Sensitivity Analysis of the Set of Sustainable Thresholds

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Abstract

In the context of constrained control-systems, the Set of Sustainable Thresholds plays in a sense the role of a dual object to the so-called Viability Kernel, because it describes all the thresholds that must be satisfied by the state of the system along a time interval, for a prescribed initial condition. This work aims at analyzing the sensitivity of the Set of Sustainable Thresholds, when it is seen as a set-valued map that depends on the initial position. In this regard, we investigate semicontinuity and Lipschitz continuity properties of this mapping, and we also study several contexts when the Set of Sustainable Thresholds is convex-valued.

Keywords: Set-valued maps, Set of Sustainable Thresholds, Discrete-time systems, Semicontinuity, Lipschitz continuity, Convexity

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1 Introduction

This paper aims at contributing in the understanding of the so called *Set of Sustainable Thresholds*, which roughly speaking corresponds to the set of all possible parameters $c \in \mathbb{R}^m$, called the *thresholds*, for which the evolution of a dynamical system remains viable for a prescribed set of constraints, which in our setting will take the form

$$g(k, x, u) \leq c, \quad \forall k \in [\![0 : T]\!] := \{0, \dots, T\},$$

for a given mapping $g : [\![0 : T]\!] \times \mathbb{R}^d \times \mathbf{U} \longrightarrow \mathbb{R}^m$ and a given nonempty set $\mathbf{U}$. Here and in the sequel, the inequalities are understood in a component-wise sense.

The Set of Sustainable Thresholds plays in a sense the role of a dual object to the so-called *Viability Kernel*. To be more precise, in Viability Theory (see, e.g. [1]) one is concerned with the question of finding all the possible initial positions for which the constraints are satisfied for at least one control; in that setting the thresholds are fixed. The Set of Sustainable Thresholds is an object that focuses on a converse question: assume the initial position is known, find all the possible thresholds for which the system is viable for at least one control. In some contexts, such as in natural resources management, the sustainability question is more appropriate than the viability one, because usually one has an estimate of how many resources are in an ecosystem, so the initial position is already prescribed and cannot be modified at will. However, in the same setting, decision-makers can fix different values for payoffs, scores or quality indicators, and put them in the form of the set of constraints described above. In this case, the decision-makers have more freedom for choosing these parameters, i.e. the thresholds.

The Set of Sustainable Thresholds has been studied in the literature from several points of view. For example, in [2], motivated by an epidemiology model, a method for computing this set for controlled cooperative models was investigated; see also [3–6]. In [7], a detailed study of the Set of Sustainable Thresholds was reported concerning the problem of characterizing their Pareto boundaries, and so forth, the set itself. This study was done considering the initial position as given; see also [8] for an extension to dynamics with uncertainty. We also mention [6] where another approach to compute the Pareto boundaries of the Set of Sustainable Thresholds was presented.

Our task in this paper is to study how the Set of Sustainable Thresholds changes with respect to the initial position. In mathematical terms this means studying the continuity properties of the set-valued map associated with the Set of Sustainable Thresholds. We focus on lower semicontinuity and Lipschitz continuity, and we prove that under mild condition both features can be ensured. Being able to ensure these properties is relevant from a theoretical as well as from a practical point of view. On the one hand, it allows us to infer that sustainable thresholds do not vary abruptly when the initial position changes; in the Lipschitz case we can actually quantify the variation. Moreover, and as it can be observed in the proof of Theorem 5 and Theorem 6, it allows us to establish a rule useful for decision-makers: *a given control produces similar sustainable thresholds for similar initial conditions*. This, in a manner, allows to handle measurement error on the initial position. On the other hand, it opens to path for studying the stability of numerical schemes to compute the Set of Sustainable Thresholds, such as the one reported in [7], and also to provide error estimates.

In this paper we also discuss cases when the Set of Sustainable Thresholds is convex-valued. It is well-known that the viability kernel has convex images in the linear-convex setting. Analogously, we prove that it is also the case for the Set of

Sustainable Thresholds. Furthermore, we prove that that fact also holds if some generalized notions of convexity and monotonicity are considered. The fact that the Set of Sustainable Thresholds is convex-valued could potentially be an important tool for developing new numerical schemes for computing its Pareto fronts. Notice that the results reported in [7] are rather general and do not take advantage of convexity.

This paper is organized as follows: In section 2, we describe the dynamical systems studied in this paper, and present basic properties of the images of the Set of Sustainable Thresholds (closedness and convexity). In section 3, we focus on the sensitivity analysis, and study the continuity properties mentioned above. In section 4 we introduce a new object, called the *Set of Sustainable and Attainable Thresholds*, which can be thought as the core of the Set of Sustainable Thresholds and we study its regularity properties. Finally, in Section 5 we provide some numerical experiments to complement the theoretical analysis and in Section 6 we make a short discussion on possible extensions to continuous-time dynamical systems.

2 Mathematical background and basic properties

The focus of this paper will be on discrete-time dynamical systems. Feasible controls in our setting are all the functions $\mathbf{u} = [\![0:T]\!] \longrightarrow \mathbf{U}$, which can be seen as *sequences of inputs* $\mathbf{u}(0), \ldots, \mathbf{u}(T) \in \mathbf{U}$. Here and in what follows, the time horizon $T > 0$ is a given nonnegative fixed integer, $\mathbf{U}$ is a given nonempty set, $[\![p:q]\!] := \{p, p+1, \ldots, q\}$, stands for the collection of all integers between p and q (inclusive), assuming always that $p < q$. The collection of all feasible controls will be denoted by $\mathcal{U}$.

For a given dynamics $F : [\![0:T]\!] \times \mathbb{R}^d \times \mathbf{U} \longrightarrow \mathbb{R}^d$, a given initial condition $\xi \in \mathbb{R}^d$ and a given control $\mathbf{u} \in \mathcal{U}$, we are concerned with functions $x : [\![0:T+1]\!] \longrightarrow \mathbb{R}^d$ solution of the dynamical system

$$x(k+1) = F(k, x(k), \mathbf{u}(k)), \qquad \forall k \in [\![0:T]\!], \quad x(0) = \xi. \tag{$D_\xi^{\mathbf{u}}$}$$

A solution of this system is uniquely determined by the initial position and the control, and therefore it is denoted by $\mathbf{x}_\xi^{\mathbf{u}}$.

Mathematically speaking, the Set of Sustainable Thresholds (SST for short), denoted by $\mathbb{S}(\xi)$ in the sequel, is defined in the following way:

$$\mathbb{S}(\xi) := \left\{ c \in \mathbb{R}^m \mid \exists \mathbf{u} \in \mathcal{U} \text{ such that } g\left(k, \mathbf{x}_\xi^{\mathbf{u}}(k), \mathbf{u}(k)\right) \leq c, \quad \forall k \in [\![0:T]\!] \right\}$$

where, as pointed out in the introduction, $g : [\![0:T]\!] \times \mathbb{R}^d \times \mathbf{U} \longrightarrow \mathbb{R}^m$ is given.

Our task in this paper is to provide a sensitivity analysis for the SST. In other words, we aim at understanding how the set $\mathbb{S}(\xi)$ changes with respect to the variable ξ. Therefore, the SST will be deemed as a set-valued map $\mathbb{S} : \mathbb{R}^d \rightrightarrows \mathbb{R}^m$ and we will study continuity and regularity properties of this set-valued map. Properties such as convexity, closedness, semi-continuity and Lipschitz continuity will be investigated.

In this work, and in compliance with [7], we assume that the data of the dynamical system with mixed constraints satisfy the following basic conditions, which we term *standing assumptions*:

(H1) $\mathbf{U}$ is a nonempty compact metric space;

(H2) $F(k, \cdot, \cdot)$ is continuous on $\mathbb{R}^d \times \mathbf{U}$ for any $k \in [\![0 : T]\!]$;
(H3) $g(k, \cdot, \cdot)$ is lower semicontinuous on $\mathbb{R}^d \times \mathbf{U}$ for any $k \in [\![0 : T]\!]$.

In the next section, these assumptions may be accordingly strengthened to get stronger regularity conditions.

2.1 Basic properties

In this part we present some basic regularity properties that are satisfied by the graph and the images of the set-valued map $\mathbb{S} : \mathbb{R}^d \rightrightarrows \mathbb{R}^m$. We also show that, under suitable additional assumptions, this set-valued map also satisfies a monotonicity property.

2.1.1 Closedness

The first property we study is closedness.
Proposition 1. *Assume that the standing assumptions are satisfied. Then, the set-valued map $\mathbb{S} : \mathbb{R}^d \rightrightarrows \mathbb{R}^m$ has closed graph with closed and nonempty images.*

Proof. Consider two sequences, $\{\xi_n\}_n \subset \mathbb{R}^d$ and $\{c_n\}_n \subset \mathbb{R}^m$, so that $c_n \in \mathbb{S}(\xi_n)$ for any $n \in \mathbb{N}$, with $\xi_n \to \xi$ and $c_n \to c$.

