0) = \sup_{(x,u)\in\Omega(t)} \{\langle A_t^* p + r, x\rangle + \langle B_t^* p + q, u\rangle - \ell_t(x, u)\} + \langle p, \varphi_t\rangle.$$

The dual problem to (P) we consider in this paper is then defined as follows:

$$\text{(D)} \qquad \min J^M(x,y), \qquad \text{over all the pair } (p,q) \in BV_{n+m},$$

where, for a given pair $(p, q) \in BV_{n+m}$ we have

$$J^M(p,q) = \int_0^1 M_t(p(t), q(t), \dot{p}(t), \dot{q}(t))dt + \int_0^1 R^M(t, \xi_p(t), \xi_q(t))d\theta$$

and, as before, $R^M(t,\cdot,\cdot)$ is the recession function given by

$$R^M(t,d_p,d_q) := \sup_{(p,q,r,0)\in \mathrm{dom}(M_t)} M_t(p,q,r+d_p,d_q) - M_t(p,q,r,0).$$

3.1. **State constraints.** Let us point out that the primal problem (P) has some implicit (time dependent) state constraints associated, which are given by

$$\mathcal{S}^L(t) := \{(x,y) \colon \exists (v,u) \text{ such that } L_t(x,y,v,u) < \infty\}, \qquad \forall t \in [0,1].$$

As we have already discussed, we have the relation

$$\mathbf{X}(t) \times \mathbb{R}^m = \mathcal{S}^L(t), \qquad \forall t \in [0,1].$$

Under the assumption we have done so far, it follows that the set-valued map $\mathcal{S}^L$ has closed graph.

Lemma 3.1. *The multifunction $\mathbf{X} : [0,1] \rightrightarrows \mathbb{R}^n$ has closed graph. In particular, $\mathrm{gr}\left(\mathcal{S}^L\right)$ is closed as well.*

Proof. Take a convergent sequence $(t_k,x_k) \to (t,x)$ with $x_k \in \mathbf{X}(t_k)$ for all k. Then, by the definition of $\mathbf{X}$, there exists a sequence $u_k \in U(t_k)$ satisfying $f(t_k,x_k,u_k) \leq 0$. Thanks to (A5), it follows that

$$|u_k| \leq \psi(t_k,x_k), \qquad \forall k \in \mathbb{N}.$$

Since, ψ is upper semicontinuous, it follows that the sequence u_k is bounded, and so, taking a subsequence if necessary, we can find $u \in \mathbb{R}^m$ such that $(t_k,u_k) \to (t,u)$. Since U has closed graph, we conclude that $u \in U(t)$. Finally, the continuity of f and ζ together with the continuity of the matrix-valued maps $s \mapsto C(s)$ and $s \mapsto D(s)$, imply that (t,x) does belong to $\mathrm{gr}\,(\mathbf{X})$. □ □

In a general setting, the dual problem (D) may have as well some (non-trivial) implicit state constraints associated, which in this case would be

$$\mathcal{S}^M(t) := \{(p,q) \colon \exists (r,s) \text{ such that } M_t(p,q,r,s) < \infty\}.$$

Nevertheless, under the compactness assumption (A6) we are considering in this work, the *dual state constraints* turns out to be the whole state space $\mathbb{R}^{n+m}$ and can be disregarded as the next results shows.

Lemma 3.2. *$\mathcal{S}^M(t) = \mathbb{R}^{n+m}$ for all $t \in [0,1]$ fixed.*

Proof. Notice first that for any $(p,q) \in \mathbb{R}^{n+m}$, by (7) we have

$$M_t(p,q,-A_t^*p,0) = \sup_{(x,u)\in\Omega(t)} \{\langle B_t^*p+q,u\rangle - \ell_t(x,u)\} + \langle p,\varphi_t\rangle.$$

Let us check that the supremum above is finite for any $(p,q) \in \mathbb{R}^{n+m}$. Suppose by contradiction that $M_t(p,q,-A_t^*p,0) = +\infty$ for some $(p,q) \in \mathbb{R}^{n+m}$. Then, there is a sequence $(x_k,u_k) \in \Omega(t)$ such that

$$\ell_t(x_k,u_k) - \langle B_t^*p+q,u_k\rangle \to -\infty.$$

Notice that x_k is uniformly bounded. Indeed, since $(x_k,u_k) \in \Omega(t)$ for any $k \in \mathbb{N}$ and $\ell_t(x_k,u_k) - \langle B_t^*p+q,u_k\rangle \leq \kappa_0$ for $k \in \mathbb{N}$ large enough, by (A6) we get for $k \in \mathbb{N}$ large enough

$$|x_k| \leq \rho(t, B_t^*p+q).$$

Thus, by (A5) we get that u_k is uniformly bounded because we also have for $k \in \mathbb{N}$ large enough that $|u_k| \leq \psi(t,x_k)$. Thus, passing into a subsequence if necessary, we can assume there is $(\bar{x}, \bar{u}) \in \mathbb{R}^{n+m}$ such that $(x_k, u_k) \to (\bar{x}, \bar{u})$. Since $\Omega(t)$ is a closed set, we actually get that $(\bar{x}, \bar{u}) \in \Omega(t)$ and by continuity of ℓ we get that

$$\ell_t(x_k, u_k) - \langle B_t^* p + q, u_k \rangle \to \ell_t(\bar{x}, \bar{u}) - \langle B_t^* p + q, \bar{u} \rangle \in \mathbb{R},$$

which is not possible. Therefore, $M_t(p, q, -A_t^* p, 0) \in \mathbb{R}$, which leads to the conclusion. □ □

Remark 3.1. *It is well known that $R^L(t, d_x, d_y) = \sigma_{\mathcal{S}^M(t)}(d_x, d_y)$, where σ_S stands for the support function of a set S; see for instance* [27, Theorem 13.3]. *This means that $R^L(t, d_x, d_y) \in \mathbb{R}$ if and only if $(d_x, d_y) = 0$, and so, any feasible arc for* (P) *must be an absolutely continuous arc, because an arc in order to be feasible needs to satisfy*

$$R^L(t, \xi_x(t), \xi_y(t)) = 0, \qquad d\theta - a.e. \text{ on } [0, 1].$$

The latter being a consequence of the fact that $\sigma_{\mathcal{S}^M(t)}$ is either 0 or $+\infty$.

3.2. **Optimality Conditions.** The following definitions taken from [29] are important concepts that arise in the study of optimality conditions for (P) and (D). These definitions have been adapted for the case we are treating in this paper, that is, absolutely continuous solutions for (P) and no state constraints on the dual problem (D).

