# ON THE CONSTRUCTION OF NEARLY TIME OPTIMAL CONTINUOUS FEEDBACK LAWS AROUND SWITCHING MANIFOLDS

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Abstract. In this paper we address the question of the construction of a nearly time optimal feedback law for a minimum time optimal control problem, which is robust with respect to internal and external perturbations. For this purpose we take as starting point an optimal synthesis, which is a suitable collection of optimal trajectories. The construction we exhibit depends exclusively on the initial data obtained from the optimal feedback which is assumed to be known.

## Introduction

This paper is concerned with state constrained minimum time problems, that is, we consider the problem of finding the smallest $T \geq 0$ such that a trajectory of a given control-affine system

$$\dot{y} = f_0(y) + \sum_{i=1}^{m} u_i f_i(y), \quad u(t) \in [-1,1]^m, \quad \text{for a.e. } t \geq 0,$$

reaches at time $t = T$ a given closed set $\Theta \subseteq \mathbb{R}^N$ (the *target*) from a given initial state $x \in \mathcal{K}$ at time $t = 0$, while satisfying the path constraint $y(t) \in \mathcal{K}$ on $[0, T]$ for a prescribed closed set $\mathcal{K} \subseteq \mathbb{R}^N$ (the *state constraint*). We are mainly interested in optimal solutions in feedback form, that is, mappings $U : \mathcal{K} \to [-1,1]^m$ that satisfy $u_x^*(t) = U(y_x^*(t))$ for a suitable collection of optimal trajectory-control pairs $\{(y_x^*, u_x^*)\}_{x \in \Omega}$, where $\Omega \subseteq \mathcal{K}$.

It is well-known that the ordinary differential equation (ODE) induced by the closed-loop system

$$\dot{y} = f_0(y) + \sum_{i=1}^{m} U_i(y) f_i(y) \tag{1}$$

may not be well-posed and solutions in the Carathéodory sense may not even exist. This is because time optimal feedback laws are in general discontinuous functions on the state; see for instance the discussion in [14]. Actually, simple examples, such as the double integrator, show that even for linear systems, without state constraints, it is likely that optimal feedbacks are discontinuous. There are indeed topological

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obstructions that block the existence of continuous feedbacks such as the Brockett's condition introduced in [9]; see also the discussion in [14]. The latter was firstly conceived for stabilization problems (to reach the target asymptotically on time), but it can be applied to some classes of optimal problems as well. Topological obstructions, such as the Brockett's condition, are so significant that they may even preclude the existence of continuous nearly time optimal strategies. One possible way to deal with this issue is to consider weaker notions of solutions such as Filippov or Krasovskii solutions, by using a regularizing procedure on the vector field of the right handside; see for instance [5, Chapter 1]. Let us point out that, in this case, the regularized vector field may not have a correlation with the original ODE, because it could be introducing new velocities that are not part of the original dynamical system.

Other techniques well-suited for the closed loops system (1) have been investigated in the literature. These methodologies avoid the possible lost of information caused by the regularizing schemes. Depending on the purpose at hand, we can classify these methods into two types: (i) generalized notions of solutions and (ii) sufficiently regular discontinuities on the feedback control. In the first case we find the *sample-and-hold* solutions (see [14] and the references therein) and in the second one the *Patchy feedbacks* (see for example [1, 2, 3]).

In this paper we follow similar ideas as in [3], that is, we focus on feedback controls that are regular enough such that the notion of Carathéodory solution can still be well-defined even though the feedback is discontinuous.

Our starting point are the so-called regular syntheses. The notion of regular synthesis was introduced in [7] and subsequently generalized to broader settings by many authors; see for instance [10, 29, 25, 28]. The main idea is that instead of working with optimal feedback directly and facing trajectories that may not be optimal, we only deal with a collection of extremals that cover the whole state space and fit together in an appropriate way. Any extremal is associated with a piecewise continuous feedback, but a trajectory of the closed-loop system (1) doesn't necessarily belong to an optimal synthesis.

The purpose of this paper is to point out that by slightly modifying a time optimal feedback law around some of its singularities we can obtain a nearly time optimal feedback that is locally Lipschitz continuous around the corresponding singularity; the time optimal feedback law is the one provided by an optimal synthesis. The type of discontinuities we have in mind are those that occur in presence of a switching manifold (also called switching locus sometimes). This kind of singular sets are in many cases trajectories of the system, and so, they can also be seen as integral manifolds.

Let us point out that, in the case that no state constraints are considered, some authors have shown that it is possible to construct nearly time optimal strategies that enjoy robustness properties without further requirements; see for example [3, 6, 35]. In the presence of state constraints, the issue has also been addressed but imposing beforehand an inward pointing condition (IPC); see for instance [16, 22, 30]. To the best of our knowledge, there are no works that, taking advantage of optimal syntheses, propose a construction of nearly time optimal strategies consistent in the sense we have described earlier.

The construction we propose does not require any type of IPC, and, as a matter of fact, it is based on a rather *natural* idea that uses convex combinations between smooth closed-loop controls. Details of the construction are given later on (see Theorem 2.1) and demonstrated on the double integrator problem (see (6) in §1.2 )

Finally, let us emphasize that the construction we propose focuses on *closed-loop* controls, which differs in nature from other regularization methods concerned with auxiliary approximated problems designed for constructing smooth *open-loop* (nearly time optimal) controls. For example, in [4] a modification on the cost is considered whereas in [38, 36, 24] a perturbation on the dynamics is investigated. The underlying idea of these works is that, by modifying the Hamiltonian of the problem, one can obtain nearly time optimal smooth approximations of an open-loop optimal control by means of the Pontryagin maximum principle.

**Notation and mathematical definitions.** Throughout this paper, $\mathbb{R}$ denotes the set Real numbers, $N$ and $m$ are given Natural numbers which remain fixed all along the exposition. We use $|\cdot|$ for the Euclidean norm and $\langle\cdot,\cdot\rangle$ for the Euclidean inner product on $\mathbb{R}^N$. The unit open ball $\{x\in\mathbb{R}^N \mid |x|<1\}$ is indicated by $\mathbb{B}$ and with a slight abuse of notation we write $\mathbb{B}(x,r)=x+r\mathbb{B}$. For a set $S\subseteq\mathbb{R}^N$, $\mathrm{int}(S)$ and $\overline{S}$ denote its interior and closure, respectively. The distance function to $S$ is $\mathrm{dist}_S(x)=\inf\{|x-y| \mid \ y\in S\}$.

For a given locally closed set $S\subseteq\mathbb{R}^N$ we write $\mathcal{T}_S^B(x)$ and $\mathcal{T}_S^C(x)$ for the Bouligand and generalized tangent cones to $S$ at $x\in S$, which are defined via

$$\mathcal{T}_S^B(x)=\left\{v\in\mathbb{R}^N \;\middle|\; \liminf_{t\to0^+}\frac{\mathrm{dist}_S(x+tv)}{t}\le 0\right\}$$

and

$$\mathcal{T}_S^C(x)=\left\{v\in\mathbb{R}^N \;\middle|\; \limsup_{\tilde{x}\to x,\ t\to0^+}\frac{\mathrm{dist}_S(\tilde{x}+tv)}{t}\le 0\right\}.$$

A set $\mathcal{M}\subseteq\mathbb{R}^N$ is a $d$-dimensional embedded manifold of $\mathbb{R}^N$ if for any $x\in\mathcal{M}$ there is an open set $\mathcal{O}$ so that

$$\mathcal{M}\cap\mathcal{O}=\{\tilde{x}\in\mathcal{O} \mid \ h_1(\tilde{x})=\ldots=h_{N-d}(\tilde{x})=0\},$$

where $h:\mathbb{R}^N\to\mathbb{R}^{N-d}$ is a smooth function whose derivative $Dh(\tilde{x})$ is surjective at any $\tilde{x}\in\mathcal{O}$. The function $h$ is called a local defining map for $\mathcal{M}$ around $x$. Furthermore, the tangent space to $\mathcal{M}$ at $x$, which we denote by $\mathcal{T}_{\mathcal{M}}(x)$, can be identified with the set

$$\{v\in\mathbb{R}^N \mid \ \langle\nabla h_1(x),v\rangle=\ldots=\langle\nabla h_{N-d}(x),v\rangle=0\}.$$

## 1. Setting of the problem

Consider the control-affine system with input constraints

(2)

$$\dot{y}(t)=f(y(t),u(t)):=f_0(y(t))+\sum_{i=1}^m u_i(t)f_i(y(t)),\quad u(t)\in[-1,1]^m,\quad \text{for a.e. } t\ge0,$$

where $f_0,\ldots,f_m:\mathbb{R}^N\to\mathbb{R}^N$ are locally Lipschitz continuous vector fields that satisfy

$$(H_f)\qquad \exists c_f>0 \text{ such that } \ |f_i(x)|\le c_f(1+|x|),\qquad \forall i\in\{0,\ldots,m\},\ \forall x\in\mathbb{R}^N$$

It is well-known that under these conditions for any measurable function (*control*) $u : [0, +\infty) \to [-1, 1]^m$ and $x \in \mathbb{R}^N$, there is a unique absolutely continuous curve $y_x^u : [0, +\infty) \to \mathbb{R}^N$ that satisfies (2) and $y_x^u(0) = x$.

In order to take into account physical or economical constraints that may appear in mathematical modeling, consider a nonempty closed sets $\mathcal{K} \subseteq \mathbb{R}^N$ (the *state constraint*), and define, for any $\tau > 0$ given, the set of *admissible controls* on $[0, \tau]$ as follows

$$\mathbb{U}_{\mathcal{K}}^{\tau}(x) := \{u : [0, +\infty) \to [-1, 1]^m \text{ measurable such that } y_x^u(t) \in \mathcal{K},\ \forall t \in [0, \tau]\}.$$

Given a nonempty closed set $\Theta \subseteq \mathbb{R}^N$ such that $\Theta \cap \mathcal{K} \neq \emptyset$, we are concerned with the minimum time problem to reach the target $\Theta$ while being feasible on $\mathcal{K}$, that is, the problem of finding the smallest $T \geq 0$ and an admissible control $u \in \mathbb{U}_{\mathcal{K}}^{T}(x)$ such that $y_x^u(T) \in \Theta$. The optimal value of this problem is called the *minimum time function* and is given by

$$\mathbf{T}^{\Theta}(x) := \inf \left\{ T \geq 0 \mid \ u \in \mathbb{U}_{\mathcal{K}}^{T}(x),\ y_x^u(T) \in \Theta \right\}, \quad \forall x \in \mathcal{K}. \tag{3}$$

Under the assumptions we have done so far, it follows that $\mathbf{T}^{\Theta}$ is lower semicontinuous (cf.[13, Proposition 2.1]). Furthermore, by [13, Theorem 7.2], it is the smallest positive lower semicontinuous viscosity supersolution of the Hamilton-Jacobi-Bellman (HJB) Equation

$$-1 + H(x, \nabla\varphi(x)) = 0, \qquad x \in \mathcal{K} \cap \operatorname{dom}(\varphi).$$

Also, it is a viscosity bilateral subsolution of the HJB Equation on $\operatorname{int}\left(\operatorname{dom}\left(\mathbf{T}^{\Theta}\right)\right)$; cf. [21, Theorem 4.2].