By definition, there is a sequence $\{\mathbf{u}_n\}_n \subset \mathcal{U}$ such that

$$g\left(k, \mathbf{x}_{\xi_n}^{\mathbf{u}_n}(k), \mathbf{u}_n(k)\right) \leq c_n, \quad \forall k \in [\![0 : T]\!],\ \forall n \in \mathbb{N}.$$

Since $\mathcal{U} \cong \mathbf{U}^{T+1}$, by **(H1)** and Tychonoff's theorem we have that $\mathcal{U}$ is compact, and so, passing into a subsequence (which we do not relabel), we can assume that there is $\mathbf{u} \in \mathcal{U}$ such that $\mathbf{u}_n \to \mathbf{u}$; that is $\mathbf{u}_n(k) \to \mathbf{u}(k)$ for any $k \in [\![0 : T]\!]$. Since we also have that $\mathbf{x}_{\xi_n}^{\mathbf{u}_n}(0) = \xi_n$, it follows that $\mathbf{x}_{\xi_n}^{\mathbf{u}_n}(0) \to \xi$. Thus, by induction on k and by **(H2)** (the continuity of the dynamics), it is not difficult to see that for any $k \in [\![0 : T]\!]$ fixed we have

$$\mathbf{x}_{\xi_n}^{\mathbf{u}_n}(k+1) = F\left(k, \mathbf{x}_{\xi_n}^{\mathbf{u}_n}(k), \mathbf{u}_n(k)\right) \to F\left(k, \mathbf{x}_{\xi}^{\mathbf{u}}(k), \mathbf{u}(k)\right) = \mathbf{x}_{\xi}^{\mathbf{u}}(k+1).$$

Consequently, in the light of **(H3)** and the fact that $c_n \to c$, we get

$$g\left(k, \mathbf{x}_{\xi}^{\mathbf{u}}(k), \mathbf{u}(k)\right) \leq c, \quad \forall k \in [\![0 : T]\!],$$

or in other words, $c \in \mathbb{S}(\xi)$, and so the graph of $\mathbb{S}$ is closed. It is then a straightforward matter to check that $\mathbb{S}(\xi)$ is closed for each $\xi \in \mathbb{R}^d$ fixed.

Finally, notice that for any $\xi \in \mathbb{R}^d$ and any $\mathbf{u} \in \mathcal{U}$, the threshold $c = (c_1 \ldots, c_m)$ given by

$$c_i = \max_{k \in [\![0:T]\!]} \left(g_i\left(k, \mathbf{x}_{\xi}^{\mathbf{u}}(k), \mathbf{u}(k)\right)\right),$$

belongs to $\mathbb{S}(\xi)$. This means that for any $\xi \in \mathbb{R}^d$, we have that $\mathbb{S}(\xi) \neq \emptyset$. $\square$

2.1.2 Convexity

Similarly as for the viability kernel, when the system is linear-convex, it can be proven that the set-valued mapping $\mathbb{S}$ verifies some convexity properties, as described below.

Proposition 2. *Assume that the standing assumptions are strengthened as follows:*

($\tilde{H}$1) $\mathbf{U} \subset \mathbb{R}^n$ *is a nonempty convex subset;*

($\tilde{H}$2) *for any* $k \in [\![0 : T]\!]$ *fixed,* $F(k, x, u) = A(k)x + B(k)u$, *where* $A(k)$ *and* $B(k)$ *are matrices of dimension* $d \times d$ *and* $d \times n$, *respectively;*

($\tilde{H}$3) *for any* $k \in [\![0 : T]\!]$ *and* $i \in [\![1 : m]\!]$ *fixed,* $g_i(k, \cdot)$ *is convex on* $\mathbb{R}^d \times \mathbf{U}$ *.*

Then, the set-valued map $\mathbb{S} : \mathbb{R}^d \rightrightarrows \mathbb{R}^m$ *has convex graph with convex images.*

Proof. Let $\xi_1, \xi_2 \in \mathbb{R}^d$, and $c_1, c_2 \in \mathbb{R}^m$, such that $c_1 \in \mathbb{S}(\xi_1)$ and $c_2 \in \mathbb{S}(\xi_2)$. Let $\lambda \in [0, 1]$. Let $\mathbf{u}_1, \mathbf{u}_2 \in \mathcal{U}$, be such that

$$g\left(k, \mathbf{x}_{\xi_i}^{\mathbf{u}_i}(k), \mathbf{u}_i(k)\right) \leq c_i, \quad \forall k \in [\![0 : T]\!],\ \forall i = 1, 2.$$

By convexity of the constraints mapping we have

$$g\left(k, \lambda\mathbf{x}_{\xi_1}^{\mathbf{u}_1}(k) + (1-\lambda)\mathbf{x}_{\xi_2}^{\mathbf{u}_2}(k), \lambda\mathbf{u}_1(k) + (1-\lambda)\mathbf{u}_2(k)\right) \leq \lambda c_1 + (1-\lambda)c_2, \quad \forall k \in [\![0 : T]\!].$$

By convexity of $\mathbf{U}$, we have that $\lambda\mathbf{u}_1(k) + (1-\lambda)\mathbf{u}_2(k) \in \mathbf{U}$ for any $k \in [\![0 : T]\!]$, and since the dynamics is linear, we also have that

$$\mathbf{x}_{\lambda\xi_1+(1-\lambda)\xi_2}^{\lambda\mathbf{u}_1+(1-\lambda)\mathbf{u}_2}(k) = \lambda\mathbf{x}_{\xi_1}^{\mathbf{u}_1}(k) + (1-\lambda)\mathbf{x}_{\xi_2}^{\mathbf{u}_2}(k), \qquad \forall k \in [\![0 : T+1]\!].$$

From here we conclude that

$$\lambda c_1 + (1-\lambda)c_2 \in \mathbb{S}\left(\lambda\xi_1 + (1-\lambda)\xi_2\right).$$

Therefore, the graph of $\mathbb{S}$ is convex. In particular, this implies that the images of $\mathbb{S}$ are convex as well. □

In Proposition 2 it is fundamental that the control space $\mathbf{U}$ is convex as the following example demonstrates.

Example 1. *Take* $F(k, x, u) = u$, $g_1(k, x, u) = -u$ *and* $g_2(k, x, u) = x$. *Notice that* ***($\tilde{H}$2)*** *and* ***($\tilde{H}$3)*** *hold. Take* $\mathbf{U} = \{-1, 1\}$, *which is not convex, the final horizon* $T > 2$ *and the initial position* $\xi > -1$. *There are three types of admissible controls that we need to analyze:*

- $\mathbf{u}(k) = -1$ *for any* $k \in [\![0 : T]\!]$*: Here we have that the minimal value that* c_1 *can take is* $c_1^* = 1$, *and since* $\mathbf{x}_\xi^{\mathbf{u}}(k+1) = -1$ *for any* $k \in [\![0 : T]\!]$ *the minimal value that* c_2 *can take is* $c_2^* = \xi$, *because* $\mathbf{x}_\xi^{\mathbf{u}}(0) = \xi > -1$. *Therefore,* $(1, \xi) \in \mathbb{S}(\xi)$.
- $\mathbf{u}(k) = 1$ *for any* $k \in [\![0 : T]\!]$*: Here we have that the minimal value that* c_1 *can take is* $c_1^* = -1$, *and since* $\mathbf{x}_\xi^{\mathbf{u}}(k+1) = 1$ *for any* $k \in [\![0 : T]\!]$ *the minimal value that* c_2 *can take is* $c_2^* = 1$. *Thus,* $(-1, 1) \in \mathbb{S}(\xi)$.

- $\mathbf{u}$ *is not constant: : Here we have that the minimal value that* c_1 *can take is* $c_1^* = 1$*, because* $\mathbf{u}(k) = -1$ *for some* $k \in [\![0 : T]\!]$*. Notice that if* $\mathbf{u}(j) = 1$ *for some* $j \in [\![0 : T-1]\!]$*, the minimal value that* c_2 *can take is* $c_2^* = 1$*, otherwise it is* $c_2^* = \xi$*.*

Notice that $(-1, 1), (1, 0) \in \mathbb{S}(0)$*, but* $(0, \frac{1}{2}) \notin \mathbb{S}(0)$*. If it was the case, then the associated control must satisfy* $\mathbf{u}(k) \geq 0$ *for any* $k \in [\![0 : T]\!]$*, and so* $\mathbf{u}$ *will be the control constantly equal to* 1*. But, we have seen that in this case the minimal value that* c_2 *can take is* $c_2^* = 1$*. It follows then that* $\mathbb{S}(0)$ *is not a convex set.*

As a matter of fact, one can check that for $\xi > -1$ *we have*

$$\mathbb{S}(\xi) = \left\{(c_1, c_2) \in \mathbb{R}^2 \mid c_1 \geq -1,\ c_2 \geq 1\right\} \bigcup \left\{(c_1, c_2) \in \mathbb{R}^2 \mid c_1 \geq 1,\ c_2 \geq \xi\right\}.$$

Therefore, $\mathbb{S}(\xi)$ *is not a convex set for any* $\xi > -1$*.*

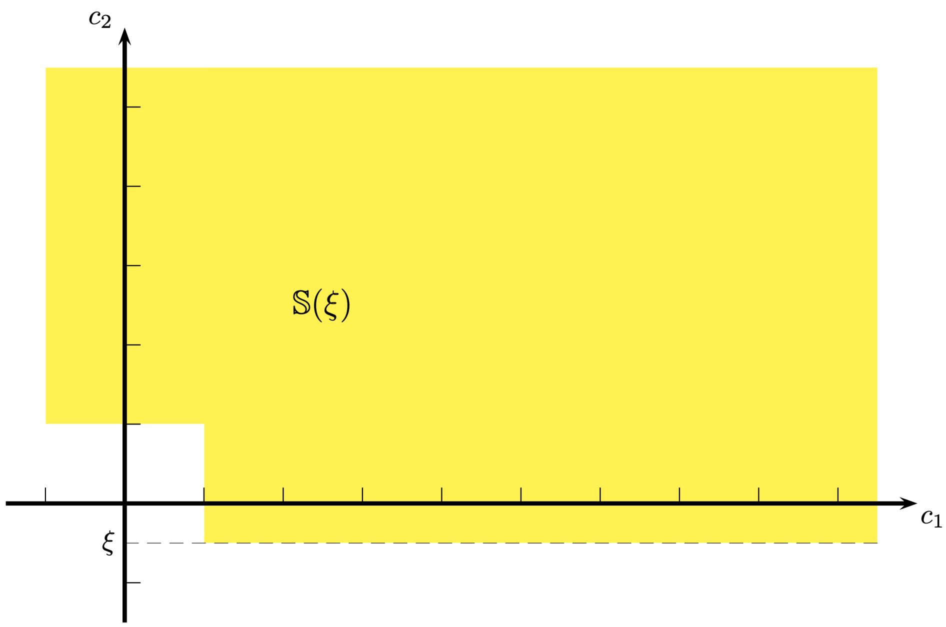


Fig. 1: Sketch of the SST of Example 1.