Definition 3.1. *A pair $(x, y) \in AC_{n+m}$ is called an* extremal *for L if there is a pair $(p, q) \in BV_{n+m}$, called a* coextremal *for (x, y), such that they satisfy:*

(i) $x(t) \in \mathbf{X}(t)$ *for all* $t \in [0, 1]$;
(ii) $(\xi_p(t), \xi_q(t)) \in N_{\mathbf{X}(t)}(x(t)) \times \{0\}$ *for* $d\theta$*-a.e.* $t \in [0, 1]$;
(iii) $(-\dot{p}(t), -\dot{q}(t), \dot{x}(t), \dot{y}(t)) \in \partial H_t(x(t), y(t), p(t), q(t))$, *where the function H is the Hamiltonian given by*

$$H(t, x, y, p, q) := \sup_{v,u} \{\langle p, v \rangle + \langle q, u \rangle - L_t(x, y, v, u)\},$$

and ∂H_t stands for the concave-convex subdiferential (see [27, Chapter 35]*) of the function*

$$((x, y), (p, q)) \mapsto H_t(x, y, p, q).$$

Remark 3.2. *According to the definition of the Lagrangian* (1)*, the Hamiltonian associated with problem* (P) *is given by*

$$H_t(x, y, p, q) = \sup_u \{\langle B_t^* p + q, u \rangle - \ell_t(x, u) \colon (x, u) \in \Omega(t)\} + \langle p, A_t x + \varphi_t \rangle.$$

Thus, if $x \notin K_t$, then $\{u \colon (x, u) \in \Omega(t)\} = \emptyset$ and so $H_t(x, y, p, q) = -\infty$.

Define now the value functions ϑ^L, $\vartheta^M : \mathbb{R}^{n+m} \times \mathbb{R}^{n+m} \to \mathbb{R} \cup \{+\infty\}$ associated with (P) and (D) respectively, by

$$\vartheta^L(x_0, y_0, x_1, y_1) := \inf_{(x,y) \in AC_{n+m}} \left\{ J^L(x, y) \colon \begin{matrix} (x(0), y(0)) = (x_0, y_0) \\ (x(1), y(1)) = (x_1, y_1) \end{matrix} \right\}$$

and

$$\vartheta^M(p_0, q_0, p_1, q_1) := \inf_{(p,q) \in BV_{n+m}} \left\{ J^M(p, q) \colon \begin{matrix} (p(0), q(0)) = (p_0, q_0) \\ (p(1), q(1)) = (p_1, q_1) \end{matrix} \right\}.$$

Notice that an optimal solution (x, y) for problem (P) satisfies

$$J^L(x, y) = \vartheta^L(x(0), y(0), x(1), y(1)).$$

Similarly, an optimal solution (p,q) for problem (D) satisfies

$$J^M(p,q) = \vartheta^M(p(0),q(0),p(1),q(1)).$$

Definition 3.2. *We say that* (x_0,y_0,x_1,y_1) *and* (p_0,q_0,p_1,q_1) *are* endpoints in duality *if* $x_0 \in \mathbf{X}(0)$, $x_1 \in \mathbf{X}(1)$ *and it also holds that*

$$\vartheta^L(x_0,y_0,x_1,y_1) + \vartheta^M(p_0,q_0,p_1,q_1) = \langle x_1,p_1\rangle + \langle y_1,q_1\rangle - \langle x_0,p_0\rangle - \langle y_0,q_0\rangle.$$

Theorem 2 in [29] derives necessary and sufficient conditions for (x,y) and (p,q) to be optimal pairs for problem (P) and (D), respectively. This result will be our vehicle to derive sufficient optimality conditions for problem (P_{χ_0}).

Lemma 3.3. *Suppose that $J^L(x,y)$ and $J^M(p,q)$ are not oppositely infinite. Then, the following statements are equivalent:*

(i) (x,y) *is an extremal for* L *with coextremal* (p,q)*;*
(ii) (x,y) *is optimal for* (P), (p,q) *is optimal for* (D) *and* $(x(0),y(0),x(1),y(1))$ *and* $(p(0),q(0),p(1),q(1))$ *are endpoints in duality.*

Proof. This is a direct consequence of [29, Theorem 2] and the fact that the $\mathcal{S}^L$ and $\mathcal{S}^M$ have both closed graph by Lemma 3.1 and Lemma 3.2, respectively. □ □

3.3. **The Hamiltonian inclusion.** The following lemma proves that the *Hamiltonian inclusion* in Definition 3.1 is equivalent to the existence of measurable multipliers satisfying a *costate equation* involving the subdifferential of a *pre-Hamiltonian* function and the normal cone of the multifunction U and *complementary slackness* conditions.

Lemma 3.4. *Suppose that* (H) *holds and let us consider some measurable arcs* $x,v,p,r:[0,1]\to\mathbb{R}^n$ *and* $y,u,q,s:[0,1]\to\mathbb{R}^m$. *Then*

$$(8)\qquad (-r(t),-s(t),v(t),u(t)) \in \partial H_t(x(t),y(t),p(t),q(t)), \qquad a.e.\ t\in[0,1]$$

if and only if $s(t)=0$ *a.e. on* $[0,1]$ *and there exist a nonnegative measurable function* $\mu:[0,1]\to[0,+\infty[$ *and a measurable arc* $\lambda:[0,1]\to\mathbb{R}^l$ *satisfying:*

(1) $\mu(t)f(t,x(t),u(t))=0$ *a.e. on* $[0,1]$*;*
(2) for a.e. $t\in[0,1]$

$$\begin{pmatrix} r(t)+A_t^*p(t)-C_t^*\lambda(t)\\ q(t)+B_t^*p(t)-D_t^*\lambda(t)\end{pmatrix} \in \partial\ell_t(x(t),u(t)) + \mu(t)\partial f_t(x(t),u(t)) + \begin{pmatrix} N_{K_t}(x(t))\\ N_{U_t}(u(t))\end{pmatrix}$$