Since $\Theta$ and $\mathcal{K}$ are closed sets, thanks to $(H_f)$, the set of admissible trajectories starting at $x \in \mathcal{K}$ fixed is compact (possibly empty) in the space of continuous functions and so, whenever $x \in \operatorname{dom}(\mathbf{T}^{\Theta})$, that is, $\mathbf{T}^{\Theta}(x) < +\infty$ we can find a control $u_x \in \mathbb{U}_{\mathcal{K}}^{T}(x)$ which realizes the infimum in (3); the proof is essentially the same as in [20, Proposition 3.2]. Hence, a time optimal synthesis is a function $U : \operatorname{dom}(\mathbf{T}^{\Theta}) \to [-1, 1]^m$ that satisfies

$$U(y_x^{u_x}(t)) = u_x(t), \quad \text{whenever } \mathbf{T}^{\Theta}(x) \in \mathbb{R} \text{ and for a.e. } t \in [0, \mathbf{T}^{\Theta}(x)].$$

For practical purposes, in many cases it is enough to find a synthesis that is almost optimal in the sense that for any $\varepsilon > 0$ given, there is a set $\mathcal{K}_\varepsilon \subseteq \operatorname{dom}(\mathbf{T}^{\Theta})$ and a mapping $U^\varepsilon : \mathcal{K}_\varepsilon \to [-1, 1]^m$ such that trajectories of the closed-loop system (1) with $U = U^\varepsilon$ are well defined, reach the target $\Theta_\varepsilon := \Theta + \mathbb{B}(0, \varepsilon)$ and satisfy

$$\tau_\varepsilon(x) := \min\{T \geq 0 \mid \ y_x^\varepsilon(T) \in \Theta_\varepsilon\} \leq \mathbf{T}^{\Theta}(x) + \varepsilon, \quad \forall x \in \mathcal{K}_\varepsilon,$$

where $y_x^\varepsilon$ is any curve that solves (1) with $U = U^\varepsilon$, which remains in $\mathcal{K}_\varepsilon$ (i.e. $y_x^\varepsilon(t) \in \mathcal{K}_\varepsilon$ for any $t \in [0, \tau_\varepsilon(x)]$) and verifies $y_x^\varepsilon(0) = x$. A feedback control such as $U^\varepsilon$ will be called in the sequel *nearly time optimal*.

Notwithstanding the fact that feedback laws are usually discontinuous functions on the state, they are likely to have rather regular singularities, in the sense that in some regions of the state-space, the feedback is smooth and the notion of Carathéodory solutions can still be defined; we refer for example to [18, 10, 11, 39, 27, 8]. The set of points where an optimal strategy behaves like that is often an embedded manifold. The latter motivates the following definition.

**Definition 1.1.** *We say that $\mathcal{M}$, an embedded manifold of $\mathbb{R}^N$, is a cell related to a feedback control law strategy $U : \mathbb{R}^N \to [-1,1]^m$ provided that*

(4)
$$x \mapsto U|_{\mathcal{M}}(x) \text{ is locally Lipschitz continuous on } \mathcal{M} \text{ and } f(x,U(x)) \in \mathcal{T}_{\mathcal{M}}(x),\ \forall x \in \mathcal{M}.$$

**Remark 1.1.** *Notice that for any initial condition in a cell, there exist $\tau_{\max} > 0$ and a unique smooth curve $y : [0,\tau_{\max}) \to \mathcal{M}$ that verifies* (1)*. This is a consequence of Nagumo Theorem and the Lipschitz character of the vector fields; see for instance* [19].

1.1. **Framework and assumptions.** The purpose of this paper is to point out that by slightly modifying a time optimal feedback law around some of its singularities we can obtain a nearly time optimal feedback that is locally Lipschitz continuous around the corresponding singularity, and thus robust with respect to external and internal perturbations. We are interested in the circumstances where there exist $\mathcal{M}_{\text{ini}}$ and $\mathcal{M}_{\text{end}}$, both being cells associated with an optimal feedback and verifying some structural conditions. Hence, from this point onwards, we assume that

$$(H_0)\ \begin{cases} (i) \quad U_0 : \mathcal{K} \to [-1,1]^m \text{ is a given optimal feedback control law.} \\ (ii) \ \ \mathcal{M}_{\text{ini}} \text{ and } \mathcal{M}_{\text{end}} \text{ are cells related to } U_0. \\ (iii)\ \Theta \cap \overline{\mathcal{M}_{\text{end}}} \neq \emptyset, \quad \mathcal{M}_{\text{end}} \subseteq \overline{\mathcal{M}_{\text{ini}}} \quad \text{and} \quad \mathcal{M}_{\text{ini}} \cup \mathcal{M}_{\text{end}} \subseteq \text{dom}(\mathbf{T}^{\Theta}). \end{cases}$$

In this setting, we are concerned with the cases in which the minimum time function to reach $\Theta$ starting from $\mathcal{M}_{\text{ini}}$ can be decomposed in the following fashion:

(5)
$$\mathbf{T}^{\Theta}(x) = \min_{\tau>0} \left\{ \mathbf{T}^{\Theta}(y_x^{u_x}(\tau)) + \tau \,\middle|\, \begin{array}{ll} y_x^{u_x}(t) \in \mathcal{M}_{\text{ini}}, & \forall t \in [0,\tau) \\ y_x^{u_x}(t) \in \mathcal{M}_{\text{end}}, & \forall t \in [\tau, \mathbf{T}^{\Theta}(x)) \end{array} \right\}, \quad \forall x \in \mathcal{M}_{\text{ini}}.$$

In other words, the optimal strategy is the concatenation of two smooth feedbacks so that the path followed by a time-minimizing curve is contained in the corresponding cell; it starts at $\mathcal{M}_{\text{ini}}$, then reaches $\mathcal{M}_{\text{end}}$, and afterwards, it hits the target. This class of singularities is exactly the one described in [18] for a synthesis around the origin for normal linear models and it also agrees with some of the generic singularities of a 2D system exhibited in [8]. In this context, we refer to $\mathcal{M}_{\text{end}}$ as a *switching manifold*.

In Figure 1 we show an illustration in order to give an idea of what is expected to happen; the optimal curve $y_x^{u_x}(\cdot)$ (drawn in black) hits $\Theta$ whereas the nearly time optimal trajectory $y_x^{\varepsilon}(\cdot)$ (the blue arc) does not reach the target but a neighborhood of it.

The basic tool we use in our analysis is the Value Function itself, which we assume fulfills the following conditions:

$$(H_0) \qquad \begin{cases} i) \ \exists\ \mathcal{Q} \subseteq \mathbb{R}^N \text{ open with } \mathcal{M}_{\text{ini}} \cup \mathcal{M}_{\text{end}} \subseteq \mathcal{Q}. \\ ii) \ \exists\ \omega : \mathcal{Q} \to \mathbb{R} \text{ of class } \mathcal{C}^2 \text{ so that } \omega|_{\mathcal{M}_{\text{ini}}} = \mathbf{T}^{\Theta}|_{\mathcal{M}_{\text{ini}}} \text{ on } \mathcal{M}_{\text{ini}}. \end{cases}$$

**Remark 1.2.** *Note that the assumption $(H_0)$ is only a local property of the value function around $\mathcal{M}_{ini} \cup \mathcal{M}_{end}$, which is rather natural in the framework of optimal syntheses we are considering in this paper. In that setting value functions are assumed to be piecewise smooth; cf.* [10, 29, 28]. *Let us point out that this hypothesis*

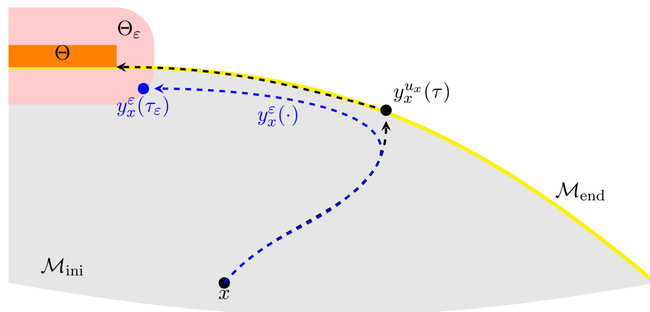


FIGURE 1. An illustration of the construction of a nearly time optimal feedback

*is independent of global properties of the minimum time function, such as continuity. For this reason, assumptions such as the* Petrov's condition *or* small-time controllability*, are not considered in this manuscript.*

Since we are interested in the circumstances when the flows from $\mathcal{M}_{\text{ini}}$ are transversal to $\mathcal{M}_{\text{end}}$ we suppose in addition that, if $U_0(\cdot)$ is an optimal synthesis given by $(H_0)$, then

$$(H_1) \qquad \begin{cases} \exists U_{\text{ini}} : \overline{\mathcal{M}_{\text{ini}}} \to [-1,1]^m \text{ locally Lipschitz continuous so that} \\ U_{\text{ini}} = U_0|_{\mathcal{M}_{\text{ini}}} \text{ on } \mathcal{M}_{\text{ini}} \text{ and } f(x, U_{\text{ini}}(x)) \notin \mathcal{T}_{\mathcal{M}_{\text{end}}}(x),\ \forall x \in \mathcal{M}_{\text{end}}. \end{cases}$$

In the foregoing hypothesis, the existence of an extension of the feedback is immediately verified if the feedback is uniformly continuous on $\mathcal{M}_{\text{ini}}$, which can be provided using density arguments.