Example 2. *The assumption on convexity over the constraints mapping* ***(H̃3)*** *may seem rather strong at first sight. However, it appears somewhat naturally in renewable resource management models. Consider for instance a discrete-time version of an example taken from [9], where each* x_k *represents the stock of a renewable resource at a time period* k *and the control* $\mathbf{u}(k)$ *is the catch. In this model a regulatory agency has the social objective of ensuring a minimal stock and a minimal catch. In mathematical terms this means that for some thresholds* $x^{\lim}$ *and* $h^{\lim}$ *we must have:*

$$-x_k \leq -x^{\lim} \qquad \text{and} \qquad -\mathbf{u}(k) \leq -h^{\lim}, \qquad \forall k \in [\![0 : T]\!].$$

*Notice that this example lies within the setting of **($\tilde{\mathbf{H}}$3)**.*

On the other hand, the assumption over the dynamics **($\tilde{\mathbf{H}}$2)** may indeed rule out several interesting cases of study; it is not very often that dynamics in natural resource management are linear. Fortunately, the convexity of the STT can still be preserved if linearity is replaced with some notions of generalized convexity and monotonicity for vector fields.

We now present a criterion, which covers several cases of interest, that allows to ensure that the set-valued map $\mathbb{S} : \mathbb{R}^d \rightrightarrows \mathbb{R}^m$ has convex graph and images. The underlying idea is the following. From [7], we have that $\mathbb{S}(\xi) = \{c \in \mathbb{R}^m \mid \omega_\xi(c) \le 0\}$, where

$$\omega_\xi(c) = \min_{\mathbf{u}\in\mathcal{U}} \max_{k\in[\![0:T]\!]} \max_{i\in[\![1:m]\!]} \left(g_i\left(k, \mathbf{x}_\xi^{\mathbf{u}}(k), \mathbf{u}(k)\right) - c_i\right), \qquad \forall c \in \mathbb{R}^m.$$

Notice that we also have

$$\mathrm{gr}\,(\mathbb{S}) = \left\{(\xi, c) \in \mathbb{R}^d \times \mathbb{R}^m \mid c \in \mathbb{S}(\xi)\right\} = \left\{(\xi, c) \in \mathbb{R}^d \times \mathbb{R}^m \mid \omega_\xi(c) \le 0\right\} \tag{1}$$

Therefore, the convexity of the SST can be obtained from studying the convexity of the function $(\xi, c) \mapsto \omega_\xi(c)$. To do so in a nonlinear dynamics framework, we need to introduce some definitions.

Definition 1. *For a nonempty set $K \subset \mathbb{R}^d$, let us consider the relation $\preceq_K$ on $\mathbb{R}^d$ defined as follows*

$$x \preceq_K y \qquad \Longleftrightarrow \qquad y - x \in K.$$

- *Given a nonempty convex set $S \subset \mathbb{R}^p$, we say that a vector field $\Psi : S \to \mathbb{R}^d$ is K-convex if*

  $$\Psi(\lambda x + (1-\lambda)y) \preceq_K \lambda\Psi(x) + (1-\lambda)\Psi(y), \qquad \forall x, y \in S,\ \lambda \in [0, 1].$$

  Similarly, we say that it is K-concave if

  $$\lambda\Psi(x) + (1-\lambda)\Psi(y) \preceq_K \Psi(\lambda x + (1-\lambda)y), \qquad \forall x, y \in S,\ \lambda \in [0, 1].$$

- *A mapping $\Phi : \mathbb{R}^d \to \mathbb{R}^d$ is said to be a K-*monotone vector field *if*

  $$\Phi(x) \preceq_K \Phi(y), \qquad \forall x, y \in \mathbb{R}^d,\ \textit{such that } x \preceq_K y.$$

- *A function $\varphi : \mathbb{R}^d \to \mathbb{R}$ is said to be a K-*monotone function *if*

  $$\varphi(x) \le \varphi(y), \qquad \forall x, y \in \mathbb{R}^d,\ \textit{such that } x \preceq_K y.$$

Remark 1. *K-monotonicity of functions is preserved under* inf *and* sup *operations. Indeed, let $\{\varphi_\alpha\}_{\alpha\in\Lambda}$ be a family of K-monotone functions, and define*

$$\varphi_-(x) := \inf_{\alpha\in\Lambda} \varphi_\alpha(x) \quad \textit{and} \quad \varphi^+(x) := \sup_{\alpha\in\Lambda} \varphi_\alpha(x), \qquad \forall x \in \mathbb{R}^d.$$

For any $x, y \in \mathbb{R}^d$ such that $x \preceq_K y$, it follows that

$$\varphi_-(x) \leq \varphi_\alpha(x) \leq \varphi_\alpha(y) \leq \varphi^+(y), \qquad \forall \alpha \in \Lambda.$$

In particular,

$$\varphi_-(x) \leq \varphi_\alpha(y) \quad \text{and} \quad \varphi_{\tilde{\alpha}}(x) \leq \varphi^+(y), \qquad \forall \alpha, \tilde{\alpha} \in \Lambda. \tag{2}$$

Therefore, taking infimum over $\alpha \in \Lambda$ and supremum over $\tilde{\alpha} \in \Lambda$ in (2), we obtain $\varphi_-(x) \leq \varphi_-(y)$ and $\varphi^+(x) \leq \varphi^+(y)$, and so φ_- and φ^+ are K-monotone functions.

The monotonicity character of the dynamics is a rather common property encountered in applications as the following examples demonstrate.

Example 3. *Consider first a particular case of the renewable resource management model mentioned in Example 2, where the dynamical system is then given by*

$$x(k+1) = f_{BH}(x(k)) - \mathbf{u}(k), \qquad \forall k \in [\![0:T]\!],$$

where f is the so-called Beverton-Holt population dynamics, *which is given by*

$$f_{BH}(x) = (1+r)x\left(1 + \frac{r}{\kappa}x\right)^{-1}, \qquad \forall x \in \mathbb{R} \setminus \left\{-\frac{\kappa}{r}\right\}. \tag{3}$$

The parameters r and κ are both positive, the first one being the intrinsic growth and the second one the carrying capacity κ. It is not difficult to see that f_{BH} is strictly increasing and concave on $(-\frac{\kappa}{r}, +\infty)$. In particular, since $f'_{BH}(0) = 1 + r$, the Beverton-Holt population dynamics can be re-defined on $(-\infty, 0)$ in the way described below, so that it is strictly increasing and concave on $\mathbb{R}$:

$$\tilde{f}_{BH}(x) = \begin{cases} f_{BH}(x) & \text{if } x \geq 0 \\ (1+r)x & \text{if } x < 0. \end{cases} \tag{4}$$

Consequently, the dynamics

$$F(k, x, u) = \tilde{f}_{BH}(x) - u, \qquad \forall x, u \in \mathbb{R},$$

satisfies the conditions of Theorem 4 stated below for $K = \mathbb{R}_+$.

Remark 2. *Note that the modification of the Beverton-Holt population dynamics done in Example 3 does not have a true impact on renewable resource management models, because in general one is concerned with the case where the state satisfies $x \geq 0$.*

Example 4. *Let us consider now a single species age-classified model of fishing (see e.g. [10, §2.6]), where the state variable $x = (x_1, \ldots, x_d)$ corresponds to the abundance population, that is, each x_j represents the number of individuals of age between $j-1$ and j. The evolution of the abundance of the age structured model presented in [10, §2.6] can be written as follows:*

$$x(k+1) = F(x(k), \mathbf{u}(k)), \qquad \forall k \in [\![0:T]\!],$$

where

$$F_1(x,u) = \tilde{f}_{BH}\left(\sum_{j=1}^{d} \gamma_j v_j x_j\right) \quad \textit{and} \quad F_{j+1}(x,u) = x_j e^{-(M_j + uF_j)}, \quad \forall x \in \mathbb{R}^d,\ u \in \mathbb{R}.$$

Here for a given age j, $\gamma_j \geq 0$ is the proportion of mature individuals, $v_j >$ is the weight at age j, $M_j \geq 0$ is the mortality rate of individuals of age j and $F_j \geq 0$ is the exploitation pattern at age j. The control u represents the fishing effort. It can be readily checked that the dynamics associated with this model complies with the conditions of Theorem 4 stated below for $K = \mathbb{R}^d_+$.