Proof. Notice first that if $x,v,p,r\in\mathbb{R}^n$ and $y,u,q,s\in\mathbb{R}^m$ are given vectors, then [27, Theorem 37.5] implies that

$$(-r,-s,v,u)\in\partial H_t(x,y,p,q)$$

is equivalent to

$$(r,s,p,q)\in\partial L_t(x,y,v,u).$$

Since ℓ_t is continuous and $\Omega(t)\neq\emptyset$, by [27, Theorem 23.8] we have

$$\partial L_t(x,y,v,u) = \big\{(r,0,0,q)\colon (r,q)\in\partial\ell_t(x,u)+N_{\Omega(t)}(x,u)\big\} + N_{\Gamma(t)}(x,y,v,u),$$

where $\Gamma(t):=\{(x,y,v,u)\colon v=A_tx+B_tu+\varphi_t\}$. It is not difficult to see that

$$N_{\Gamma(t)}(x,y,v,u) = \{(A_t^*z,0,-z,B_t^*z)\colon z\in\mathbb{R}^n\}, \qquad \forall (x,y,v,u)\in\Gamma(t).$$

Also, since for any $t\in[0,1]$, there is $(\bar{x},\bar{u})\in \mathrm{ri}\,(K_t)\times\mathrm{ri}\,(U_t)$ such that $f_t(\bar{x},\bar{u})<0$, by [27, Corollary 23.8.1] we have that

$$N_{\Omega(t)}(x,u) = N_{K_t}(x)\times N_{U_t}(u) + N_{\{f_t\le 0\}}(x,u) + \{(C_t^*\lambda, D_t^*\lambda)\colon \lambda\in\mathbb{R}^l\}.$$

Notice too that thanks to [27, Corollary 23.7.1] we have

$$N_{\{f_t\le 0\}}(x,u) = \{(r,q)\in \mu\partial f_t(x,u)\colon \mu\ge 0,\ \mu f_t(x,u)=0\}.$$

Therefore, (8) is equivalent to $s(t)=0$ for a.e. $t\in[0,1]$ and

$$\begin{pmatrix} r(t)\\ q(t)\end{pmatrix} \in \partial\ell_t(x(t),u(t)) - \begin{pmatrix} A_t^*p(t)\\ B_t^*p(t)\end{pmatrix} + \begin{pmatrix} N_{K_t}(x(t))\\ N_{U_t}(u(t))\end{pmatrix} \\ + N_{\{f_t\le 0\}}(x(t),u(t)) + \left\{\begin{pmatrix} C_t^*\lambda\\ D_t^*\lambda\end{pmatrix} : \lambda\in\mathbb{R}^l\right\},\quad \text{a.e. } t\in[0,1].$$

From here it is easy to see that the *if* implication of the lemma holds true. Let us now focus on the *only if* implication, which essentially requires us to justify that some suitable measurable selections exist.

Let us consider the set

$$\mathcal{U}(t) := \partial\ell_t(x(t),u(t)) \times N_{K_t\times U_t}(x(t),u(t)) \times [0,+\infty[\times\partial f_t(x(t),u(t))\times\mathbb{R}^l,$$

the measurable function

$$\nu(t) := \begin{pmatrix} r(t)+A_t^*p(t)\\ q(t)+B_t^*p(t)\end{pmatrix},$$

and, for $t\in[0,1]$, $\mu\in[0,+\infty[$, $\xi,\eta,\gamma\in\mathbb{R}^{n+m}$ and $\lambda\in\mathbb{R}^l$, the continuous mapping

$$g(t,(\xi,\eta,\mu,\gamma,\lambda)) := \xi+\eta+\mu\gamma+\begin{pmatrix} C_t^*\lambda\\ D_t^*\lambda\end{pmatrix}$$

and the Caratheodory mapping

$$h(t,(\xi,\eta,\mu,\gamma,\lambda)) := \mu f_t(x(t),u(t)).$$

Notice that if (8) holds true, then for a.e. $t\in[0,1]$ we also have

$$\mathcal{U}'(t) := \left\{(\xi,\eta,\mu,\gamma,\lambda)\in\mathcal{U}(t)\colon \begin{array}{l} g(t,(\xi,\eta,\gamma,\lambda)) = \nu(t)\\ h(t,(\xi,\eta,\gamma,\lambda)) = 0\end{array}\right\}\neq\emptyset.$$

Moreover, by [30, Theorem 14.26] and [30, Theorem 14.56], we have that $\mathcal{U}$ defines a measurable set-valued map with closed images. Thus, by the Generalized Filippov Selection Theorem [31, Theorem 2.3.13], there are measurable functions $\xi,\eta,\gamma : [0,1]\to\mathbb{R}^{n+m}$, $\mu:[0,1]\to[0,+\infty[$ and $\lambda:[0,1]\to\mathbb{R}^l$ such that the following holds for a.e. $t\in[0,1]$

$$\xi(t)\in\partial\ell_t(x(t),u(t)),\quad \eta(t)\in N_{K_t\times U_t}(x(t),u(t))\quad\text{and}\quad \gamma(t)\in\partial f_t(x(t),u(t)),$$

$$\begin{pmatrix} r(t)+A_t^*p(t)\\ q(t)+B_t^*p(t)\end{pmatrix} = \xi(t)+\eta(t)+\mu(t)\gamma(t)+\begin{pmatrix} C_t^*\lambda(t)\\ D_t^*\lambda(t)\end{pmatrix}.$$

and $\mu(t)f_t(x(t),u(t))=0$. This completes the proof of the lemma. □ □

4. Proof of the main results

4.1. **Proof of Theorem 2.1.** We have now all the tools needed for proving Theorem 2.1.

Theorem 2.1. Our vehicle to prove this result will be Lemma 3.3. First, we show that the existence of multipliers (p,μ,λ) satisfying the conditions in Theorem 2.1 implies that (x,y) is an extremal for L with coextremal (p,q), in the sense of Definition 3.1, where $y(t)=\int_0^t u(s)ds$ and $q(t)=0$.

Notice that condition (i) in Definition 3.1 is a direct consequence of (x, u) being admissible for (P_{χ_0}) and (ii) in Definition 3.1 is straightforward from the assumptions and the definition of q. On the other hand, the Hamiltonian condition (iii) in Definition 3.1 follows directly from Lemma 3.4.

Now, let us check that $J^L(x, y)$ and $J^M(p, q)$ are not oppositely infinite. It will be enough to see that $J^L(x, y) \in \mathbb{R}$. First, since (x, u) is admissible for (P_{χ_0}), it follows that for a.e. $t \in [0, 1]$ we have

$$\dot{y}(t) = u(t), \quad \dot{x}(t) = A(t)x(t) + B(t)u(t) + \varphi(t) \quad \text{and} \quad (x(t), u(t)) \in \Omega(t).$$

Therefore, it is not difficult to see that

$$J^L(x, y) = \int_0^1 L_t(x(t), y(t), \dot{x}(t), \dot{y}(t))dt = \int_0^1 \ell_t(x(t), u(t))dt.$$

Now, thanks to (A5), it follows that

$$|u(t)| \leq \psi(t, x(t)), \qquad \text{for a.e. } t \in [0, 1].$$

Since, ψ is upper semicontinuous and $x \in AC_n$, it follows that $u \in L^\infty_m$, and so due to the continuity of ℓ we get that $t \mapsto \ell_t(x(t), u(t))$ is integrable, which implies that $J^L(x, y) \in \mathbb{R}$.