1.2. **An example: the double integrator problem.** Before presenting the theoretical development, we exhibit first an explicit example to enlighten the technique to be used in the rest of the paper. We consider the double integrator problem:

$$\min T \quad \text{s.t.} \quad \begin{pmatrix} \dot y_1 \\ \dot y_2 \end{pmatrix} = \begin{pmatrix} y_2 \\ u \end{pmatrix}, \quad u(t) \in [-1,1] \quad \text{a.e. on } [0,T], \quad y(0) = (x_1, x_2), \quad y(T) = (0,0).$$

In this example, the target is $\Theta = \{(0,0)\}$ and the switching manifolds are contained in the curve given by

$$2x_1 + \text{sign}(x_2)x_2^2 = 0.$$

In Figure 2, this set is represented by the black curve and the red ball is the $\varepsilon$-neighborhood of the origin we want to reach. Note that outside the gray zone the optimal policy is already locally Lipschitz continuous, so a nearly time optimal continuous feedback $U^\varepsilon$ only needs to differ from the optimal one in the gray zone.

We recall that in this situation, the minimum time function to reach the origin can be computed explicitly and it is given by

$$\mathbf{T}^\Theta(x) = \begin{cases} -x_2 + \sqrt{2x_2^2 - 4x_1} & 2x_1 + \text{sign}(x_2)x_2^2 < 0, \\ x_2 + \sqrt{2x_2^2 + 4x_1} & 2x_1 + \text{sign}(x_2)x_2^2 > 0, \\ |x_2| & 2x_1 + \text{sign}(x_2)x_2^2 = 0. \end{cases}$$

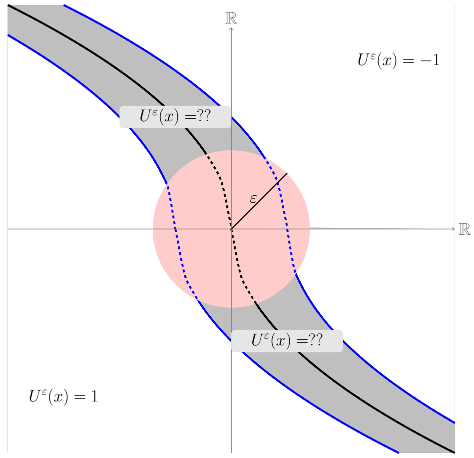


FIGURE 2. double integrator example

Let us focus on the construction around the manifolds

$$\mathcal{M}_{\text{ini}} = \{x \in \mathbb{R}^2 \mid \; 2x_1 + \text{sign}(x_2)x_2^2 > 0\} \quad \text{and} \quad \mathcal{M}_{\text{end}} = \{x \in (0, +\infty) \times (-\infty, 0) \mid \; h(x) = 0\},$$

where

$$h(x) := 2x_1 - x_2^2, \quad \forall x = (x_1, x_2) \in \mathbb{R}^2.$$

We check that $(H_0)$ is verified with $\mathcal{Q} = \{2x_1 + x_2^2 > 0\}$ and $\omega(x) = x_2 + \sqrt{2x_2^2 + 4x_1}$.

Let $\varepsilon > 0$ given and take $\delta \in (0, \varepsilon)$ to be fixed. Consider the curve

$$\mathcal{M}_{\text{end}}^{\delta} = \{x \in \mathbb{R}^2 \mid \; 0 < x_1, \; x_2 < 0, \; h(x) = 2\delta\}.$$

The region of interest, where the optimal control is going to be modified is depicted in Figure 3. It tallies with the area between the curves $\mathcal{M}_{\text{end}}$ and $\mathcal{M}_{\text{end}}^{\delta}$. Outside of this zone, there is no real need to alter it, because, as aforementioned, the feedback is continuous outside of the switching curve. Therefore, we can set

$$\mathcal{K}_{\varepsilon} = \{x \in \mathbb{R}^2 \mid \; 2x_1 + \text{sign}(x_2)x_2^2 \geq 0\}.$$

Let $\Omega_\delta$ be the zone where it is desired to modify the feedback, that is,

$$\Omega_\delta = \{x \in \mathcal{O} : \; 0 \leq h(x) \leq 2\delta\}, \quad \text{where } \mathcal{O} := \mathbb{R} \times (-\infty, 0).$$

We consider as well the locally Lipschitz continuous function $\lambda : \Omega_\delta \to [0, 1]$ defined via

$$\lambda(x) = \left(1 - \frac{1}{2\delta} h(x)\right), \quad x \in \Omega_\delta.$$

Notice that $\lambda(x) = 1$ if and only if $x \in \mathcal{M}_{\text{end}}$. Hence the prototype nearly time optimal strategy is

$$U^\delta(x) = \begin{cases} 1 & x \in \mathcal{O}, \; h(x) = 0, \\ -1 + 2\lambda(x) & x \in \text{int}(\Omega_\delta), \\ -1 & \text{otherwise}, \end{cases} \quad , \quad x \in \mathcal{K}_\varepsilon \setminus \mathbb{B}(0, \varepsilon). \tag{6}$$

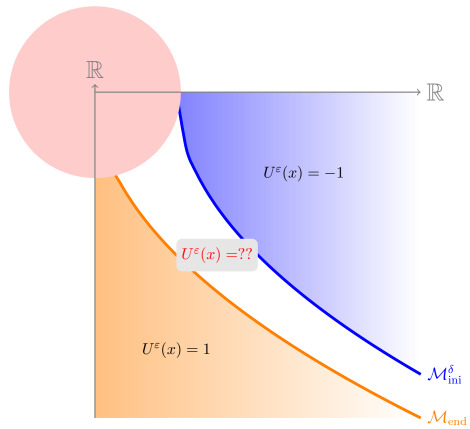


FIGURE 3. Zone of interest

Clearly, $U^\delta$ is continuous and therefore the next ordinary differential equation always admits solutions in the classical sense for any initial condition on $\mathcal{K}_\varepsilon \setminus \mathbb{B}(0,\varepsilon)$:

$$\begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \end{pmatrix} = \begin{pmatrix} y_2 \\ U^\delta(y) \end{pmatrix}. \tag{7}$$

**Remark 1.3.** *Note that $U^\delta$ is actually locally Lipschitz continuous. Indeed, we have*

$$|U^\delta(x) - U^\delta(y)| = 2\lambda(x) = 2 - \frac{1}{\delta}h(x) \leq \frac{1}{\delta}(h(y) - h(x)), \qquad \forall x \in \mathit{int}(\Omega_\delta),\ y \in \mathcal{O} \setminus \mathit{int}(\Omega_\delta),$$

*where the last inequality comes from the fact that $h(y) > 2\delta$. This shows that the Lipschitz constant of the feedback depends on $h$, but more importantly, it is inversely proportional to $\delta$, and so, blows up as $\delta \to 0$.*

Let $y$ be a solution to (7) lying on $\Omega_\delta$ with initial condition $x \in \operatorname{int}(\Omega_\delta)$. Let $[0,\tau)$ be the maximal interval of time for which $y$ belongs to $\operatorname{int}(\Omega_\delta)$, that is

$$\tau = \inf\{t > 0 \mid\ y(t) \in \operatorname{int}(\Omega_\delta)\}.$$

Define $\rho(t) := h(y(t))$ for any $t \in [0,\tau)$ and note that this function is differentiable on $(0,\tau)$. Whereupon, setting $u = U^\delta(y)$ on $(0,\tau)$ we get:

$$\dot{\rho}(t) = 2y_2(t)(1 - u(t)), \quad \forall t \in (0,\tau). \tag{8}$$

Therefore, as $y_2(t) < 0$ for any $t \in (0,\tau)$, the sign of $\dot{\rho}$ is negative on $(0,\tau)$, which means that the function $\rho(\cdot)$ is strictly decreasing on $(0,\tau)$. Using an argument of density, this affirmation can be extended to any arc solution to (7) that starts from $\mathcal{M}^\delta_{\text{end}}$.

**Remark 1.4.** *Let us point out that $\tau$ is finite. Actually, if it is not the case, we can assume that there is $\alpha \in (0,1)$ so that $u(t) \leq 1 - \alpha$ for any $t \geq 0$. Otherwise, since $\rho(\cdot)$ is decreasing we would have that $\dot{y}_2(t) = u(t) > 1 - \alpha$, which implies that $\tau$ is finite.*

*We might also assume that $y_2(t) \leq -\alpha$ for any $t > 0$, and therefore $\dot{\rho}(t) \leq -2\alpha^2$. This inequality yields to a contradiction because for some $t > 0$, $\rho(t) = 0$ but $u(t) < 1$.*

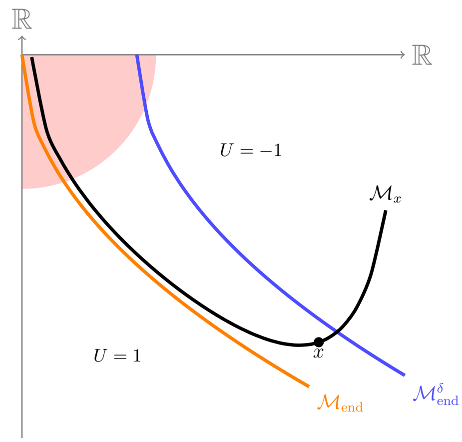


FIGURE 4. Manifold associated with the perturbed feedback.

A simple computation shows that, since $\rho$ is strictly decreasing on $(0, \tau)$, if $\tau_\varepsilon(x)$ stands for the time required to hit the target $\Theta_\varepsilon$ starting from $x$, the following estimate holds true

$$y_2(t) \leq -\alpha_\varepsilon(\delta), \quad \forall x \in \Omega_\delta,\ t \in [0, \tau_\varepsilon(x)],$$

where $\alpha_\varepsilon(\delta) := \sqrt{2}\sqrt{\sqrt{1+\varepsilon^2+2\delta} - (1+\delta)}$; the bound is obtained by finding the intersection point between $\mathcal{M}^{\delta}_{\text{end}}$ and the circle of radius $\varepsilon$.