Remark 3. *If $K = \mathbb{R}^d_+$, then K-convexity, K-concavity and K-monotonicity mean component-wise convexity, concavity and monotonicity in the usual sense on $\mathbb{R}$, respectively. Note as well that in the case $K = \{0\}$, the K-monotonicity is trivial.*

It is well-known that if K is a closed convex cone, $\preceq_K$ is a reflexive and transitive relation, which is also antisymmetric if for instance K is pointed (K contains no lines). However, for our purposes, none of these features are required.

Let us now present the main results of this section.

Theorem 3. *Assume that the standing assumptions are satisfied. Let $K \subset \mathbb{R}^d$ be a given nonempty set and assume in addition that*

(A1) $\mathbf{U} \subset \mathbb{R}^n$ *is a convex subset;*

(A2) *for any $k \in [\![0 : T]\!]$ fixed, $F(k,\cdot)$ is K-convex and $F(k,\cdot,u)$ is a K-monotone vector field for any $u \in \mathbf{U}$ fixed;*

(A3) *for any $k \in [\![0 : T]\!]$ and $i \in [\![1 : m]\!]$ fixed, $g_i(k,\cdot)$ is convex on $\mathbb{R}^d \times \mathbf{U}$ and $g_i(k,\cdot,u)$ is a K-monotone function for any $u \in \mathbf{U}$ fixed.*

Then, $\mathrm{gr}\,(\mathbb{S})$ *is a convex subset of $\mathbb{R}^d \times \mathbb{R}^m$, for each $\xi \in \mathbb{R}^d$ the set $\mathbb{S}(\xi)$ is convex and $\mathbb{S}(\xi) \subset \mathbb{S}(\xi')$ whenever $\xi' \preceq_K \xi$.*

Proof. The conclusion follows from proving, thanks to (1), that $(\xi, c) \mapsto \omega_\xi(c)$ is convex and that $\omega_{\xi'}(c) \leq \omega_\xi(c)$ for any $c \in \mathbb{R}^m$ fixed and $\xi', \xi \in \mathbb{R}^d$ such that $\xi' \preceq_K \xi$.

Let us begin by recalling that the value function ω_ξ can be computed by means of the Dynamic Programming Principle (see, e.g., [7, Proposition 3]). Indeed, define for any $n \in [\![0 : T]\!]$

$$V_n^c(\xi) = \min_{\mathbf{u} \in \mathcal{U}} \left\{ \max_{k=n,\dots,T} \Phi^c(k, x_k, \mathbf{u}(k)) \mid x_{k+1} = F(k, x_k, \mathbf{u}(k)),\ k \in [\![n : T]\!],\ x_n = \xi \right\},$$

where

$$\Phi^c(k,x,u) = \max_{i \in [\![1:m]\!]} g_i(k,x,u) - c_i.$$

Notice that $V_0^c(\xi) = \omega_\xi(c)$ and $V_T^c(\xi) = \min_{u \in \mathbf{U}} \Phi^c(T, \xi, u)$. From [7, Proposition 3] we have

$$V_n^c(\xi) = \min_{u \in \mathbf{U}} \max \left\{ V_{n+1}^c\left(F(n,\xi,u)\right), \Phi^c(n,\xi,u) \right\}. \tag{5}$$

Therefore, to get the conclusion, we use the Principle of Mathematical Induction backward on the time variable n to prove the following claim: given $n \in [\![0:T]\!]$

$$(c,\xi) \mapsto V_n^c(\xi) \text{ is convex and } \xi \mapsto V_n^c(\xi) \text{ is a } K\text{-monotone function.} \tag{6}$$

Before going further, let us state two affirmations that will be helpful in the sequel:

(Af1) for any $k \in [\![0:T]\!]$ fixed, the mapping $(c,x,u) \mapsto \Phi^c(k,x,u)$ is convex;

(Af2) for any $k \in [\![0:T]\!]$, $c \in \mathbb{R}^m$ and $u \in \mathbf{U}$ fixed, $x \mapsto \Phi^c(k,x,u)$ is K-monotone.

Notice that **(Af1)** is a consequence of **(A3)**, because we have that the mapping $(c,x,u) \mapsto \Phi^c(k,x,u)$ is the maximum of a family of the convex functions. Moreover, **(Af2)** is a consequence of Remark 1. Indeed, in this case we need to observe that $x \mapsto \Phi^c(k,x,u)$ is the maximum of a family of K-monotone functions.

Base case: the fact that for $n = T$ the claim (6) holds true is a rather direct consequence of **(Af1)** and **(Af2)**. Indeed, we can deduce first that $(c,\xi) \mapsto V_T^c(\xi)$ is convex, since it is the marginal function of a convex function, and second that $\xi \mapsto V_T^c(\xi)$ is K-monotone, because it is the infimum of a family of K-monotone functions; see Remark 1.

Induction step: given $n \in [\![0:T-1]\!]$, assume that the mapping $(c,\xi) \mapsto V_{n+1}^c(\xi)$ is convex and $\xi \mapsto V_{n+1}^c(\xi)$ is a K-monotone function. We divide this part of the proof into two steps.

1. We first prove that $(c,\xi) \mapsto V_n^c(\xi)$ is convex. Notice that, thanks to (5) and **(Af1)**, we just need to prove

$$(c,\xi,u) \mapsto V_{n+1}^c\left(F(n,\xi,u)\right) \text{ is convex.} \tag{7}$$

   Indeed, this will imply that $(c,\xi,u) \mapsto \left\{V_{n+1}^c\left(F(n,\xi,u)\right), \Phi^c(n,\xi,u)\right\}$ is convex, and so is $(c,\xi) \mapsto V_n^c(\xi)$ since it is the marginal of a convex function. The fact that (7) holds true follows from the fact that $(\xi,u) \mapsto F(n,\xi,u)$ is K-convex and from the induction hypothesis: Observe that for any $c \in \mathbb{R}^m$, $x,y \in \mathbb{R}^d$, $u,v \in \mathbf{U}$ and $\lambda \in [0,1]$ we have that

$$V_{n+1}^c\left(F(n,\lambda x+(1-\lambda)y,\lambda u+(1-\lambda)v)\right) \le V_{n+1}^c\left(\lambda F(n,x,u)+(1-\lambda)F(n,y,v)\right).$$

   Evaluating at $c = \lambda a + (1-\lambda)b$ for $a,b \in \mathbb{R}^m$ and using the convexity of the mapping $(c,\xi) \mapsto V_{n+1}^c(\xi)$, we get (7).

2. Let us prove now that $\xi \mapsto V_n^c(\xi)$ is a K-monotone function. Notice first that $\xi \mapsto V_{n+1}^c\left(F(n,\xi,u)\right)$ is a K-monotone function. Indeed, since $\xi \mapsto F(n,\xi,u)$ is a K-monotone vector field, by the induction hypothesis we get:

$$\xi' \preceq_K \xi \Longrightarrow F(n,\xi',u) \preceq_K F(n,\xi,u) \Longrightarrow V_{n+1}^c\left(F(n,\xi',u)\right) \le V_{n+1}^c\left(F(n,\xi,u)\right).$$

   From Remark 1 and **(Af2)** we get that $\xi \mapsto \left\{V_{n+1}^c\left(F(n,\xi,u)\right), \Phi^c(n,\xi,u)\right\}$ is a K-monotone function. Thus, by (5) and again Remark 1, the latter implies that $\xi \mapsto V_n^c(\xi)$ is a K-monotone function.

Therefore, the conclusion follows from the Induction Principle. □

Remark 4. *Notice that the data in Example 1 satisfy the assumptions **(A2)** and **(A3)** with $K=\{0\}$. Thus the convexity assumption **(A1)** is also essential for the validity of Theorem 3.*

Similar arguments as the ones presented above can be used to prove the following result. Notice that in this theorem, the monotonicity behavior of the mapping $\xi \mapsto \mathbb{S}(\xi)$ changes with respect to Theorem 3.

Theorem 4. *Assume that the standing assumptions are satisfied. Let $K \subset \mathbb{R}^d$ be a given nonempty set and assume that*

(A1) $\mathbf{U} \subset \mathbb{R}^n$ *is a convex subset;*

(A2) *for any $k \in [\![0:T]\!]$ fixed, $F(k,\cdot)$ is K-concave and $F(k,\cdot,u)$ is a K-monotone vector field for any $u \in \mathbf{U}$ fixed;*

(A3) *for any $k \in [\![0:T]\!]$ and $i \in [\![1:m]\!]$ fixed, $g_i(k,\cdot)$ is convex on $\mathbb{R}^d \times \mathbf{U}$ and $-g_i(k,\cdot,u)$ is a K-monotone function for any $u \in \mathbf{U}$ fixed.*

Then, $\mathrm{gr}\,(\mathbb{S})$ is a convex subset of $\mathbb{R}^d \times \mathbb{R}^m$, for each $\xi \in \mathbb{R}^d$ the set $\mathbb{S}(\xi)$ is convex and $\mathbb{S}(\xi') \subset \mathbb{S}(\xi)$ whenever $\xi' \preceq_K \xi$.

Proof. The idea in this case is to prove as well that $c \mapsto \omega_\xi(c)$ is convex, but now that $\omega_\xi(c) \le \omega_{\xi'}(c)$ for any $c \in \mathbb{R}^m$ fixed and $\xi', \xi \in \mathbb{R}^d$ such that $\xi' \preceq_K \xi$.