This observation allows the use of Lemma 3.3 to deduce that (x, y) is optimal for L, (p, q) is optimal for M and the endpoints are in duality, that is,

$$\begin{aligned} &\vartheta^L(x(0), y(0), x(1), y(1)) + \vartheta^M(p(0), q(0), p(1), q(1)) \\ &\qquad = \langle x(1), p(1)\rangle + \langle y(1), q(1)\rangle - \langle x(0), p(0)\rangle - \langle y(0), q(0)\rangle. \end{aligned}$$

Since, by assumption $p(1) = 0$ and by definition $y(0) = q(0) = q(1) = 0$ and $x(0) = \chi_0$, the above equality implies

$$J^L(x, y) + J^M(p, q) = -\langle \chi_0, p(0)\rangle. \tag{9}$$

Next, define the functions

$$\begin{aligned} g^L(x_0, y_0) &:= \inf\{J^L(z, v) \colon (z, v) \in BV_{n+m},\ (z(0), v(0)) = (x_0, y_0)\}, \\ g^M(p_0, q_0) &:= \inf\{J^M(r, s) \colon (r, s) \in BV_{n+m},\ (r(0), s(0)) = (-p_0, -q_0)\}. \end{aligned}$$

Thanks to [16, Proposition 3.1] it follows that

$$g^L(x_0, y_0) + g^M(p_0, q_0) \geq \langle x_0, p_0\rangle + \langle y_0, q_0\rangle$$

for all $(x_0, y_0), (p_0, q_0) \in \mathbb{R}^{n+m}$. Thus, by (9) and the above inequality we get

$$g^L(\chi_0, 0) + g^M(-p(0), 0) \geq -\langle \chi_0, p(0)\rangle = J^L(x, y) + J^M(p, q),$$

which implies that

$$g^L(\chi_0, 0) + g^M(-p(0), 0) = J^L(x, y) + J^M(p, q)$$

or, equivalently,

$$0 \geq g^L(\chi_0, 0) - J^L(x, y) = J^M(p, q) - g^M(-p(0), 0) \geq 0.$$

We conclude that $g^L(\chi_0, 0) = J^L(x, y)$ and $g^M(-p(0), 0) = J^M(p, q)$. Hence

$$J^L(x, y) = \inf\{J^L(z, v) \colon (z, v) \in BV_{n+m},\ (z(0), v(0)) = (\chi_0, 0)\}. \tag{10}$$

Finally, to prove that (x, u) is indeed a solution for (P_{χ_0}), take any admissible pair (z, v) for this problem and set $w = \int_0^t v(t)dt$. Then, $(z(0), w(0)) = (\chi_0, 0)$ and

$J^L(z,w) = \int_0^1 \ell_t(z(t),v(t))dt$ by the admissibility and absolute continuity of (z,w). Thus, by (10) we have

$$\int_0^1 \ell_t(x(t),u(t)) = J^L(x,y) \leq J^L(z,w) = \int_0^1 \ell_t(z(t),v(t))dt$$

proving that (x,u) is in fact optimal. □ □

4.2. **Necessary Conditions.** Let us now focus on the necessary conditions. Let (x^*,u^*) be an optimal solution for (P_{χ_0}), and let us define $y^*(t) = \displaystyle\int_0^t u^*(s)ds$ for any $t \in [0,1]$.

One possible strategy to prove this is using Lemma 3.3. Notice that if (x,y) is optimal for (P), to use Lemma 3.3 we need to prove the existence of $(p,q) \in BV_{n+m}$, an optimal solution for (D), such that $(x(0),y(0),x(1),y(1))$ and $(p(0),q(0),p(1),q(1))$ are endpoints in duality. Our main tool for doing so will be [16, Theorem 3.8] applied to the Bolza problem

$$(\mathrm{P}_{(x_0,y_0)}) \qquad \begin{cases} \text{Minimize} \quad J^\Lambda(x,y) + g(x(1),y(1)) \text{ over all } (x,y) \in BV_{n+m} \\ \qquad\qquad\quad \text{such that } (x(0),y(0)) = (x_0,y_0), \end{cases}$$

where

$$g(x,y) := \frac{1}{2}|x - x^*(1)|^2 + \frac{1}{2}\left|y - y^*(1)\right|^2$$

and $\Lambda : [0,1] \times \mathbb{R}^{n+m} \times \mathbb{R}^{n+m} \to \mathbb{R} \cup \{+\infty\}$ given by

$$\Lambda(t,x,y,v,u) := L(t,x,y,v,u) + \frac{1}{2}\left|y - y^*(t)\right|^2. \tag{11}$$

Recall that, since L is a lower semicontinuous function thanks to assumption (A1) to (A3), it is a *Lebesgue-normal integrand*, and so it is Λ. Also, it is not difficult to see that Hamiltonian and the *dual* Lagrangian associated with Λ are given by the expression

$$(t,p,q,r,s) \mapsto H(t,x,y,p,q) - \frac{1}{2}|y - y^*(t)|^2$$

and

$$(t,p,q,r,s) \mapsto M(t,p,q,r,0) + \frac{1}{2}|s^2| + \langle s, y^*(t)\rangle,$$

respectively. Let us also point out that Lemmas 3.1, 3.2 and assumption (A1) to (A6) guarantee that Hypotheses 2.1 and 2.2[1] in [16] are verified; see also Remark 2.2. Notice also that if $C_t = 0$ for any $t \in [0,1]$ and (H') holds, then [16, Hypotheses 3.2] is verified with $(\bar{x},\bar{y})$ for the primal problem and $(\bar{p},\bar{q}) = (0,0)$ for the dual problem, where $\bar{y}(t) = \displaystyle\int_0^t \bar{u}(s)ds$ for any $t \in [0,1]$. Indeed, if g^* denotes the Fenchel conjugate of g, then it is enough to note that $\mathrm{dom}\,(g) = \mathrm{dom}\,(g^*) = \mathbb{R}^{n+m}$, $\bar{x}(t) \in \mathrm{int}\,(\mathbf{X}(t))$ for any $t \in [0,1]$,

$$J^\Lambda(\bar{x},\bar{y}) = \int_0^1 \ell(t,\bar{x}(t),\bar{u}(t))dt + \int_0^1 \left|\bar{y}(t) - y^*(t)\right|^2 dt \quad \text{and} \quad J^M(0,0) = 0.$$

Moreover, the following lemma shows that [16, Hypotheses 3.4] also holds.