Note also that $\tau > \tau_\varepsilon(x)$ is due to $\delta < \varepsilon$. Accordingly, thanks to Remark 1.4, $\tau_\varepsilon(x)$ is a finite number likewise $\tau$. Furthermore, it is not difficult to see that

$$\dot\rho(t) = \frac{2}{\delta} y_2(t)\rho(t) = \frac{2}{\delta}\dot y_1(t)\rho(t), \quad \forall t \in (0, \tau),$$

which implies that

$$\rho(t) = h(x)\exp\left(\frac{2}{\delta}(y_1(t) - x_1)\right) \quad \forall t \in [0, \tau).$$

In particular, $\rho(t) > 0$ and $x_2(t) > 0$ for any $t \in [0, \tau]$, and so $y_2(\tau) = 0$. Indeed, any trajectory of the modified feedback that begins at $x \in \Omega_\delta$, belongs to the manifold $\mathcal{M}_x$ that has been portrayed in Figure 4 and whose analytic expression is

$$\mathcal{M}_x = \left\{ \tilde x \in \mathcal{O} \mid\ h(\tilde x) = h(x)\exp\left(\frac{2}{\delta}(\tilde x_1 - x_1)\right)\right\},\ x \in \Omega_\delta.$$

On the other hand, on the interval $(0, \tau_\varepsilon(x))$ the next inequality holds:

$$\dot\rho(t) \leq -\frac{2\alpha_\varepsilon(\delta)}{\delta}\rho(t), \quad \forall t \in (0, \tau_\varepsilon).$$

So, $\rho(t) \leq h(x)\exp\left(-\frac{2\alpha_\varepsilon(\delta)}{\delta}t\right)$ for any $t \in [0, \tau_\varepsilon(x)]$. Whereupon,

$$\dot y_2(t) = 1 - \frac{1}{\delta}\rho(t) \geq 1 - \frac{h(x)}{\delta}\exp\left(-\frac{2\alpha_\varepsilon(\delta)}{\delta}t\right).$$

This yields, integrating the inequality between $t = 0$ and $t = \tau_\varepsilon(x)$, to

$$y_2(t) - x_2 \geq \tau_\varepsilon(x) - \frac{h(x)}{\delta}\int_0^{\tau_\varepsilon(x)} \exp\left(-\frac{2\alpha_\varepsilon(\delta)}{\delta}t\right) dt \geq \tau_\varepsilon(x) - \frac{h(x)}{2\alpha_\varepsilon(\delta)}.$$

Remark that $\mathbf{T}^\Theta(x) = x_2 + \sqrt{4x_2^2 + 2h(x)}$, and so,

$$y_2(t) - x_2 \leq -\alpha_\varepsilon(\delta) + \mathbf{T}^\Theta(x), \quad t \in [0, \tau_\varepsilon(x)].$$

Consequently, we have found out that

$$\frac{h(x)}{2\alpha_\varepsilon(\delta)} - \alpha_\varepsilon(\delta) + \mathbf{T}^\Theta(x) \geq \tau_\varepsilon(x), \quad \forall x \in \Omega_\delta.$$

Besides, due to $h(x) \leq 2\delta$ we get

$$\frac{h(x)}{2\alpha_\varepsilon(\delta)} - \alpha_\varepsilon(\delta) \leq \frac{\delta - \alpha_\varepsilon(\delta)^2}{\alpha_\varepsilon(\delta)} \leq \frac{3\delta}{\sqrt{2}\sqrt{\sqrt{1+\varepsilon^2+2\delta} - (1+\delta)}}.$$

We readily see that the righthand side can be as close of zero as wanted, so we can find $\delta > 0$ small enough which makes the bound in the preceding inequality not greater than $\varepsilon$. In particular, we obtain the next result.

**Proposition 1.1.** *For any $\varepsilon > 0$, there exists $\delta_0 \in (0, \varepsilon)$ which makes, for any $\delta \in (0, \delta_0]$, the feedback $U^\delta$ given by* (6) *nearly time optimal on $\mathcal{K}_\varepsilon = \{x \in \mathbb{R}^2 \mid 2x_1 + sign(x_2)x_2^2 \geq 0\}$, that is,*

$$\mathbf{T}^\Theta(x) \leq \tau_\varepsilon(x) \leq \mathbf{T}^\Theta(x) + \varepsilon, \quad \forall x \in \mathcal{K}_\varepsilon \setminus \mathbb{B}(0, \varepsilon).$$

1.2.1. *A numerical test.* From the analysis recently exposed we can see that, given $\varepsilon > 0$ if we pick $\delta_0 > 0$ to be a zero of the function $\chi_\varepsilon : [0, \varepsilon] \to [0, +\infty)$ defined via

$$\chi_\varepsilon(\delta) = 9\delta^2 - 2\varepsilon^2\left(\sqrt{1+\varepsilon^2+2\delta} - (1+\delta)\right), \quad \forall \delta \in [0, \varepsilon],$$

then the feedback $U^\delta$ is nearly time optimal. In Figure 5 we have represented in blue the curve of zeros of the function $\varepsilon \mapsto \chi_\varepsilon$ for $\varepsilon \in [0, 1]$. We empirically observe that this function is of order $o(\varepsilon^2)$; in the same figure, the red curve portrays the function $\delta = \frac{1}{4}\varepsilon^2$.

Using the above-described fashion to choose $\delta_0$ we have tested the feedback for the values $\varepsilon \in \{0.1, 0.05, 0.01\}$ from 100 random initial conditions lying on

$$\{x \in [0, 55] \times [-10, 0] \mid h(x) \geq 0\}.$$

Using the solver ode45 in Matlab we have obtained the following results for $\mathbf{T}^\Theta(x) - \tau_\varepsilon(x)$

| $\mathbf{T}^\Theta(x) - \tau_\varepsilon(x)$ | $\varepsilon$=0.1 $\delta_0$=0.027 | $\varepsilon$=0.05 $\delta_0$=0.0016 | $\varepsilon$=0.01 $\delta_0$=0.0013 |
|---|---|---|---|
| worst case | 0.0960 | 0.0463 | 0.0070 |
| best case | 0.0999 | 0.0495 | 0.0099 |
| average | 0.0978 | 0.0476 | 0.0082 |

The last table provides an empirical support to the procedure we have exposed. Indeed, in any case, we have a much stronger result, that is, $\mathbf{T}^\Theta(x) \geq \tau_\varepsilon(x)$. This fact can be explained by noticing that the optimal time to reach the target, from the circle of radius $\varepsilon$ is of order $\varepsilon$ as well. We also mention that, as it can also

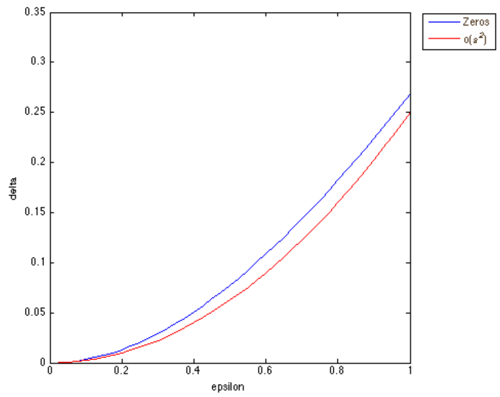


FIGURE 5. The zeros of the $\chi_\varepsilon$.

be inferred from the exposition, the choice of $\delta_0$ is not at all sharp, which makes suitable to continue looking for better bounds related to the closed-loop control.

1.2.2. *Further extensions.* Instead of considering the particular choice of function $\lambda(x) = \left(1 - \frac{1}{2\delta}h(x)\right)$ we might consider any other the continuous functions verifying $\lambda : \Omega_\delta \to [0, 1]$ and

$$\lambda|_{\mathcal{M}_{\text{end}}} \equiv 1 \quad \text{and} \quad \lambda|_{\mathcal{M}_{\text{ini}}}(x) = 0, \quad \text{as long as } h(x) = \delta. \tag{9}$$

Under these circumstances, we are able to prove the existence of a $\delta > 0$ which makes the strategy given by (6) nearly time optimal on compacts sets of $\mathcal{K}_\varepsilon$.

**Proposition 1.2.** *For any $\varepsilon > 0$ and $r > 0$, there exists $\delta_0 \in (0, \varepsilon)$ such that for any continuous functions $\lambda : \Omega_\delta \to [0, 1]$ that verifies (9) with $\delta \in (0, \delta_0)$, the feedback $U^\delta$ given by (6) is nearly time optimal on*

$$\mathcal{K}_\varepsilon^r := \{x \in \mathbb{R}^2 \mid 2x_1 + sign(x_2)x_2^2 \geq 0\} \cap \overline{\mathbb{B}}(0, r).$$

*Proof.* Notice that since $\rho(t) := h(y_x^\varepsilon(t)) > 0$ for any $t \in (0, \tau)$, then $\mathbf{T}^\Theta$ is differentiable along the arc $t \mapsto y(t) := y_x^\varepsilon(t)$ and so, for any $t \in (0, \tau)$

$$\begin{aligned}\frac{d}{dt}\mathbf{T}^\Theta(y(t)) &= -1 + 2\lambda(y_x^\varepsilon(t))\left(1 + \frac{2y_2(t)}{\sqrt{4y_2^2(t) + 2\rho(t)}}\right)\\ &\leq -1 + 2\lambda(y_x^\varepsilon(t))\left(\frac{\rho(t)}{4y_2^2(t) + 2\rho(t)}\right)\\ &\leq -1 + \frac{\delta}{\alpha_\varepsilon(\delta)^2}\end{aligned}$$

Thereby, integrating between $t = 0$ and $t = \tau_\varepsilon(x)$ we get

$$-\mathbf{T}^\Theta(x) \leq \mathbf{T}^\Theta(y(\tau_\varepsilon(x))) - \mathbf{T}^\Theta(x) \leq -\left(1 - \frac{\delta}{\alpha_\varepsilon(\delta)^2}\right)\tau_\varepsilon(x). \tag{10}$$

Since $x \mapsto \mathbf{T}^\Theta(x)$ is continuous on $\mathcal{K}_\varepsilon^r$, the foregoing inequality implies that $\tau_\varepsilon(x)$ is uniformly bounded from above on $\mathcal{K}_\varepsilon^r$. Let $t_r$ be its minimal upper bound and take $\delta_0 \in (0,\varepsilon)$ so that

$$\frac{\delta_0}{\alpha_\varepsilon(\delta_0)^2} = \frac{\delta_0}{2(\sqrt{1+\varepsilon^2+2\delta_0}-(1+\delta_0))} \le \frac{\varepsilon}{t_r}.$$

The choice of $\delta_0$ is possible inasmuch as $\delta \mapsto \frac{\delta}{\alpha_\varepsilon(\delta)^2}$ ranges between 0 and $+\infty$ on $(0,\varepsilon)$. Finally, we get the desired result from (10). □

## 2. Control-affine systems

One of the goal of the previous section was to show an explicit example in which the construction of a continuous nearly time optimal feedback was plausible. Now, we look for similar constructions for a broader class of problems, that is, we focus on the control-affine system (1) under the assumption that $(H_f)$ holds.