The proof is essentially the same as for Theorem 3, however noticing that in this case for any given $u \in \mathbf{U}$ and $c \in \mathbb{R}^m$, the function $\xi \mapsto -\Phi^c(k,\xi,u)$ is K-monotone, which implies that the function $\xi \mapsto -V_T^c(\xi)$ is K-monotone for any given $c \in \mathbb{R}^m$.

Therefore, the Induction Hypothesis in this case says that for $n \in [\![0:T-1]\!]$ the mapping $(c,\xi) \mapsto V_{n+1}^c(\xi)$ is convex and $\xi \mapsto -V_{n+1}^c(\xi)$ is a K-monotone function. The conclusion then is obtained arguing as in the proof of Theorem 3.

□

3 Sensitivity analysis of the SST

In this part we focus on the sensitivity analysis of the SST, by studying several notions of continuity for the set-valued map $\mathbb{S} : \mathbb{R}^d \rightrightarrows \mathbb{R}^m$.

3.1 Lower semicontinuity

Recall that a set-valued map $\Psi : \mathbb{R}^d \rightrightarrows \mathbb{R}^m$ is said to be *lower semicontinuous* at $x \in \mathbb{R}^d$ if and only if for every $y \in F(x)$ and for every sequence $\{x_n\}_n$ converging to x, there exists a sequence $\{y_n\}_n$ converging to y such that $y_n \in F(x_n)$, for any $n \in \mathbb{N}$.

Theorem 5. *Assume that **(H2)** holds and suppose that $g(k,\cdot,u)$ is continuous on $\mathbb{R}^d$ for any $u \in \mathbf{U}$ and $k \in [\![0:T]\!]$ fixed. Then, $\mathbb{S} : \mathbb{R}^d \rightrightarrows \mathbb{R}^m$ is lower semicontinuous.*

Proof. Let $c \in \mathbb{S}(\xi)$ and $\mathbf{u} \in \mathcal{U}$ the corresponding control given by the definition of the SST. Take a sequence $\{\xi_n\}_n$ such that $\xi_n \longrightarrow \xi$. Notice first that for every $k \in [\![0:T]\!]$ fixed, we have $\mathbf{x}^{\mathbf{u}}_{\xi_n}(k) \longrightarrow \mathbf{x}^{\mathbf{u}}_{\xi}(k)$; this is a straightforward consequence of **(H2)**. Also, by assumption, since each mapping $g(k,\cdot,\mathbf{u}(k))$ is continuous at $x = \mathbf{x}^{\mathbf{u}}_{\xi}(k)$, we have that

$$\max_{k \in [\![0:T]\!]} g_i\left(k, \mathbf{x}^{\mathbf{u}}_{\xi_n}(k), \mathbf{u}(k)\right) \longrightarrow \max_{k \in [\![0:T]\!]} g_i\left(k, \mathbf{x}^{\mathbf{u}}_{\xi}(k), \mathbf{u}(k)\right), \qquad \forall i \in [\![1:m]\!].$$

Define now

$$c_{n,i} = c_i - \max_{k\in[\![0:T]\!]} g_i\left(k, \mathbf{x}_\xi^{\mathbf{u}}(k), \mathbf{u}(k)\right) + \max_{k\in[\![0:T]\!]} g_i\left(k, \mathbf{x}_{\xi_n}^{\mathbf{u}}(k), \mathbf{u}(k)\right)$$

Clearly, $c_{n,i} \longrightarrow c_i$ for any $i \in [\![1:m]\!]$ and also since $c_i \geq \max_{k\in[\![0:T]\!]} g_i(k, \mathbf{x}_\xi^{\mathbf{u}}(k), \mathbf{u}(k))$, we have that $(c_{n,1}, \ldots, c_{n,m}) \in \mathbb{S}(\xi_n)$, which completes the proof. $\square$

Another concept that may be worth studying is the *continuity* of the mapping $\xi \mapsto \mathbb{S}(\xi)$. Recall that a set valued-map $\Psi : \mathbb{R}^d \rightrightarrows \mathbb{R}^m$ is said to be continuous at x if it is lower semicontinuous at x and upper semicontinuous at x in the sense that for every neighborhood V of $\Psi(x)$, there exists $\eta > 0$ such that:

$$\Psi(x') \subset V, \qquad \forall x' \in \mathbb{R}^d \text{ such that } \|x - x'\| < \eta.$$

Upper semicontinuity for set-valued maps is a notion that does not fit well with non-compact valued maps, as can be inferred from [11, Theorem 1.1.2]. Notice that the SST is by definition unbounded because

$$\mathbb{S}(\xi) + \mathbb{R}_+^m \subset \mathbb{S}(\xi), \qquad \forall \xi \in \mathbb{R}^d.$$

This suggests that upper semicontinuity is not a property commonly satisfied by the SST, even for very simple cases. Indeed, it is not difficult to see that in Example 1, the mapping $\xi \mapsto \mathbb{S}(\xi)$ is not upper semicontinuous at $\xi = 0$.

Therefore, in general the mapping $\xi \mapsto \mathbb{S}(\xi)$ will not be continuous. However, contrarily to the single-valued case, continuity of a set-valued map is not mandatory for that mapping to be Lipschitz continuous. Recall that a set valued-map $\Psi : \mathbb{R}^d \rightrightarrows \mathbb{R}^m$ is said to be *Lipschitz continuous* if there is a constant $\kappa_\Psi \geq 0$ such that

$$\sup_{y\in\Psi(x)} \operatorname{dist}(y, \Psi(\bar{x})) \leq \kappa_\Psi \|x - \bar{x}\|, \qquad \forall x, \bar{x} \in \mathbb{R}^d,$$

where $\operatorname{dist}(z, S) := \inf_{s\in S} \|s - z\|$ is the so-called distance function to a set S.

Remark 5. *A set-valued map can be Lipschitz continuous without being upper semicontinuous. For example, one might consider the set-valued map $\Psi : \mathbb{R}^d \rightrightarrows \mathbb{R}^d$, such that $\Psi(x) = \{y \in \mathbb{R}^d \mid y = f(x) + v, v \in \mathbb{R}_+^d\}$, where the mapping $f : \mathbb{R}^d \to \mathbb{R}^d$ is Lipschitz continuous on $\mathbb{R}^d$.*

3.2 Lipschitz continuity

As can be inferred from the discussion above, Lipschitz continuity of the mapping $\xi \mapsto \mathbb{S}(\xi)$ can be examined despite the fact that this mapping is not continuous in general. Indeed, it is possible to demonstrate that the SST depends in a Lipschitz way on the initial position variable, provided the data of the problem is regular enough.

Theorem 6. *Suppose that there exist κ_F, $\kappa_g \geq 0$ such that $F(k, \cdot, u)$ and $g(k, \cdot, u)$ are Lipschitz continuous on $\mathbb{R}^d$ of modulus κ_F and κ_g, respectively, for any $u \in \mathbf{U}$ and $k \in [\![0:T]\!]$ fixed. Then, $\xi \mapsto \mathbb{S}(\xi)$ is Lipschitz continuous on $\mathbb{R}^d$.*

Proof. Let $\kappa_{\mathbb{S}} := \sqrt{m}\ \kappa_g \max\left\{1, \kappa_F^T\right\}$. We are going to prove that

$$\sup_{\hat{c}\in\mathbb{S}(\xi)} \operatorname{dist}\left(\hat{c}, \mathbb{S}\left(\bar{\xi}\right)\right) \leq \kappa_{\mathbb{S}}\|\xi - \bar{\xi}\|, \quad \forall \bar{\xi}, \xi \in \mathbb{R}^d.$$

Let $\bar{\xi}, \xi \in \mathbb{R}^d$ be given and take any $c \in \mathbb{S}(\xi)$ (fixed but arbitrary). Let $\mathbf{u} \in \mathcal{U}$ be a control such that

$$g\left(k, \mathbf{x}_{\xi}^{\mathbf{u}}(k), \mathbf{u}(k)\right) \leq c, \qquad \forall k \in [\![0:T]\!].$$

Let $\bar{c} \in \mathbb{R}^m$ be given by

$$\bar{c}_i = c_i + \max_{k\in[\![0:T]\!]} \left(g_i\left(k, \mathbf{x}_{\bar{\xi}}^{\mathbf{u}}(k), \mathbf{u}(k)\right) - g_i\left(k, \mathbf{x}_{\xi}^{\mathbf{u}}(k), \mathbf{u}(k)\right)\right)^+, \quad \forall i \in [\![1:m]\!]$$

where a^+ stands for the positive part of $a \in \mathbb{R}$.