[1] It is important to highlight that in [16, Hypotheses 2.2] the upper semicontinuity is understood in the same sense as in [29], that is, the multifunctions have closed graphs.

Lemma 4.1. *For every* $(x,y) \in \text{int}\,(\mathbf{X}(t)) \times \mathbb{R}^m$ *and* $(p,q) \in \mathbb{R}^{n+m}$*, the function* $t \mapsto H(t,x,y,p,q)$ *is integrable in* $[0,1]$.

Proof. Take a fixed element $(x,y) \in \text{int}\,(\mathbf{X}(t)) \times \mathbb{R}^m$ and $(p,q) \in \mathbb{R}^{n+m}$. In particular, $x \in K_t$ and so, by Remark 3.2 we have

$$H_t(x,y,p,q) = \sup_u \{\langle B_t^* p + q, u\rangle - \ell_t(x,u) \colon (x,u) \in \Omega(t)\} + \langle p, A_t x + \varphi_t\rangle.$$

Furthermore, by (A5) we have

$$\sup\{|u| \colon (x,u) \in \Omega(t)\} \leq \sup\{|u| \colon u \in U_t, f_t(x,u) \leq 0\} \leq \psi(t,x).$$

Since ψ is upper semicontinuous, the mapping $t \mapsto \psi(t,x)$ is bounded on $[0,1]$. Therefore, the inequality above, together with (A1) and (A2), implies that the mapping $t \mapsto H(t,x,y,p,q)$ is bounded in $[0,1]$, and therefore, it is integrable. □
□

Having checked that the assumptions in [16, Theorem 3.8] are satisfied, we are now in a position to derive the necessary conditions.

Theorem 2.2. For any $(x_0,y_0) \in \mathbb{R}^{n+m}$, let $V(x_0,y_0)$ be the value of the problem ($\mathrm{P}_{(x_0,y_0)}$). Notice that by Remark 3.1, any feasible arc for ($\mathrm{P}_{(x_0,y_0)}$) must belong to AC_{n+m}. Also, since (x^*,u^*) is an optimal solution for (P_{χ_0}), it follows that (x^*,y^*) is the unique solution to ($\mathrm{P}_{(x_0,y_0)}$) when $(x_0,y_0) = (\chi_0, 0)$. Moreover

$$V(\chi_0,0) = \int_0^1 \ell_t(x^*(t),u^*(t))dt \in \mathbb{R}.$$

Notice that if $\partial V(\chi_0,0) \neq \emptyset$, then [16, Theorem 3.8] combined with Lemma 3.4 yields the existence of $p \in BV_n$ with the desired properties. Indeed, by [16, Theorem 3.8] one get that if $(\eta,\gamma) \in \partial V(\chi_0,0)$ then there exists an extremal $(x,y) \in AC_{n+m}$ for L and a coextremal $(p,q) \in BV_{n+m}$ such that

$$-p(1) = x(1) - x^*(1) \quad \text{and} \quad -q(1) = y(1) - y^*(1).$$

It is worth pointing out that, thanks to Lemma 3.3 and the transversality condition, it follows that (x,y) is also an optimal solution ($\mathrm{P}_{(x_0,y_0)}$). From where we deduce that $(x,y) = (x^*,y^*)$, and so $p(1) = q(1) = 0$.

Now, by classical arguments in convex analysis (see for instance [27, Theorem 23.4]) we get that $\partial V(\chi_0,0) \neq \emptyset$ if $(\chi_0,0) \in \text{int}\,(\text{dom}\,(V))$. Since $(x_0,y_0) \mapsto V(x_0,y_0)$ is a convex function and finite at $(\chi_0,0)$, it is enough to see that V is uniformly bounded on a neighborhood of $(\chi_0,0)$.

Let $\Psi(t)$ be the standard fundamental matrix for the non autonomous linear system $\dot{x}(t) = A(t)x(t)$. Recall that, for any $u \in L^\infty([0,1];\mathbb{R}^m)$ and $x_0 \in \mathbb{R}^n$, the unique solution to

$$\dot{x}(t) = A(t)x(t) + B(t)u(t) + \varphi(t) \quad \text{a.e. on } [0,1], \qquad x(0) = x_0$$

is given by

$$x(t) = \Psi(t)x_0 + \Psi(t)\int_0^t \Psi^{-1}(s)[B(s)u(s) + \varphi(s)]ds, \quad \forall t \in [0,1].$$

Following ideas introduced in [15, Proposition 3.3], we claim that for any $\varepsilon > 0$ there exists a feasible process $(x_\varepsilon, u_\varepsilon)$ such that $\mathbb{B}_{\mathbb{R}^n}(x_\varepsilon(t), \varepsilon r_0) \subseteq K_t$ for each $t \in [0,1]$,

$$f_t(x_\varepsilon(t), u_\varepsilon(t)) \leq \varepsilon f_t(\bar{x}(t), \bar{u}(t)) < 0, \qquad \text{for a.e. } t \in [0,1],$$

where $(\bar{x},\bar{u})$ and $r_0>0$ are given by (H'), and

$$V(\chi_0,0)+\varepsilon\kappa_\ell\left(\|\bar{x}-x^*\|_\infty+\|\bar{u}-u^*\|_{L^\infty}\right)\geq\int_0^1\ell_t(x_\varepsilon(t),u_\varepsilon(t))dt.$$

To see this, define for $\alpha\in]0,1]$ the process

$$(x_\alpha,u_\alpha)=\alpha(\bar{x},\bar{u})+(1-\alpha)(x^*,u^*).$$

Then by convexity and (H'), it is not difficult to see that

$$\begin{aligned}
&\dot{x}_\alpha(t)=A(t)x_\alpha(t)+B(t)u_\alpha(t)+\varphi(t) && \text{a.e. on }[0,1],\\
&f_t(x_\alpha(t),u_\alpha(t))\leq\alpha f_t(\bar{x}(t),\bar{u}(t))<0 && \text{a.e. on }[0,1],\\
&\zeta(t)=D(t)u_\alpha(t) && \text{a.e. on }[0,1],\\
&\mathbb{B}_{\mathbb{R}^n}(x_\alpha(t),\alpha r_0)\subseteq K_t && \text{on }[0,1],\\
&u_\alpha(t)\in U_t && \text{a.e. on }[0,1].
\end{aligned}$$