Under these circumstances we assume that $\mathcal{M}_{\text{ini}}$ is an open set and $\mathcal{M}_{\text{end}}$ is a smooth surface of codimension 1 (at least of class $\mathcal{C}^2$). Since, the analysis we propose is merely local (on bounded sets), we can always find a local defining map for $\mathcal{M}_{\text{end}}$ whose domain is a neighborhood of $\mathcal{M}_{\text{end}} \setminus \Theta_\varepsilon$; this can be achieved by using a partition of the unity. Accordingly, for sake of simplicity we may rather assume that there is a continuous function $\rho : \mathcal{M}_{\text{end}} \to (0,+\infty)$ that makes $\mathcal{O} \subseteq \mathbb{R}^N$ a tubular neighborhood of $\mathcal{M}_{\text{end}}$; see [23, Theorem 6.24] for further details. Therefore, the map $\pi_{\mathcal{M}_{\text{end}}} : \mathcal{O} \to \mathcal{M}_{\text{end}}$, the projection over $\mathcal{M}_{\text{end}}$ is well defined on $\mathcal{K}$ and locally Lipschitz continuous. In addition, we also suppose that we can find $h : \mathbb{R}^N \to \mathbb{R}$ continuous which is a $\mathcal{C}^k$ submersion on $\mathcal{O}$ so that

$$\mathcal{M}_{\text{end}} = \{x \in \mathcal{O} \mid h(x) = 0\} \text{ and } \mathcal{M}_{\text{ini}} \cap \mathcal{O} \subseteq \{x \in \mathcal{O} \mid h(x) > 0\}. \tag{11}$$

With a slight abuse of notation, let us write $\partial\mathcal{M}_{\text{end}}$ for $\overline{\mathcal{M}_{\text{end}}} \setminus \mathcal{M}_{\text{end}}$, and for any $r > 0$ and $\delta > 0$ we set

$$\Sigma^{r,\sigma} = \{x \in \mathcal{O} \mid |x| \le r,\ \text{dist}_{\partial\mathcal{M}_{\text{end}}}(\pi_{\mathcal{M}_{\text{end}}}(x)) \ge \sigma\}.$$

These subsets of $\mathcal{M}_{\text{end}}$ are introduced in order to localize the area where the feedback is going to be modified. This plays the same role as the ball of radius $r$ used in Section 1.2.2 but well-suited for the case $\mathcal{M}_{\text{end}}$ is bounded.

Let $U_{\text{ini}}$ be the extension of $U_0|_{\mathcal{M}_{\text{ini}}}$ up to $\overline{\mathcal{M}_{\text{ini}}}$ given by $(H_1)$ and consider as well

$$U_{\text{end}}(x) = U_0|_{\mathcal{M}_{\text{end}}}(\pi_{\mathcal{M}_{\text{end}}}(x)), \quad \forall x \in \mathcal{O}.$$

The main result of this paper is described below.

**Theorem 2.1.** *Assume $(H_f)$, $(H_0)$ and $(H_1)$ hold along with*

$$x \in \partial\mathcal{M}_{end} \Rightarrow \textit{Either } x \in \Theta \textit{ or } \exists \mu > 0\ f(x, U_{ini}(x)) = \mu f(x, U_{end}(x)). \tag{12}$$

*Let $\varepsilon > 0$, $r > 0$, then, there exists $\delta_0 > 0$ such that for any $\delta \in (0,\delta_0)$ and any function $\lambda : h^{-1}([0,\delta]) \to [0,1]$ locally Lipschitz continuous that satisfies $\lambda(x) = 0$ if $h(x) = \delta$ and $\lambda(x) = 1$ if $h(x) = 0$, the feedback control $U^\delta : \mathcal{M}_{ini} \cup \mathcal{M}_{end} \to [-1,1]^m$ defined as*

$$U^\delta(x) = \begin{cases} U_{ini}(x) & h(x) \ge \delta \\ U_{ini}(x) + \lambda(x)(U_{end}(x) - U_{ini}(x)) & 0 < h(x) < \delta\,, \\ U_{end}(x) & h(x) = 0 \end{cases} \qquad \forall x \in \mathcal{M}_{ini} \cup \mathcal{M}_{end},$$

*is continuous on an arbitrary large neighborhood of $(\mathcal{M}_{ini} \cup \mathcal{M}_{end}) \cap \mathbb{B}(0,r) \setminus (\partial\mathcal{M}_{end} + \varepsilon\mathbb{B})$ and nearly time optimal on $(\mathcal{M}_{ini} \cup \mathcal{M}_{end}) \cap \mathbb{B}(0,r)$.*

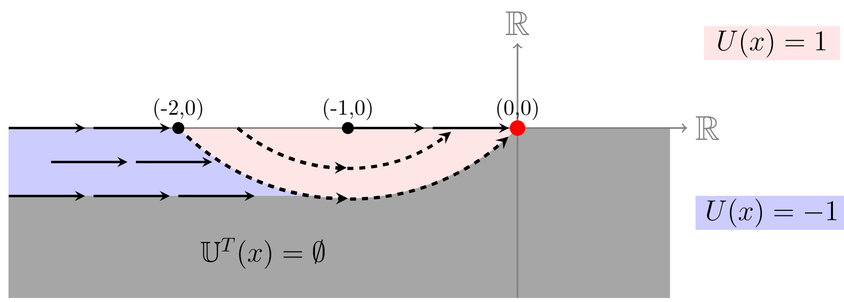


FIGURE 6. The optimal strategy for the problem of Example [29, Example 5.3] with state constraints.

**Remark 2.1.** *Similarly as done for the double integrator, suppose that* $\lambda(x) = 1 - \frac{1}{\delta}h(x)$. *Note then that*

$$|U^\delta(x) - U^\delta(y)| = \lambda(x)|U_{end}(x) - U_{ini}(x)| \leq \frac{2^m}{\delta}|h(x) - h(y)|$$

*for any* $x, y \in \mathcal{M}_{ini} \cup \mathcal{M}_{end}$ *with* $0 \leq h(x) \leq \delta < h(y)$.

*As in Remark 1.3, we get that* $U^\delta$ *is locally Lipschitz continuous, and its Lipschitz constant is inversely proportional to* $\delta$, *and so, blows up as* $\varepsilon \to 0$ *(because in practice* $\delta_0 \to 0$ *as* $\varepsilon \to 0$*). This is an expected result of the trade-off between precision (*$\varepsilon \approx 0$*) and regularization (continuity of the nearly time optimal feedback).*

Before proving Theorem 2.1 let us illustrate it with an example of a minimum time problem with state constraints. Let us consider the minimum time problem exhibited in [29, Example 5.3]: Find the minimum $T \geq 0$ such that

$$\begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \end{pmatrix} = \begin{pmatrix} 1 - y_2 \dfrac{u+1}{2} \\ \\ (y_1 + 1)\dfrac{u+1}{2} \end{pmatrix}, \quad u(t) \in [-1, 1] \quad \text{a.e. on } [0, T] \quad y(0) = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \; y(T) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Consider the state constraint $\mathcal{K} = \{x \in \mathbb{R}^2 \mid \; x_2 \leq 0\}$. Notice that under these circumstances, the IPC is not verified. Furthermore, it is not difficult to see that the optimal feedback for this problem is given by Figure 6.

Let us recall that the construction we propose does not require any type of IPC, and it is based on convex combinations between the feedback laws of one stratum $\mathcal{M}_{\text{ini}}$ and another stratum $\mathcal{M}_{\text{end}} \subseteq \overline{\mathcal{M}_{\text{ini}}} \setminus \mathcal{M}_{\text{ini}}$.

The framework of the present paper allows us to treat the singularity of the feedback at the points on the $x$-axis. The construction for this case might be focused on the strata

$$\mathcal{M}_{\text{ini}} = \mathbb{B}((-1, 1), \sqrt{2}) \cap \text{int}(\mathcal{K}) \quad \text{and} \quad \mathcal{M}_{\text{end}} = \{(x, 0) \mid -1 < x < 0\}.$$

The procedure consists in modifying the feedback around $\mathcal{M}_{\text{end}}$ in such a way it changes in a continuous way. Actually, in this case we can take $h(x, y) = -y$ and for some $\delta > 0$ small $\lambda(x, y) = 1 + \frac{y}{\delta}$. Note as well that $(H_0)$ and (12) are satisfied, and so Theorem 2.1 can be applied. By doing such modification the time required to hit a neighborhood of the target is almost optimal. In Figure 7 we show an

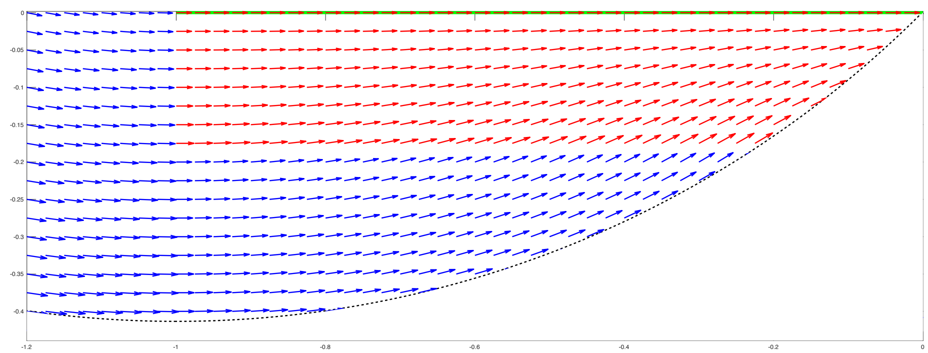

FIGURE 7. A nearly time optimal feedback for the problem of Example [29, Example 5.3] with state constraints. In this picture the parameter is $\delta = 0.2$.

illustration of the modified nearly time optimal feedback for $\delta = 0.2$. The vectors in red correspond to the modified feedback, while the ones in blue are the provided by the optimal synthesis.