It follows by construction that $\bar{c} \in \mathbb{S}\left(\bar{\xi}\right)$ and in particular

$$\operatorname{dist}\left(c, \mathbb{S}\left(\bar{\xi}\right)\right) = \inf\left\{\|c - \hat{c}\| \mid \hat{c} \in \mathbb{S}\left(\bar{\xi}\right)\right\} \leq \|c - \bar{c}\|.$$

Let us prove first that

$$\|c - \bar{c}\| \leq \kappa_{\mathbb{S}}\|\xi - \bar{\xi}\|. \tag{8}$$

Notice that if

$$g_i\left(k, \mathbf{x}_{\bar{\xi}}^{\mathbf{u}}(k), \mathbf{u}(k)\right) \leq g_i\left(k, \mathbf{x}_{\xi}^{\mathbf{u}}(k), \mathbf{u}(k)\right), \qquad \forall k \in [\![0:T]\!],\ i \in [\![1:m]\!],$$

then $c = \bar{c}$ and therefore (8) holds. So, let us assume that there is $\hat{k} \in [\![0:T]\!]$ and $i \in [\![1:m]\!]$ such that

$$g_i\left(\hat{k}, \mathbf{x}_{\bar{\xi}}^{\mathbf{u}}(\hat{k}), \mathbf{u}(\hat{k})\right) > g_i\left(\hat{k}, \mathbf{x}_{\xi}^{\mathbf{u}}(\hat{k}), \mathbf{u}(\hat{k})\right).$$

In particular, for some $k \in [\![0:T]\!]$

$$|\bar{c}_i - c_i| = \bar{c}_i - c_i = g_i\left(k, \mathbf{x}_{\bar{\xi}}^{\mathbf{u}}(k), \mathbf{u}(k)\right) - g_i\left(k, \mathbf{x}_{\xi}^{\mathbf{u}}(k), \mathbf{u}(k)\right).$$

Notice that

$$\|g(0, \xi, \mathbf{u}(0)) - g(0, \bar{\xi}, \mathbf{u}(0))\| \leq \kappa_g\|\xi - \bar{\xi}\|.$$

Fix now $j \in [\![0:T-1]\!]$. Then, we have that

$$\begin{aligned}
\|g(j+1, \mathbf{x}_{\xi}^{\mathbf{u}}(j+1), \mathbf{u}(j+1)) - g(j+1, \mathbf{x}_{\bar{\xi}}^{\mathbf{u}}(j+1), \mathbf{u}(j+1))\| \\
&\leq \kappa_g\|\mathbf{x}_{\xi}^{\mathbf{u}}(j+1) - \mathbf{x}_{\bar{\xi}}^{\mathbf{u}}(j+1)\| \\
&\leq \kappa_g\|F(j, \mathbf{x}_{\xi}^{\mathbf{u}}(j), \mathbf{u}(j)) - F(j, \mathbf{x}_{\bar{\xi}}^{\mathbf{u}}(j), \mathbf{u}(j))\| \\
&\leq \kappa_g\kappa_F\|\mathbf{x}_{\xi}^{\mathbf{u}}(j) - \mathbf{x}_{\bar{\xi}}^{\mathbf{u}}(j)\|.
\end{aligned}$$

It is not difficult then to get by the Induction Principle that for any $j \in [\![0:T]\!]$

$$\|g(j, \mathbf{x}_\xi^{\mathbf{u}}(j), \mathbf{u}(j)) - g(j, \mathbf{x}_{\bar{\xi}}^{\mathbf{u}}(j), \mathbf{u}(j))\| \leq \kappa_g \kappa_F^j \|\xi - \bar{\xi}\|.$$

Consequently,

$$|\bar{c}_i - c_i| \leq \kappa_g \kappa_F^k \|\xi - \bar{\xi}\|.$$

Therefore,

$$\|c - \bar{c}\| \leq \sqrt{m}\ \kappa_g \kappa_F^k \|\xi - \bar{\xi}\|.$$

Thus, since $\kappa_F^k \leq \max\left\{1, \kappa_F^T\right\}$, we get that inequality (8) holds.

Recall now that $\operatorname{dist}\left(c, \mathbb{S}\left(\bar{\xi}\right)\right) \leq \|c - \bar{c}\|$ because $\bar{c} \in \mathbb{S}\left(\bar{\xi}\right)$. Then, from (8) we get

$$\operatorname{dist}\left(c, \mathbb{S}\left(\bar{\xi}\right)\right) \leq \kappa_{\mathbb{S}} \|\xi - \bar{\xi}\|. \tag{9}$$

Finally, observe that c in (9) could be any threshold in $\mathbb{S}(\xi)$ and the righthand side in (9) does not depend on c. From this remark, we can conclude that

$$\operatorname{dist}\left(\hat{c}, \mathbb{S}\left(\bar{\xi}\right)\right) \leq \kappa_{\mathbb{S}} \|\xi - \bar{\xi}\|, \qquad \forall \hat{c} \in \mathbb{S}(\xi). \tag{10}$$

Therefore, taking supremum over $\hat{c} \in \mathbb{S}(\xi)$ in (10), we get the desired conclusion. □

Remark 6. *If in the preceding result we assume that the standing assumptions are satisfied and if we change Lipschitz continuity with* local Lipschitz continuity*, the result holds as well, with $\xi \mapsto \mathbb{S}(\xi)$ being now* locally Lipschitz continuous*. This is due to the fact that the set of admisible trajectories is locally bounded with respect to the initial position; recall that the dynamics is continuous and the control space is compact.*

4 The attainable thresholds

As we have discussed in the previous section, upper semicontinuity of the SST is unlikely to hold, even in very simple cases as in Example 1. This is mainly due to the fact that the SST is by definition unbounded. Now, this unboundedness is not necessarily due to the data of the problem, but to the structure of the inequalities, because we have that if $c \in \mathbb{S}(\xi)$, then $c + v \in \mathbb{S}(\xi)$ for any $v \in \mathbb{R}_+^m$. This hints that some useful information can also be obtained from the thresholds that can be achieved by some control.

Accordingly, let us define the *Set of Sustainable and Attainable Thresholds* (SSAT for short) as all the possible thresholds in the SST that can be realized by some control:

$$\mathbb{S}^A(\xi) := \left\{ c \in \mathbb{S}(\xi) \mid \exists \mathbf{u} \in \mathcal{U},\ \forall i \in [\![1:m]\!] \text{ such that } \max_{k \in [\![0:T]\!]} g_i\left(k, \mathbf{x}_\xi^{\mathbf{u}}(k), \mathbf{u}(k)\right) = c_i \right\}.$$

It is not difficult to see that

$$\mathbb{S}(\xi) = \mathbb{S}^A(\xi) + \mathbb{R}_+^m, \qquad \forall \xi \in \mathbb{R}^d.$$

4.1 Lower semicontinuity and Lipschitz continuity

Similar arguments as the ones used to prove Theorem 5 and Theorem 6 can be developed for proving continuity properties of the SSAT. Here we provide the statement and the sketch of the proofs. Notice that the assumptions of Theorem 5 and Theorem 6 agree with the assumptions in the first and second items of Theorem 7, respectively.

Theorem 7. *The following statements are true:*

1. *Suppose that **(H2)** holds and that $g(k,\cdot,u)$ is continuous on $\mathbb{R}^d$ for any $u \in \mathbf{U}$ and $k \in [\![0:T]\!]$ fixed. Then, $\mathbb{S}^A : \mathbb{R}^d \rightrightarrows \mathbb{R}^m$ is lower semicontinuous.*
2. *Suppose that there exist $\kappa_F,\ \kappa_g \geq 0$ such that $F(k,\cdot,u)$ and $g(k,\cdot,u)$ are Lipschitz continuous on $\mathbb{R}^d$ of modulus κ_F and κ_g, respectively, for any $u \in \mathbf{U}$ fixed and $k \in [\![0:T]\!]$ fixed. Then, $\xi \mapsto \mathbb{S}^A(\xi)$ is Lipschitz continuous on $\mathbb{R}^d$.*

Proof.

1. Take $c \in \mathbb{S}^A(\xi)$. Then, there is $\mathbf{u} \in \mathcal{U}$ such that $c_i = \max_{k\in[\![0:T]\!]} g_i(k, \mathbf{x}^{\mathbf{u}}_{\xi}(k), \mathbf{u}(k))$. As done in Theorem 5, for any sequence $\{\xi_n\}_n$ such that $\xi_n \to \xi$, we have that

$$c_{n,i} := \max_{k\in[\![0:T]\!]} g_i\left(k, \mathbf{x}^{\mathbf{u}}_{\xi_n}(k), \mathbf{u}(k)\right) \longrightarrow \max_{k\in[\![0:T]\!]} g_i\left(k, \mathbf{x}^{\mathbf{u}}_{\xi}(k), \mathbf{u}(k)\right), \quad \forall i \in [\![1:m]\!].$$

   By setting $c_n = (c_{n,1}, \ldots, c_{n,m})$, we get that $c_n \to c$ and by definition $c_n \in \mathbb{S}^A(\xi_n)$. This completes the proof of this item.

2. Take $\bar{\xi}, \xi \in \mathbb{R}^d$ be given and any $c \in \mathbb{S}^A(\xi)$ (fixed but otherwise arbitrary). As in Theorem 6, to conclude, we need to prove that there is $\kappa_{\mathbb{S}^A} > 0$ (independent of c, ξ and $\bar{\xi}$) such that

$$\operatorname{dist}\left(c, \mathbb{S}\left(\bar{\xi}\right)\right) \leq \kappa_{\mathbb{S}^A}\|\xi - \bar{\xi}\|. \tag{11}$$

   Let $\mathbf{u} \in \mathcal{U}$ be such that $c_i = \max_{k\in[\![0:T]\!]} g_i(k, \mathbf{x}^{\mathbf{u}}_{\xi}(k), \mathbf{u}(k))$ and set $\bar{c} \in \mathbb{R}^m$ as

$$\bar{c}_i = \max_{k\in[\![0:T]\!]} g_i\left(k, \mathbf{x}^{\mathbf{u}}_{\bar{\xi}}(k), \mathbf{u}(k)\right).$$

   Notice that

$$|\bar{c}_i - c_i| \leq \max_{k\in[\![0:T]\!]} |g_i\left(k, \mathbf{x}^{\mathbf{u}}_{\bar{\xi}}(k), \mathbf{u}(k)\right) - g_i\left(k, \mathbf{x}^{\mathbf{u}}_{\xi}(k), \mathbf{u}(k)\right)|.$$

   Recall that from the proof of Theorem 6, it follows that for any $j \in [\![0:T]\!]$

$$\|g(j, \mathbf{x}^{\mathbf{u}}_{\xi}(j), \mathbf{u}(j)) - g(j, \mathbf{x}^{\mathbf{u}}_{\bar{\xi}}(j), \mathbf{u}(j))\| \leq \kappa_g \kappa_F^j \|\xi - \bar{\xi}\|.$$

   Consequently, proceeding as done for proving Theorem 6, we get

$$\|c - \bar{c}\| \leq \sqrt{m}\ \kappa_g \kappa_F^k \|\xi - \bar{\xi}\|.$$

   Thus, inequality (11) holds with $\kappa_{\mathbb{S}^A} := \sqrt{m}\ \kappa_g \max\{1, \kappa_F^T\}$.