Observe that $x_\alpha(t)-x^*(t)=\alpha(\bar{x}(t)-x^*(t))$ for any $t\in[0,1]$ and also that $u_\alpha(t)-u^*(t)=\alpha(\bar{u}(t)-u^*(t))$ for a.e. $t\in[0,1]$. Thus,

$$\begin{aligned}
\int_0^1\ell_t(x_\alpha(t),u_\alpha(t))dt&\leq V(\chi_0,0)+\kappa_\ell\int_0^1\left(|x_\alpha(t)-x^*(t)|+|u_\alpha(t)-u^*(t)|\right)dt\\
&\leq V(\chi_0,0)+\alpha\kappa_\ell\left(\|\bar{x}-x^*\|_\infty+\|\bar{u}-u^*\|_{L^\infty}\right).
\end{aligned}$$

Therefore, taking $\alpha=\min\{1,\varepsilon\}$ we get what we have claimed above.

Now, take $\delta>0$ such that $\mathbb{B}_{\mathbb{R}^n}(\chi_0;\delta)\subseteq K_0$; recall from Remark 2.5 that $\chi_0\in\operatorname{int}(K_0)$. Let $\varepsilon>0$, $x_0\in\mathbb{B}_{\mathbb{R}^n}(\chi_0,\delta)$ and $\bar{x}_\varepsilon$ be the solution of the ODE

$$\dot{x}(t)=A(t)x(t)+B(t)u_\varepsilon(t)+\varphi(t)\quad\text{a.e. on }[0,1],\qquad x(0)=x_0.$$

It is not difficult to see that

$$\|\bar{x}_\varepsilon-x_\varepsilon\|_\infty\leq\|\Psi\|_\infty|x_0-x_\varepsilon(0)|\leq e^{\|A\|_\infty}(\delta+\varepsilon|\chi_0-\bar{x}(0)|).$$

Therefore, since $e^{\|A\|_\infty}|\chi_0-\bar{x}(0)|<r_0$, for $\delta,\varepsilon>0$ small enough we get that $\|\bar{x}_\varepsilon-x_\varepsilon\|_\infty\leq\varepsilon r_0$, and so, $\bar{x}_\varepsilon(t)\in K_t$ for each $t\in[0,1]$ and moreover, since f is uniformly continuous on compacts set containing

$$\{(t,x^*(t),u)\colon t\in[0,1],\ u\in U_t\text{ such that }|u|\leq\psi(t,x^*(t))\},$$

and $|u^*(t)|\leq\psi(t,x^*(t))$ a.e. on $[0,1]$, we can also assume the following

$$f_t(\bar{x}_\varepsilon(t),u_\varepsilon(t))\leq\frac{\varepsilon}{2}f_t(\bar{x}(t),\bar{u}(t)),\quad\text{for a.e. }t\in[0,1].$$

Now, setting $\bar{y}_\varepsilon(t)=y_0+\int_0^t u_\varepsilon(s)ds$ and $\kappa^*=\|\bar{x}-x^*\|_\infty+\|\bar{u}-u^*\|_{L^\infty}$ we get that

$$\begin{aligned}
V(x_0,y_0)&\leq\int_0^1\ell_t(\bar{x}_\varepsilon(t),u_\varepsilon(t))dt+\frac{1}{2}\int_0^1|\bar{y}_\varepsilon(t)-y^*(t)|^2dt\\
&+\frac{1}{2}|\bar{x}_\varepsilon(1)-x^*(1)|^2+\frac{1}{2}|\bar{y}_\varepsilon(1)-y^*(1)|^2\\
&\leq\int_0^1\ell_t(x_\varepsilon(t),u_\varepsilon(t))dt+\kappa_\ell\|\bar{x}_\varepsilon-x_\varepsilon\|_\infty\\
&+|\bar{x}_\varepsilon(1)-x_\varepsilon(1)|^2+|x_\varepsilon(1)-x^*(1)|^2+2|y_0|^2+2\|u^*-u_\varepsilon\|_{L^\infty}^2\\
&\leq V(\chi_0,0)+2\varepsilon\kappa_\ell\kappa^*+\|\Psi\|_\infty^2(\delta+\varepsilon)^2+3\varepsilon^2+2|y_0|^2.
\end{aligned}$$

It follows then that V is uniformly bounded on a neighborhood of $(\chi_0, 0)$, and so the conclusion follows. □ □

5. Discussion of the Main Result and Example

Let us conclude this paper by doing a short discussion about the results we have obtained.

5.1. **Multiple mixed-constraints.** Notice that Theorem 2.1 and Theorem 2.2 can also be stated for problems with multiple mixed-constraints, that is, the case where there are $k \in \mathbb{N}$ continuous function $f^1, \dots f^k : [0,1] \times \mathbb{R}^{n+m} \to \mathbb{R}$ such that each f_t^i is a convex function for each $t \in [0,1]$ and $i \in \{1, \dots, k\}$ fixed. In this case the condition (H) is

$$(\tilde{H}) \qquad \mathrm{ri}\,(K_t) \times \mathrm{ri}\,(U_t) \bigcap \bigcap_{i=1}^{k} \{(x,u) \colon f_t^i(x,u) < 0\} \neq \emptyset, \qquad \forall t \in [0,1].$$

From [27, Corollary 23.8.1], we have also have that

$$N_{\{f_t^1 \leq 0, \dots, f_t^k \leq 0\}}(x,u) = \sum_{i=1}^{k} N_{\{f_t^i \leq 0\}}(x,u).$$

Moreover, since for each $i = 1, \dots k$ we have

$$N_{\{f_t^i \leq 0\}}(x,u) = \big\{(r,q) \in \mu \partial f_t^i(x,u) \colon \mu \geq 0,\ \mu f_t^i(x,u) = 0\big\}$$

it is not difficult to see that Theorem 2.1 turns into

Theorem 5.1. *Suppose that ($\tilde{H}$) holds and (x,u) be an admissible process for (P_{χ_0}) for which there exist an arc $p \in BV_n$, some nonnegative measurable functions $\mu_1, \dots, \mu_k : [0,1] \to [0,+\infty[$ and a measurable arc $\lambda : [0,1] \to \mathbb{R}^l$ satisfying the following conditions:*

(i) $p(1) = 0$;
(ii) $\mu_i(t) f^i(t, x(t), u(t)) = 0$ *for a.e.* $t \in [0,1]$ *for any* $i = 1, \dots, k$;
(iii) *for a.e.* $t \in [0,1]$

$$\begin{pmatrix} \dot{p}(t) + A_t^* p(t) - C_t^* \lambda(t) \\ B_t^* p(t) - D_t^* \lambda(t) \end{pmatrix} \in \partial \ell_t(x(t), u(t)) + \sum_{i=1}^{k} \mu_i(t) \partial f_t^i(x(t), u(t)) + \begin{pmatrix} N_{K_t}(x(t)) \\ N_{U_t}(u(t)) \end{pmatrix}.$$

(iv) $\xi_p(t) \in N_{\mathbf{X}(t)}(x(t))$ *$d\theta$-a.e.*

Then, (x,u) *is a solution for* (P_{χ_0}).