2.1. **Technical lemmas.** The proof of the Theorem 2.1 is the outcome of several lemmas which we proceed to state from this point on. We set

$$\Omega_\delta^{r,\sigma} := h^{-1}([0,\delta]) \cap \Sigma^{r,\sigma}, \quad \forall r, \sigma, \delta > 0.$$

**Lemma 2.1.** *Let $r, \sigma, \varepsilon > 0$, if $(H_f)$ and $(H_1)$ are satisfied, then we can find $\delta, \beta > 0$ so that*

$$\langle \nabla h(x), f(x, U_{ini}(x)) \rangle \leq -\beta, \quad \forall x \in \Omega_\delta^{r,\sigma} \setminus \Theta_\varepsilon.$$

*Proof.* If the statement is not true, we can construct a sequence $x_n \in \Omega_\delta^{r,\sigma} \setminus \Theta_\varepsilon$ which converges to some $x \in \mathcal{M}_{\text{end}} \cap \Sigma^{r,\sigma} \setminus \Theta_\varepsilon$ that verifies

$$\langle \nabla h(x), f(x, U_{\text{ini}}(x)) \rangle \geq 0.$$

Let $\tilde{y}$ stand for the arc associated with $U_{\text{ini}}$ which starts from some $\tilde{x} \in \mathcal{M}_{\text{ini}}$ and reaches $x$ at time $\tau(\tilde{x}) = \inf\{t > 0 \mid \tilde{y}(t) \in \mathcal{M}_{\text{end}}\}$; $\tilde{y}$ is the backward curve emerging from $x$ and by virtue of (5), it is well-defined. The Mean Value Theorem implies that $\exists t \in [0, \tau(\tilde{x})]$ for which:

$$0 > -\frac{h(\tilde{x})}{\tau(\tilde{x})} = \langle \nabla h(\tilde{y}(t)), f(\tilde{y}(t), U_{\text{ini}}(\tilde{y}(t))) \rangle.$$

The lefthand side is strictly negative and remains bounded as long as $\tilde{x} \to x$, this is because of the Gronwall's Lemma and the Mean Value Theorem imply

$$\begin{aligned} h(\tilde{x}) \leq & \sup_{s\in[0,1]} |\nabla h(x + s(\tilde{x} - x))| \int_0^{\tau(\tilde{x})} |\dot{\tilde{y}}(s)| ds \\ \leq & (1 + |\tilde{x}|)(e^{c_f \tau(\tilde{x})} - 1) \sup_{s\in[0,1]} |\nabla h(x + s(\tilde{x} - x))|. \end{aligned}$$

Hence, $\limsup_{\tilde{x}\to x} \frac{h(\tilde{x})}{\tau(\tilde{x})} \leq c_f(1+|x|)|\nabla h(x)|$. In view of the initial supposition, the former inequality yields to

$$\langle \nabla h(x), f(x, U_{\text{ini}}(x))\rangle = 0.$$

However, this final equation leads to a contradiction with $(H_1)$. So, the conclusion follows. □

**Lemma 2.2.** *For any $r > 0$ and $\sigma > 0$, there are $\Delta > 0$ and $\varrho > 0$ so that*

$$dist_{\mathcal{M}_{end}}(x) \leq \Delta|h(x)|, \quad \forall x \in \Sigma^{r,\sigma} \cap (\mathcal{M}_{end} + \varrho\mathbb{B}).$$

*Furthermore,* $\Delta \inf\{|\nabla h(x)| \mid \ \Sigma^{r,\sigma} \cap \mathcal{M}_{end}\} \geq 1$.

*Proof.* By virtue of the Grave-Lyusternik Theorem (cf. [15, Theorem 5.32]), for any $x \in \mathcal{M}_{\text{end}}$ there exist $\Delta_x > 0$ and $\varrho_x \in (0, \rho(x))$ so that

$$\text{dist}_{\mathcal{M}_{\text{end}}}(\tilde{x}) \leq \Delta_x|h(\tilde{x})|, \quad \forall \tilde{x} \in \mathbb{B}(x, \varrho_x).$$

Evaluating at $\tilde{x} = x + t\nabla h(x)$ with $t > 0$ we get

$$t|\nabla h(x)| \leq \Delta_x|h(x + t\nabla h(x)) - h(x)|.$$

Whereupon, dividing by $t$ and letting $t \to 0$ we obtain that $|\nabla h(x)|\Delta_x \geq 1$.

Since $\Sigma^{r,\sigma} \cap \mathcal{M}_{\text{end}}$ is compact and can be covered by $\{\mathbb{B}(x, \varrho_x)\}_{x\in\mathcal{M}_{\text{end}}}$, we can take $x_1, \ldots, x_p \in \mathcal{M}_{\text{end}}$ so that $\{\mathbb{B}(x_i, \varrho_{x_i})\}_{i=1}^p$ covers $\Sigma^{r,\sigma} \cap \mathcal{M}_{\text{end}}$. Consequently, setting $\varrho = \min_{i=1\ldots,p} \varrho_{x_i}$ and $\Delta = \max_{i=1\ldots,p} \Delta_{x_i}$ we get the conclusion. □

**Lemma 2.3.** *Suppose that $(H_f)$, $(H_0)$ and $(H_1)$ are verified, and let $r, \sigma > 0$. Then, there exist $C > 0$ and $\varrho > 0$ (the same as in Lemma 2.2) so that*

$$|\langle \nabla \mathbf{T}^\Theta(x), f(x, U_{end}(x)) - f(x, U_{ini}(x))\rangle| \leq C|h(x)|, \ \forall x \in \Sigma^{r,\sigma} \cap (\mathcal{M}_{end} + \varrho\mathbb{B}).$$

*Proof.* Let $\omega$ be given by $(H_0)$. Note that the minimum time function is a classical solution of the HJB equation on $\mathcal{M}_{\text{ini}}$ (because of $(H_0)$), and so

$$-1 + H(x, \nabla\omega(x)) = 0, \quad x \in \mathcal{M}_{\text{ini}}.$$

Due to the optimality of $U_{\text{ini}}$ on $\mathcal{M}_{\text{ini}}$ we have, for any $x \in \mathcal{M}_{\text{ini}} \cap \mathcal{O}$

$$\langle \nabla\omega(x), f(x, U_{\text{ini}}(x))\rangle = -H(x, \nabla\omega(x)) \leq \langle \nabla\omega(x), f(x, U_{\text{end}}(x))\rangle.$$

Thus, by density we find out that

$$\alpha(x) := \langle \nabla\omega(x), f(x, U_{\text{end}}(x)) - f(x, U_{\text{ini}}(x))\rangle \geq 0, \quad \forall x \in \mathcal{M}_{\text{end}}.$$

On the other hand, by (5) we have that

$$\mathbf{T}^\Theta(x) = \mathbf{T}^\Theta_{\overline{\mathcal{M}_{\text{ini}}}}(x), \quad x \in \mathcal{M}_{\text{ini}} \cup \mathcal{M}_{\text{end}},$$

where $\mathbf{T}^\Theta_{\overline{\mathcal{M}_{\text{ini}}}}$ is the minimum time function to reach the target $\Theta$ while being feasible on $\overline{\mathcal{M}_{\text{ini}}}$. Using the standard theory of HJB with state constraints (see for instance [37, 12], [13, Theorem 7.2] or [21, Theorem 4.2]), we can easily see that $\mathbf{T}^\Theta_{\overline{\mathcal{M}_{\text{ini}}}}$ is a supersolution of the equation

$$-1 + H(x, \nabla\varphi(x)) = 0, \quad \forall x \in \overline{\mathcal{M}_{\text{ini}}}.$$

In particular, by the optimality of the feedback $U_{\text{end}}$ and due to $\omega$ is an admissible test function ($\omega \equiv \mathbf{T}^\Theta$ on $\mathcal{M}_{\text{ini}} \cup \mathcal{M}_{\text{end}}$) we have

$$-1 - \langle \nabla\omega(x), f(x, U_{\text{end}}(x))\rangle = -1 + H(x, \nabla\omega(x)) \geq 0, \quad \forall x \in \mathcal{M}_{\text{end}}.$$

By the same argument used earlier, we can show that $\alpha(x) \leq 0$ for any $x \in \mathcal{M}_{\text{end}}$. Hence, we find out that

$$\alpha(x) = 0, \quad \forall x \in \mathcal{M}_{\text{end}}.$$

Therefore, if $L > 0$ indicates the Lipschitz modulus of $\alpha$ on $\Sigma^{r,\sigma}$ we have that

$$|\alpha(x)| \leq L\text{dist}_{\mathcal{M}_{\text{end}}}(x), \quad \forall x \in \Sigma^{r,\sigma}.$$

By Lemma 2.2, the conclusion follows easily. □

2.2. **Proof of Theorem 2.1.** We are now in position to proof the main result of the paper.

*Proof of Theorem 2.1.* For sake of clarity, we split the proof in several steps. Let $\sigma = \frac{\varepsilon}{2}$ and $\tilde{r} \geq r$, consider $\tilde{\delta} > 0$ and $\varrho > 0$ given by Lemma 2.1 and 2.3 associated with $\varepsilon$, $\sigma$ and $\tilde{r}$, respectively. Let $\rho_0 \in (0, \varrho)$ be a lower bound for $\rho(\cdot)$ on $\Sigma^{\tilde{r},\sigma}$ and $\Delta > 0$ given by Lemma 2.2. We set $\delta_0 = \min\{\tilde{\delta}, \frac{\varepsilon}{2\Delta}, \frac{\rho_0}{\Delta}\}$ and take $\delta \in (0, \delta_0)$ fixed but arbitrary.