□

Proposition 1 also holds for the SSAT, however now the continuity of the constraints mapping is required.

Proposition 8. *Assume that the standing assumptions are satisfied. Moreover, suppose that $g(k,\cdot)$ is continuous on $\mathbb{R}^d \times \mathbf{U}$ for any $k \in [\![0:T]\!]$ fixed. Then, $\mathbb{S}^A : \mathbb{R}^d \rightrightarrows \mathbb{R}^m$ has closed graph with closed and nonempty images.*

Proof. Consider two sequences, $\{\xi_n\}_n \subset \mathbb{R}^d$ and $\{c_n\}_n \subset \mathbb{R}^m$, so that $c_n \in \mathbb{S}^A(\xi_n)$ for any $n \in \mathbb{N}$, with $\xi_n \to \xi$ and $c_n \to c$. By definition, we know that there is sequence of controls $\{\mathbf{u}_n\}_n$ such that $\max_{k\in[\![0:T]\!]} g_i(k, \mathbf{x}_{\xi_n}^{\mathbf{u}_n}(k), \mathbf{u}_n(k)) = c_{n,i}$ for any $i \in [\![0:m]\!]$. As in the proof Proposition 1, we may assume by compactness that there is $\mathbf{u} \in \mathcal{U}$ such that $\mathbf{u}_n \to \mathbf{u}$ with $\mathbf{x}_{\xi_n}^{\mathbf{u}_n}(k) \to \mathbf{x}_{\xi}^{\mathbf{u}}(k)$ for any $k \in [\![0:T]\!]$. Since the set $[\![0:T]\!]$ is finite, we can assume that $c_n = g(\hat{k}, \mathbf{x}_{\xi}^{\mathbf{u}_n}(\hat{k}), \mathbf{u}_n(\hat{k}))$ for some $\hat{k} \in [\![0:T]\!]$ (passing into a subsequence if necessary). Now, since $g(\hat{k},\cdot)$ is continuous on $\mathbb{R}^d \times \mathbf{U}$, we get

$$c_n = g(\hat{k}, \mathbf{x}_{\xi}^{\mathbf{u}_n}(\hat{k}), \mathbf{u}_n(\hat{k})) \to g(\hat{k}, \mathbf{x}_{\xi}^{\mathbf{u}}(\hat{k}), \mathbf{u}(\hat{k})).$$

Therefore, $c = g(\hat{k}, \mathbf{x}_{\xi}^{\mathbf{u}}(\hat{k}), \mathbf{u}(\hat{k}))$, and so, $c_i \leq \max_{k\in[\![0:T]\!]} g_i(k, \mathbf{x}_{\xi}^{\mathbf{u}}(k), \mathbf{u}(k))$. On the other hand, since for any $k \in [\![0:T]\!]$ and $i \in [\![0:m]\!]$ fixed we have

$$g_i(k, \mathbf{x}_{\xi_n}^{\mathbf{u}_n}(k), \mathbf{u}_n(k)) \leq c_{n,i},$$

it is not difficult to see that $\max_{k\in[\![0:T]\!]} g_i(k, \mathbf{x}_{\xi}^{\mathbf{u}}(k), \mathbf{u}(k)) \leq c_i$. Therefore, $c \in \mathbb{S}^A(\xi)$. $\square$

Remark 7. *Notice that, in the proof of Proposition 8, the continuity assumption on $g(k,\cdot)$ is fundamental for ensuring that c, the limit of the sequence $\{c_n\}$ belongs the limit of the sequence $\{c_n\}$ is exactly $g(k, \mathbf{x}_{\xi}^{\mathbf{u}}(k), \mathbf{u})$ for some control $\mathbf{u}$. If we assume only lower semicontinuity on $g(k,\cdot)$, as in Proposition 1, we cannot reach the same conclusion, i.e., that that c can be attained by some control.*

4.2 Upper semicontinuity

The question that rises now is what can we say about upper semicontinuity. To answer this, notice that under the standing assumptions, we have that

$$\mathfrak{m}_i(\xi) := \min_{\mathbf{u}\in\mathcal{U}} \min_{k\in[\![0:T]\!]} g_i\left(k, \mathbf{x}_{\xi}^{\mathbf{u}}(k), \mathbf{u}(k)\right) > -\infty, \qquad \forall i \in [\![1:m]\!].$$

Consequently, if $\mathfrak{m}_i(\xi) = (\mathfrak{m}_1(\xi), \ldots, \mathfrak{m}_m(\xi))$ then

$$\mathbb{S}^A(\xi) \subset \mathfrak{m}(\xi) + \mathbb{R}^m_+.$$

In a similar way, if any g_i is bounded from above or lower semicontinuity in **(H3)** is strengthened to continuity, then we would also have that

$$\mathbb{S}^A(\xi) \subset \mathfrak{M}(\xi) + \mathbb{R}^m_-,$$

where $\mathfrak{M}(\xi) = (\mathfrak{M}_1(\xi), \dots, \mathfrak{M}_m(\xi))$ and

$$\mathfrak{M}_i(\xi) := \sup_{\mathbf{u}\in\mathcal{U}} \max_{k\in[\![0:T]\!]} g_i\left(k, \mathbf{x}_\xi^{\mathbf{u}}(k), \mathbf{u}(k)\right) < +\infty, \qquad \forall i \in [\![1:m]\!].$$

This would mean that the SSAT has compact images, which can be useful to gain properties such as upper semicontinuity. This is actually the case as we show next.

Theorem 9. *Assume that the standing assumptions are satisfied. Moreover, suppose that $g(k,\cdot)$ is continuous on $\mathbb{R}^d \times \mathbf{U}$ for any $k \in [\![0:T]\!]$ fixed. Then, $\mathbb{S}^A : \mathbb{R}^d \rightrightarrows \mathbb{R}^m$ is upper semicontinuous.*

Proof. As pointed out above we have that

$$\mathbb{S}^A(\xi) \subset \mathbb{K}(\xi) := \{c \in \mathbb{R}^m \mid \mathfrak{m}_i(\xi) \leq c_i \leq \mathfrak{M}_i(\xi), \forall i \in [\![0:m]\!]\}, \qquad \forall \xi \in \mathbb{R}^d.$$

On the other hand, it also follows that $\xi \mapsto \mathfrak{m}_i(\xi)$ and $\xi \mapsto \mathfrak{M}_i(\xi)$ are lower and upper semicontinuous, respectively. This implies that for any $\bar{\xi} \in \mathbb{R}^d$ and $\delta > 0$ fixed, there is a compact set $\mathbf{K} \subset \mathbb{R}^m$ such that

$$\mathbb{K}(\xi) \subseteq \mathbf{K}, \qquad \forall \xi \in \mathbb{R}^d, \text{ such that } \|\xi - \bar{\xi}\| \leq \delta.$$

Therefore, since the map $\mathbb{S}^A : \mathbb{R}^d \rightrightarrows \mathbb{R}^m$ has closed graph (Proposition 8), in the light of [11, Corollary 1.1.1], we conclude that $\mathbb{S}^A : \mathbb{R}^d \rightrightarrows \mathbb{R}^m$ is upper semicontinuous. $\square$

5 Numerical experiments

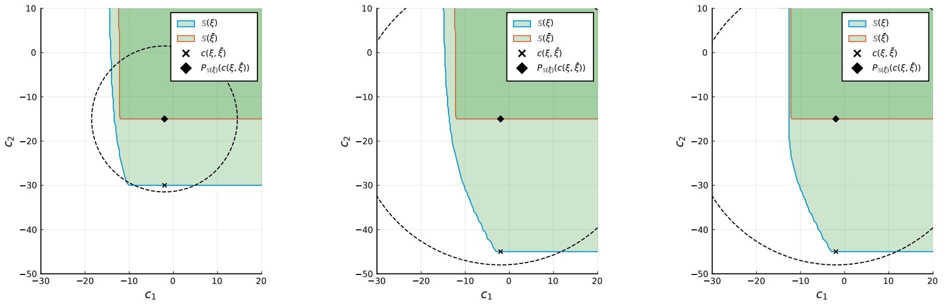
We have discussed in the previous sections various conditions ensuring continuity properties of the SST mapping $\mathbb{S} : \xi \mapsto \mathbb{S}(\xi)$. We now assume another point of view and investigate numerically these properties on two renewable resource management examples.