In a similar way, if now the second condition in (H') is replaced with

$$\operatorname*{ess-sup}_{t \in [0,1]} \max_{i=1,\dots,k} f_t^i(\bar{x}(t), \bar{u}(t)) < 0$$

then the conditions in the preceding theorem are also necessary.

5.2. **Regularity of the costate.** Let us now focus on the regularity of the costate multiplier p. In particular, let us show that under the constraint qualification version of the so called *bounded slope condition* (see [10] for example), the costate p is an absolutely continuous function and not merely a function of bounded variation.

Theorem 5.2. *Let $y \in \mathbf{X}(t)$. Then, for any $v \in \mathbb{R}^m$ such that $(y,v) \in \Omega(t)$ we have*

$$N_{\mathbf{X}(t)}(y) = \left\{\eta \in \mathbb{R}^n \colon (\eta, 0) \in N_{\Omega(t)}(y,v)\right\}. \tag{12}$$

In particular, if the the following bounded slope constraint qualification *(BSCQ) condition holds: there exists a constant $M > 0$ such that*

$$(y,v) \in \Omega(t),\ (\alpha,\beta) \in N_{\Omega(t)}(y,v) \implies |\alpha| \le M|\beta|, \tag{13}$$

then the costate p given by Theorem 2.2 is absolutely continuous.

Proof. Let $\rho_1 : \mathbb{R}^{n+m} \to \mathbb{R}^n$ be the projection onto $\mathbb{R}^n$, i.e. $\rho_1(y,v) = y$. As it was noted before, $\rho_1(\Omega(t)) = \mathbf{X}(t)$. Clearly, the map ρ_1 is strictly differentiable and $\nabla\rho_1(y,v)$ is surjective at any (y,v); in fact, $\nabla\rho_1(y,v) = \rho_1$. By [21, Theorem 1.57], ρ_1 is metrically regular at each (y,v), and hence, by [22, Theorem 4.2], whenever $\Omega(t) \neq \emptyset$ and $(y,v) \in \Omega(t)$ we have the equality

$$N_{\mathbf{X}(t)}(y) = [\rho_1^*(y)]^{-1}\left(N_{\Omega(t)}(y,v)\right)$$

where $\rho_1^* : \mathbb{R}^n \to \mathbb{R}^{n+m}$ is the adjoint operator of ρ_1. As one readily verifies, $\rho_1^*(y) = (y,0)$ for all y implying (12).

On the other hand, if (x^*,u^*) is as in Theorem 2.2, we can redefine u^* in a set of null measure in order to guarantee the inclusion $(x^*(t),u^*(t)) \in \Omega(t)$ for all $t \in [0,1]$ and not just almost everywhere. Indeed, if $t \in [0,1]$ is such that $(x^*(t),u^*(t)) \notin \Omega(t)$, we can take a sequence $t_k \to t$ such that $(x^*(t_k),u^*(t_k)) \in \Omega(t_k)$ for all $k \in \mathbb{N}$ and the sequence $u^*(t_k)$ is bounded. In this case, $(t_k,x^*(t_k),u^*(t_k))$ is bounded as well, so we can pass to a subsequence if necessary in order to guarantee the convergence of $(t_k,x^*(t_k),u^*(t_k))$ to $(t,x^*(t),u)$, for some $u \in \mathbb{R}^m$ which, by (A1),(A2) and (A3), satisfies $(x^*(t),u) \in \Omega(t)$. This u serves as our new value for $u^*(t)$. Therefore, if $\xi_p d\theta$ is any representation of the singular part of dp as before, condition (iii) in Theorem 2.2 and (12) imply

$$(\xi_p(t),0) \in N_{\Omega(t)}(x^*(t),u^*(t)) \qquad \forall t \in [0,1],$$

and consequently, by (13), ξ_p vanishes. □ □

5.3. **Endpoint cost and constraints.** Our main theorem on sufficient optimality conditions has been written for problems with a free terminal point and no endpoint cost. This explains why the transversality condition (item (i) in Theorem 2.1) is $p(1) = 0$. However, it is possible to extend our result for problems with general endpoint costs and constraints, as for instance the ones studied in [32]. Consider a function $g : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$, which is convex, proper and lower semicontinuous. This covers for instance the case in which

$$g(a,b) = \begin{cases} g_0(a,b) & \text{if } (a,b) \in S, \\ +\infty & \text{otherwise,} \end{cases}$$

where g_0 is a continuous convex function and S is a closed, convex and nonempty set. The problem to be studied now is

$$(\mathrm{P}^\sharp)\qquad \begin{cases} \text{Minimize} & \displaystyle\int_0^1 \ell(t,x(t),u(t))dt + g(x(0),x(1)),\\ \text{over all} & (x,u) \in AC([0,1];\mathbb{R}^n)\times L^\infty([0,1];\mathbb{R}^m),\\ \text{such that} & \dot{x}(t) = A(t)x(t)+B(t)u(t)+\varphi(t) \quad \text{a.e. on } [0,1],\\ & f(t,x(t),u(t)) \le 0 \quad \text{a.e. on } [0,1],\\ & \zeta(t) = C(t)x(t)+D(t)u(t) \quad \text{a.e. on } [0,1],\\ & x(t)\in K(t) \quad \text{on } [0,1],\\ & u(t)\in U(t) \quad \text{a.e. on } [0,1]. \end{cases}$$

If we replace item (i) in Theorem 2.1 with the transversality condition

$$(p(0),-p(1)) \in \partial g(x(0),x(1))$$

then, Theorem 2.1 holds still for ($\mathrm{P}^\sharp$). Indeed, just as in the proof of Theorem 2.1, conditions (ii), (iii) and (iv) in Theorem 2.1 imply that the arc (x,y) is an extremal with coextremal (p,q), where $y(t)=\int_0^t u(\tau)d\tau$ and $q(t)=0$.