**Continuity of $U^\delta$:** First of all notice that by construction, the feedback law is locally Lipschitz continuous on $\Omega_\delta^{\tilde{r},\sigma}$ for any $\tilde{r} > 0$. Moreover, due to $\rho_0 \geq \Delta\delta_0$, we have that for any $x \in \mathcal{M}_{\text{end}}$ we can find $\sigma_x \in (0, \rho_0)$ so that $h(x + \sigma_x \nabla h(x)) = \delta$. By the Implicit Function Theorem we can also see that the function $x \mapsto \sigma_x$ is continuously differentiable on $\mathcal{M}_{\text{end}}$. Now, since $U^\delta(x) = U_{\text{ini}}(x)$ whenever $h(x) \geq \delta$ we have that $\sigma \mapsto U^\varepsilon(x + \sigma\nabla h(x))$ is continuous on $[0, \rho_0)$. Therefore, $U^\delta$ is continuous on $\Omega_\delta^{\tilde{r},\sigma} \cup h^{-1}([\delta, +\infty)) \cap \mathcal{M}_{\text{ini}}$. As a matter of fact, since $U^\delta$ is separately locally Lipschitz continuous in $\Omega_\delta^{\tilde{r},\sigma}$ and in $h^{-1}([\delta, +\infty)) \cap \mathcal{M}_{\text{ini}}$, it is necessarily locally Lipschitz continuous on the union of both sets.

Let $x \in \mathcal{M}_{\text{end}}$ with $\text{dist}_{\partial\mathcal{M}_{\text{end}}}(x) < \frac{\varepsilon}{2}$, then for any $s > 0$

$$\text{dist}_{\partial\mathcal{M}_{\text{end}}}(x + s\nabla h(x)) \leq \text{dist}_{\partial\mathcal{M}_{\text{end}}}(x) + \text{dist}_{\mathcal{M}_{\text{end}}}(x + s\nabla h(x)).$$

By Lemma 2.2 and the choice of $\delta_0$, if $h(x + s\nabla h(x)) \leq \delta_0$ then we necessarily have that $\text{dist}_{\partial\mathcal{M}_{\text{end}}}(x + s\nabla h(x)) < \varepsilon$. In particular, since $\sigma = \frac{\varepsilon}{2}$ we obtain

$$(\mathcal{M}_{\text{ini}} \cup \mathcal{M}_{\text{end}}) \cap \mathbb{B}(0, r) \setminus (\partial\mathcal{M}_{\text{end}} + \varepsilon\mathbb{B}) \subseteq \Omega_\delta^{\tilde{r},\sigma} \cup \left[h^{-1}([\delta, +\infty)) \cap \mathcal{M}_{\text{ini}}\right].$$

**Invariance of $\Omega_\delta^{\tilde{r},\sigma}$:** Let $\beta > 0$ given by Lemma 2.1 and let $y$ be the solution associated with the feedback $U^\delta$ given in the statement and whose initial condition is $x \in \text{int}\,\Omega_\delta^{r,\sigma}$. Let $\tau > 0$ be the escape time of $y$ from $\Omega_\delta^{\tilde{r},\sigma}$. Thereby, setting $\rho := h \circ y$ we get for any $t \in (0, \tau)$

$$\dot{\rho}(t) = (1 - \lambda(y))\langle\nabla h(y), f(y, U_{\text{ini}}(y))\rangle + \lambda(y)\langle\nabla h(y), f(y, U_{\text{end}}(y))\rangle.$$

Recall that $\langle\nabla h(x), f(x, U_{\text{end}}(x))\rangle = 0$ on $\mathcal{M}_{\text{end}}$, so by $(H_f)$ and Lemma 2.2 there exists a constant $\tilde{C} > 0$ so that

$$\langle\nabla h(x), f(x, U_{\text{end}}(x))\rangle \leq \tilde{C}|h(x)|, \quad \forall x \in \Sigma^{\tilde{r},\sigma} \cap (\mathcal{M}_{\text{end}} + \varrho\mathbb{B}).$$

Hence, by reducing $\delta_0$ if necessary, we may assume that

$$\langle\nabla h(x), f(x, U_{\text{end}}(x))\rangle \leq \frac{\beta}{2}, \quad \forall x \in \Sigma^{\tilde{r},\sigma},$$

which leads to $\dot{\rho}(t) \leq -\frac{\beta}{2}$ on $(0, \tau)$. Furthermore, since the feedback is locally Lipschitz continuous on $\Omega_\delta^{\tilde{r},\sigma}$, $\rho(t) > 0$ for any $t \in (0, \tau)$; otherwise for some $x \in \mathcal{M}_{\text{end}}$ there are two backward solution, one reaching $\mathcal{M}_{\text{end}}$ and another remaining there. Consequently, by taking $\tilde{r}$ larger, we can assume that $\text{dist}_{\partial\mathcal{M}_{\text{end}}}(\pi_{\mathcal{M}_{\text{end}}}(y(\tau))) = \sigma$.

**Reachability of the target:** We claim that $y(\tau) \in \Theta_\varepsilon$. Indeed, let $z = \pi_{\mathcal{M}_{\text{end}}}(y(\tau))$ and suppose that $\text{dist}_\Theta(z) > \sigma = \frac{\varepsilon}{2}$, otherwise the affirmation does hold because, Lemma 2.2 leads to

$$\text{dist}_\Theta(y(\tau)) \leq \text{dist}_\Theta(z) + |z - y(\tau)| \leq \frac{\varepsilon}{2} + \Delta\delta < \varepsilon.$$

Remark that, since the optimal trajectory that starts from $z$ reach the target without leaving $\mathcal{M}_{\text{end}}$, we have that

$$f(z, U_{\text{end}}(z)) \in \text{int}\left(\mathcal{T}^C_{\Sigma^{\tilde{r},\sigma}}(z)\right).$$

Accordingly, by reducing $\delta_0$ once again if necessary and using the continuity of $f(\cdot, U_{\text{end}}(\cdot))$ on $\mathcal{O}$, we can assume that $f(y(\tau), U_{\text{end}}(y(\tau))) \in \text{int}(\mathcal{T}^C_{\Sigma^{\tilde{r},\sigma}}(y(\tau)))$ as well.

By (12), we can find $\bar{x} \in \partial\mathcal{M}_{\text{end}}$ and $\mu > 0$ so that $|\bar{x} - z| = \sigma$ with $f(\bar{x}, U_{\text{ini}}(\bar{x})) = \mu f(\bar{x}, U_{\text{end}}(\bar{x}))$. In particular, due to the continuity of the vector fields and to the fact that $f(y(\tau), U_{\text{end}}(y(\tau))) \in \text{int}(\mathcal{T}^C_{\Sigma^{\tilde{r},\sigma}}(y(\tau)))$ we can conclude that

$$f(y(\tau), U_{\text{ini}}(y(\tau))) \in \mathcal{T}^C_{\Sigma^{\tilde{r},\sigma}}(y(\tau)).$$

Therefore, by the control-affine structure of the dynamics, the convexity of the Clarke tangent cone and the Accessibility Lemma of Convex Analysis (see for instance [34, Theorem 6.1]) we get $f(y(\tau), U^\delta(y(\tau))) \in \text{int}(\mathcal{T}^C_{\Sigma^{\tilde{r},\sigma}}(y(\tau)))$ which is no possible because, since $\tau$ is an escaping time, we should have $-\dot{y}(\tau) \in \mathcal{T}^B_{\Sigma^{\tilde{r},\sigma}}(y(\tau)) = \mathcal{T}^C_{\Sigma^{\tilde{r},\sigma}}(y(\tau))$. Thus, in particular, $\tau > \tau_\varepsilon(x)$.

Moreover, by a density argument, since the dynamics is locally bounded, the same deduction is valid if the initial condition belongs to $\Omega^{r,\sigma}_\delta$.

**Nearly time optimality of the feedback:** Notice that $\mathbf{T}^\Theta$ is differentiable along the arc $t \mapsto y(t)$ and so, in view of the control-affine structure of the dynamics, for any $t \in (0, \tau)$

$$\begin{aligned}
\frac{d}{dt}\mathbf{T}^\Theta(y) &= \langle \nabla\mathbf{T}^\Theta(y), f(y, U^\delta(y))\rangle \\
&= \langle \nabla\mathbf{T}^\Theta(y), f(y, U_{\text{ini}}(y))\rangle \\
&+ \langle \nabla\mathbf{T}^\Theta(y), f(y, U^\delta(y)) - f(y, U_{\text{ini}}(y))\rangle \\
&= -1 + \lambda(y)\langle \nabla\mathbf{T}^\Theta(y), f(y, U_{\text{end}}(y)) - f(y, U_{\text{ini}}(y))\rangle \\
&\leq -1 + 2C(\varepsilon, \tilde{r})\delta
\end{aligned}$$

The last inequality and $C(\varepsilon, \tilde{r})$ are due to Lemma 2.3. Additionally, by the same argument employed in Proposition 1.2, we can prove that $\tau_\varepsilon(x)$ is finite and bounded from above on any set $\Omega^{r,\sigma}_\delta$. Therefore, reducing $\delta_0$ a last time if require, we might assume that $\tau_\varepsilon(x)C(\varepsilon, \tilde{r})\delta_0 \leq \varepsilon$ so that

$$\tau_\varepsilon(x) \leq \mathbf{T}^\Theta(x) + \varepsilon, \quad \forall x \in \Omega^{r,\sigma}_\delta.$$

Finally, since outside $\Omega^{r,\sigma}_\delta$ the optimal control has not been changed, by (5) any trajectory starting at $x \in \mathcal{M}_{\text{ini}} \cup \mathcal{M}_{\text{end}} \setminus \Omega^{r,\sigma}_\delta$ reaches $\Omega^{r,\sigma}_\delta$ within finite time, $\tilde{\tau}_\varepsilon(x)$. Consequently, if $y_x$ stands for the trajectory associated with the nearly time optimal feedback, we have

$$\mathbf{T}^\Theta(x) = \tilde{\tau}_\varepsilon(x) + \mathbf{T}^\Theta(y_x(\tilde{\tau}_\varepsilon(x))) \geq \tilde{\tau}_\varepsilon(x) + \tau_\varepsilon(y_x(\tilde{\tau}_\varepsilon(x))) - \varepsilon$$

So, since $\tilde{\tau}_\varepsilon(x) + \tau_\varepsilon(y_x(\tilde{\tau}_\varepsilon(x))) \geq \tau_\varepsilon(x)$ the conclusion follows. □

## 3. Discussion and perspectives

We finish the present paper by discussing the contribution of the development exhibited and by indicating some possible extensions regarding the type of singularities that could be treated in future works.

Before going further, let us mention that in the literature there are papers dealing with the construction of *almost everywhere* continuous stabilizing feedbacks, that is, for the case in which there is no criterion to be minimized by the control system; we refer mainly to the works of Rifford [31, 32, 33].

### 3.1. Contributions of the paper.

In this paper we have investigated the relation between optimal feedbacks with a stratified set of discontinuities and nearly time optimal continuous feedback. As reported in the introduction, this connection can be avoided if the optimal process at hand has no state constraints involved in its formulation. However, for problems with restricted state-space, it seems to be a good strategy to proceed as we have done here. This is because, as it has drawn to attention in [20], in many optimal control problems the boundary of the state constraint is relevant and the pointing-like condition are not always satisfied.

Furthermore, we believe that the construction we have proposed is rather simple to be implemented once the optimal synthesis have been known. Also, it yields automatically to full robustness around the area where the modification has taken place, which allows to eschew possible issues coming from inaccuracies in its implementation. For example, if the manifold $\mathcal{M}_{\text{end}}$ belongs to the boundary of the state constraint, the nearly time optimal feedback we have given is such that none of its Carathéodory solutions will hit $\mathcal{M}_{\text{end}}$ but will remain close to it in order to reach finally a neighborhood of the target. Consequently, a discrete scheme with step-size sufficiently small will produce curves that track the nearly time optimal one and that stay in $\mathcal{M}_{\text{ini}}$. In contrast, if the optimal strategy is used directly, once close to the boundary, any discrete scheme will produce iterations that may lie outside the state constraint, forcing the algorithm to project back over $\mathcal{K}$ and therefore producing the undesirable *Zeno effect* that could deteriorate the optimality of the curves associated with the discrete scheme.

In conclusion, the main contribution of this paper is that we have pointed out that around some types of singularities the feedback can be modified in such a way it becomes considerably more regular than it was initially, in particular, robust with respect to internal and external perturbations.

### 3.2. Further extensions.

In this paper we are dealing with minimum time problems, however the technique we propose can be used for any other type of problem provided that a proper notion of *near optimality* is considered. In particular, it can cover problems that present optimal trajectories with *one switch*; bang-singular for instance. Among these problems, a natural class are the ones with the *exact turnpike* property (cf. [17, 26]). Roughly speaking, this property occurs when optimal trajectories approach to an equilibrium point that minimizes the problem at steady state, and remain in a neighborhood of it for a large period of time. The construction we propose could in principle be adapted to this case in order to avoid *chattering* around the turnpike when reconstructing numerically an optimal trajectory. In this case, the role of $\mathcal{M}_{\text{ini}}$ would be played by a neighborhood of the turnpike and $\mathcal{M}_{\text{end}}$ by the turnpike. This as a suitable extension that needs to be investigated in more details.

On the other hand, in the analysis we have exposed, it is important that the singularity of the feedback occurs at a switching manifold. However, it is not difficult to envisage other types of singularities that can be considered. For instance, if instead of reaching the manifold $\mathcal{M}_{\text{end}}$ we are allowed to leave at any point in a transversal way, then a similar analysis can be applied by using the backwards dynamics instead of the forwards.

We finally remark that in Theorem 2.1 the result was stated for an open set and a smooth surface of codimension 1, but a similar result can be stated if the dimension of both manifolds are smaller. Nevertheless, in that case, further hypotheses may be needed in order to make the nearly time optimal trajectories feasible on $\mathcal{K}$. This is because the following condition can not be automatically taken as granted:

$$f(x, U_{\text{ini}}(x) + \lambda(x)(U_{\text{end}}(x) - U_{\text{ini}}(x))) \in \mathcal{T}_{\mathcal{M}_{\text{ini}}}(x), \quad \forall x \in \mathcal{M}_{\text{ini}} \text{ near } \mathcal{M}_{\text{end}}.$$

## References


[1] F. Ancona and A. Bressan. Patchy vector fields and asymptotic stabilization. *ESAIM: Control, Optimisation and Calculus of Variations*, 4:445–471, 1999.

[2] F. Ancona and A. Bressan. Flow stability of patchy vector fields and robust feedback stabilization. *SIAM Journal on Control and Optimization*, 41(5):1455–1476, 2002.

[3] F. Ancona and A. Bressan. Nearly time optimal stabilizing patchy feedbacks. *Annales de l'Institut Henri Poincare (C) Non Linear Analysis*, volume 24, pages 279–310. Elsevier, 2007.

[4] D. Anisi, J. Hamberg and X. Xiaoming, Nearly time optimal paths for a ground vehicle. *Journal of Control Theory and Applications*, volume 1(1):2–8, 2003.

[5] J.-P. Aubin and A. Cellina. *Differential inclusions: Set-valued maps and viability theory*, volume 264 of *Grundlehren der Mathematischen Wissenschaften*. Springer-Verlag New York, Inc., 1984.

[6] L. Berkovitz. Optimal feedback controls. *SIAM Journal on Control and Optimization*, 27(5):991–1006, 1989.

[7] V. Boltyanskii. Sufficient conditions for optimality and the justification of the dynamic programming principle. *SIAM Journal on Control and Optimization*, 4(2):326–361, 1966.

[8] U. Boscain and B. Piccoli. *Optimal syntheses for control systems on 2-D manifolds*, volume 43 of *Mathématiques & Applications*. Springer, 2004.

[9] R. Brockett. Asymptotic stability and feedback stabilization. *Differential Geometric Control Theory*, 181–191. Birkhäuser Boston, 1983.

[10] P. Brunovskỳ. Every normal linear system has a regular time optimal synthesis. *Mathematica Slovaca*, 28(1):81–100, 1978.

[11] P. Brunovskỳ. Existence of regular synthesis for general control problems. *Journal of Differential Equations*, 38(3):317–343, 1980.

[12] I. Capuzzo-Dolcetta and P.-L. Lions. Hamilton-Jacobi equations with state constraints. *Transation of the American Mathematical Society*, 318(2):643–683, 1990.

[13] P. Cardaliaguet, M. Quincampoix and P. Saint-Pierre. Optimal times for constrained nonlinear control problems without local controllability. *Applied Mathematics and Optimization*, 36:21–42, 1997.

[14] F. Clarke. Discontinuous feedback and nonlinear systems. *8th IFAC Symposium on Nonlinear Control Systems*, 2010.

[15] F. Clarke. *Functional analysis, calculus of variations and optimal control*, volume 264 of *Graduate Text in Mathematics*. Springer, 2013.

[16] F. H. Clarke, L. Rifford, and R. Stern. Feedback in state constrained optimal control. *ESAIM: Control, Optimisation and Calculus of Variations*, 7:97–133, 2002.

[17] T. Faulwasser, M. Korda, C. Jones and D. Bonvin. On turnpike and dissipativity properties of continuous-time optimal control problems. *Automatica*, 81: 297–304, 2017

[18] O. Hájek. Terminal manifolds and switching locus. *Mathematical systems theory*, 6(3):289–301, 1972.

[19] C. Hermosilla. Stratified discontinuous differential equations and sufficient conditions for robustness. *Discrete & Continuous Dynamical Systems - A*, 35(9): 4415–4437, 2015.

[20] C. Hermosilla and H. Zidani. Infinite horizon problems on stratifiable state constraints sets. *Journal of Differential Equations*, 258(4): 1430–1460, 2015.

[21] C. Hermosilla, P. Wolenski and H. Zidani. The Mayer and Minimum time problems with stratified state constraints. *Set-Valued and Variational Analysis*, 26 (3): 643–662, 2018.

[22] H. Ishii and S. Koike. On $\varepsilon$-optimal controls for state constraint problems. *Annales de l'Institut Henri Poincare (C) Non Linear Analysis*, 17(4):473–502, 2000.

[23] J. M. Lee. *Introduction to smooth manifolds*, volume 218 of *Graduate Text in Mathematics*. Springer, 2013.

[24] K. Mall and M.J. Grant. Epsilon-trig regularization method for bang-bang optimal control problems. *Journal of Optimization Theory and Applications*, 174(2):500–517, 2017.

[25] A. Marigo and B. Piccoli. Regular syntheses and solutions to discontinuous ODEs. *ESAIM: Control, Optimisation and Calculus of Variations*, 7:291–307, 2002.

[26] A. Marigo and B. Piccoli. Safety controls and applications to the Dubins' car. *Nonlinear Differential Equations and Applications NoDEA*, 11(1):73–94, 2004

[27] L. D. Meeker. Local time optimal feedback control of strictly normal two-input linear systems. *SIAM Journal on Control and Optimization*, 27(1):53–82, 1989.

[28] B. Piccoli. Optimal syntheses for state constrained problems with application to optimization of Cancer therapies. *Mathematical Control and Related Fields*, 2(4):383–398, 2012.

[29] B. Piccoli and H. J. Sussmann. Regular synthesis and sufficiency conditions for optimality. *SIAM Journal on Control and Optimization*, 39(2):359–410, 2000.

[30] F. S. Priuli. State constrained patchy feedback stabilization. *Mathematical Control and Related Fields*, 5(1):141 – 163, 2015.

[31] L. Rifford. Semiconcave control-Lyapunov functions and stabilizing feedbacks. *SIAM Journal on Control and Optimization*, 41(3):659–681, 2002.

[32] L. Rifford. Stratified semiconcave control-Lyapunov functions and the stabilization problem. In *Annales de l'Institut Henri Poincare (C) Non Linear Analysis*, volume 22, pages 343–384. Elsevier, 2005.

[33] L. Rifford. On the existence of local smooth repulsive stabilizing feedbacks in dimension three. *Journal of Differential Equations*, 226(2):429–500, 2006.

[34] R. Rockafellar. *Convex analysis*, volume 28 of *Princeton Mathematical Series*. Princeton University Press, 1970.

[35] J. Rowland and R. Vinter. Construction of optimal feedback controls. *Systems & control letters*, 16(5):357–367, 1991.

[36] C. Silva and E. Trélat. Smooth regularization of bang-bang optimal control problems. *IEEE Transactions on Automatic Control* , 55(11):2488–2499, 2010.

[37] H. Soner. Optimal control with state-space constraint I. *SIAM Journal on Control and Optimization*, 24(3):552–561, 1986.

[38] P. Spinelli and G.S. Rakotonirainy. Minimum Time Problem Synthesis. *Systems & control letters*, 10:281–290, 1988.

[39] H. Sussmann. Regular synthesis for time optimal control of single-input real analytic systems in the plane. *SIAM Journal on Control and Optimization*, 25(5):1145–1162, 1987.



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