Therefore, since $J_L(x,y)$ is finite, (x,y) is optimal in the sense that

$$J_L(x,y) = \inf\left\{ J_L(z,v) : (z,v)\in BV_n\times BV_m,\ \begin{array}{l}(z(0),v(0)) = (x(0),0)\\ (z(1),v(1)) = (x(1),y(1))\end{array} \right\}$$

and $(x(0),y(0),x(1),y(1))$ and $(p(0),q(0),p(1),q(1))$ are endpoints in duality:

$$J_L(x,y)+J_M(p,q) = \langle x(1),p(1)\rangle - \langle x(0),p(0)\rangle.$$

On the other hand, if (z,v) is an admissible process for ($\mathrm{P}^\sharp$), then setting $w(t) = \int_0^t v(\tau)d\tau$, we have by [29, Theorem 1] that

$$J_L(z,w)+J_M(p,q) \ge \langle z(1),p(1)\rangle - \langle z(0),p(0)\rangle.$$

By the transversality condition we get

$$g(z(0),z(1)) \ge g(x(0),x(1)) + \langle x(1)-z(1),p(1)\rangle - \langle x(0)-z(0),p(0)\rangle.$$

These two inequalities, with the fact that the endpoints are in duality, lead to

$$\begin{aligned} J_L(z,w)+g(z(0),z(1)) &\ge -J_M(p,q)+g(x(0),x(1)) + \langle x(1),p(1)\rangle - \langle x(0),p(0)\rangle\\ &= J_L(x,y)+g(x(0),x(1)) \end{aligned}$$

From where we deduce the optimality of (x,u) for problem ($\mathrm{P}^\sharp$).

5.4. **Example.** We end this paper with a simple example that illustrates how the previous theorems can help us discern between solutions and non-solutions.

Example 5.1. *Consider the following problem:*

$$\begin{cases} \min & J(x,u) = \displaystyle\int_0^1 x(t)dt,\\ s.t. & (x,u)\in AC\times L^\infty,\\ & \dot{x}(t) = \ln(2)[1+u(t)] \quad a.e.,\\ & |x(t)|+|u(t)| \le 1 \quad a.e.,\\ & u(t)\in[-1,1] \quad a.e,\\ & x(0)=0. \end{cases}$$

Clearly, assumptions (A1)-(A6) and (H) *are satisfied, moreover,* (H') *is also satisfied with* $(\bar{x},\bar{u}) = (0,0)$*. Thus, the sufficient conditions in Theorem 2.1 and* *the necessary conditions in Theorem 2.2 are valid. Notice that in this example* $\mathbf{X}(t) = [-1, 1]$ *for all* $t \in [0, 1]$.

First, we analyze the admissible process (y, v) *defined by*

$$y(t) = -2e^{-\ln(2)t} + 2, \quad v(t) = 2e^{-\ln(2)t} - 1.$$

Note that the mixed constraint is always active, i.e. $|y(t)| + |v(t)| = 1$ *for all* t. *Inclusion* (iii) *in Theorem 2.1 becomes*

$$(-\dot{p}(t), 0) \in (-1, \ln(2)p(t)) - (\mu(t), \mu(t)) + \{0\} \times \{0\} \quad a.e.$$

which, together with the complementary slackness, is equivalent to the pair of equations $\dot{p}(t) = 1 + \mu(t)$ *a.e. and* $0 \leq \mu(t) = \ln(2)p(t)$ *a.e.. On the other hand, computing the normal cone in* (iv) *we obtain*

$$N_{\mathbf{X}(t)}(y(t)) = \begin{cases} \{0\} & \text{if } t \in [0, 1) \\ \mathbb{R}_+ & \text{if } t = 1. \end{cases}$$

Since p *is nonnegative, its absolutely continuous part is strictly increasing and its singular part has only a nonnegative jump at* $t = 1$, *the transversality condition* $p(1) = 0$ *is not to achieve. This analysis shows that the process* (y, v) *cannot be a solution since the conditions are not satisfied. Indeed, it is not difficult to see that the solution is the process* $(x, u) = (0, -1)$; *we proceed to prove this with the help of Theorem 2.1. For this process, condition* (iii) *is*

$$(-\dot{p}(t), 0) \in (-1, \ln(2)p(t)) + [-\mu(t), \mu(t)] \times \{1\} + \{0\} \times \mathbb{R}_-$$

or, equivalently, $-\dot{p}(t) + 1 \in [-\mu(t), \mu(t)]$ *and* $-\ln(2)p(t) - 1 \leq 0$. *Both of these equations are clearly satisfied by defining* $p(t) = 0$ *and* $\mu(t) = 1$, *as well as the complementary slackness condition in* (i) *and the inclusion in* (iii). *Since the conditions in Theorem 2.1 are sufficient,* (x, u) *is indeed a solution.*

Finally, notice p *was taken absolutely continuous even though the mixed constraint does not satisfy the BSCQ due to the inclusion*

$$\mathbb{R}_+ \times \{0\} \subset N_{\Omega(t)}((1, 0)), \qquad \forall t \in [0, 1].$$

Remark 5.1. *Notice that in Example 5.1 we have* $f(t, x, u) \in [0, 2\ln(2)]$ *for any* $u \in U(t)$ *and the underlying state constraint is the set* $[-1, 1]$. *In particular, in this example the IPC is not satisfied, however, as pointed out above, the interior feasibility condition (H') holds, demonstrating that this condition is less restrictive than the IPC as claimed in Remark 2.6.*

6. Conclusion

It is well-known that when no state constraints or mixed constraints are present, any feasible process that satisfies the Maximum Principle in a problem with a linear-convex structure is actually a (global) minimizer. This is also true for problem with (pure) state constraints; see for instance [14, Section 8] or [11], and the reference therein. In this paper we have shown that this fact is also true for problems with mixed constraints; as a matter of fact, for problems that have mixed and pure state constraints combined. It is important to highlight that this has been accomplished without the need of imposing smoothness assumptions on the data of the problem or further structural conditions on the costate multiplier as for instance done in [19].

Furthermore, we have also shown that the optimality conditions we have obtained are in fact necessary under a suitable Slater type qualification condition. This condition is of different nature than the usual regularity conditions imposed in the literature (the Mangasarian-Fromovitz's one or the bounded slope condition), and in some sense, it is weaker than usual regularity conditions, because it does not force the costate multiplier to be an absolutely continuous arc, it can rather be an arc of bounded variation.

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