We first start by considering the setting of Example 3, whose dynamics is given for any time k by

$$F(k, x, u) = \tilde{f}_{\mathrm{BH}}(x) - u \quad \forall x, u \in \mathbb{R},$$

with $\tilde{f}_{\mathrm{BH}}$ defined in Equation (4) being the Beverton-Holt dynamics extended to the whole real line. We consider the same constraints mapping as in Example 2, that is, $g_1(k, x, u) = -u$ and $g_2(k, x, u) = -x$ for any $x, u \in \mathbb{R}$ and $k \in [\![0:m]\!]$. This corresponds to the competing goals of ensuring a sufficient yield while preserving a sufficient stock. We can remark that this example satisfies the hypotheses of Theorem 6, ensuring that $\mathbb{S}$ is a Lipschitz set-valued mapping. We now provide numerical simulations showcasing this fact. We have considered the parameters $K = 50$ and $r = 1.75$, and controls in $\mathbf{U} = [0, 25]$. These numerical simulations were obtained using the algorithm from [7] using an 131-points discretization for the set of attainable states and a 126-points discretization of the control space $\mathbf{U}$.

Figure 2 shows that in this example, the SST appears to be Lipschitz continuous, as predicted by Theorem 6. Indeed, we have fixed $\bar{\xi}$ and considered several values $\xi \geq \bar{\xi}$ in the experiments and highlighted a constant $\kappa_{\mathbb{S}}$ such that the distance of any



(a) $\bar{\xi} = 15$, $\xi = 30$, $T = 10$. (b) $\bar{\xi} = 15$, $\xi = 45$, $T = 10$. (c) $\bar{\xi} = 15$, $\xi = 45$, $T = 50$.

Fig. 2: Plots of the two SST $\mathbb{S}\left(\bar{\xi}\right)$ and $\mathbb{S}(\xi)$, with the points $c(\xi, \bar{\xi}) \in \mathbb{S}(\xi)$ (denoted with a cross) achieving the largest distance to $\mathbb{S}\left(\bar{\xi}\right)$, as well as its projection on $\mathbb{S}\left(\bar{\xi}\right)$ (denoted by a diamond). The plots also showcase the circle centered at $P_{\mathbb{S}\left(\bar{\xi}\right)}(c(\xi, \bar{\xi}))$ with radius $1.1|\bar{\xi} - \xi|$.

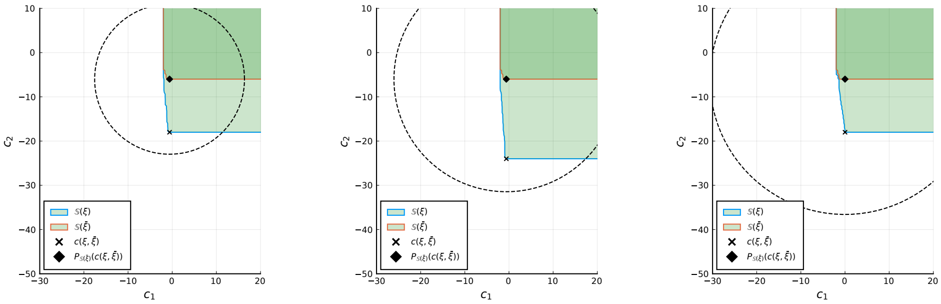
point in $\mathbb{S}(\xi)$ from $\mathbb{S}\left(\bar{\xi}\right)$ can be bounded by $\kappa_{\mathbb{S}}|\xi - \bar{\xi}|$. In the case $\xi \leq \bar{\xi}$, we have that $\mathbb{S}(\xi) \subset \mathbb{S}\left(\bar{\xi}\right)$ from Theorem 4. Note that we have used in Figure 2 the constant $\kappa_{\mathbb{S}} = 1.1$, for both $T = 10$ and $T = 50$. In comparison, Theorem 6 gives an upper bound on $\kappa_{\mathbb{S}}$ that is equal to $(1+r)^T$. Our computations seem to indicate that in this case, it is possible to have a Lipschitz constant that is uniform in T, leading to much lower Lipschitz constant, and thus better robustness of the SST in face of errors on ξ.

We now turn to simulations for the dynamics with age classes described in Example 4. We consider here only two age classes, meaning that $d = 2$. Therefore, the dynamics reads as

$$F_1(k, x, u) = \tilde{f}_{\mathrm{BH}}\left(\sum_{j=1}^{2} \gamma_j v_j x_j\right), \; F_2(k, x, u) = x_1 e^{-M+uF}, \quad \forall x \in \mathbb{R}^2, u \in \mathbb{R}.$$

We consider the constraints mapping $g_1(k, x, u) = -u$ and $g_2(k, x, u) = -(x_1 + x_2)$. Again, this example satisfies the conditions of Theorem 6, showing that $\mathbb{S}$ is Lipschitz continuous. In order to conduct the simulations, we have used the algorithm from [7], with a 2601-points discretization of the space of attainable states and a 201-point discretization of $\mathbf{U}$. We have considered the parameters $\gamma = (0.2, 0.8)$, $v = (0.3, 0.1)$, $M = 0.2$, and $F = 0.6$, controls in $\mathbf{U} = [0, 2]$, and the time horizon $T = 5$.

Figure 3 illustrates the Lipschitz continuity of the SST in this example. Again, we can observe that the maximal distance between a point on $\mathbb{S}(\xi)$ and $\mathbb{S}\left(\bar{\xi}\right)$ is bounded by $\kappa_{\mathbb{S}}\|\xi - \bar{\xi}\|$ for some constant $\kappa_{\mathbb{S}}$ (in Figure 3, we used $\kappa_{\mathbb{S}} = 2$). Contrary to the first example depicted in Figure 2, the state space is now only partially ordered. This can lead to outcomes that are hard to predict. For instance, $\xi = (9, 9)$ and $\xi' = (18, 0)$ cannot be ordered, have the same total population, and their SST are very similar, while the values $\|\bar{\xi} - \xi\|$ and $\|\bar{\xi} - \xi'\|$ are very different for $\bar{\xi} = (3, 3)$. In the case where one needs to evaluate the SST $\mathbb{S}(\xi)$ subject to measurement errors in ξ, this seems to



(a) $\bar{\xi} = (3, 3)$, $\xi = (9, 9)$. (b) $\bar{\xi} = (3, 3)$, $\xi = (12, 12)$. (c) $\bar{\xi} = (3, 3)$, $\xi = (18, 0)$.

Fig. 3: Plots of the two SST $\mathbb{S}\left(\bar{\xi}\right)$ and $\mathbb{S}(\xi)$, with the points $c(\xi, \bar{\xi}) \in \mathbb{S}(\xi)$ (denoted with a cross) achieving the largest distance to $\mathbb{S}\left(\bar{\xi}\right)$, as well as its projection on $\mathbb{S}\left(\bar{\xi}\right)$ (denoted by a diamond). The plots also showcase the circle centered at $P_{\mathbb{S}\left(\bar{\xi}\right)}(c(\xi, \bar{\xi}))$ with radius $2\|\bar{\xi} - \xi\|$.

indicate that some types of errors will have a larger impact on the resulting SST than others, depending on the direction of the perturbation.

Overall, we have illustrated the Lipschitz continuity properties of the SST through two renewable resource management examples, depicted in Figure 2 and Figure 3. In the context of population dynamics, this property is crucial as it allows to bound the perturbation of SST by the norm of the perturbation on the initial condition, which is difficult to measure. Theorem 6 gives conditions on the dynamics and the constraints mapping to ensure this property, and gives a possible Lipschitz constant. Our examples allow us to reveal intriguing phenomenon that are not captured by Theorem 6. First, we see in the example of Figure 2 that the actual Lipschitz constant of the SST can be much lower than the one predicted by Theorem 6. Second, the simulations for our second example, depicted in Figure 3 indicate that the direction of the perturbation also impacts the magnitude of the perturbation of the SST. These two phenomenon tend to make the SST more stable to perturbations than what was predicted by Theorem 6, indicating future research lines for the sensitivity analysis of the SST.

6 Discussion and perspectives

The focus of this paper has been on discrete-time dynamical systems. This type of model allows to simplify the mathematical content, while covering a wide range of application in natural resources management, as pointed out in [10]. One of the main advantage of this setting is that it allows us to treat mathematical models with mixed constraints without the need of introducing additional structural assumptions on the data of the problem, such as the *bounded slope condition*; see for instance [12].

It is not difficult to conceive that the results we have presented in this paper could be transposed to continuous-time dynamical systems. On the one hand, a continuous-time version of Theorem 6 could be obtained if a Gronwall-type lemma is available for trajectories of the dynamical system. On the other hand, if there are just state constraints, rather than mixed constraints, many of the arguments we have exposed in the preceding sections should be, *mutatis mutandis*, useful for proving the corresponding continuous-time versions. For example, it is well-known that under mild assumptions on the data, we have the compactness of trajectories theorem and the Filippov's selection theorem (see, e.g. [11, 13]). These two properties seem at first sight to be crucial for proving Proposition 1 and Theorem 5 in the context of continuous-time dynamical systems.

Concerning convexity properties, it is worthy to point out that a continuous-time version of Proposition 3 seems to be possible to obtain in the light of the results reported in [14, 15], by using the notion of *cross-nonnegativity*. It is also apparent that one needs to investigate a representation of the SST in terms of a suitable value function, that is, whether the SST can be represented as $\mathbb{S}(\xi) = \{c \in \mathbb{R}^m \mid W_\xi(c) \leq 0\}$, where for instance

$$W_\xi(c) := \min_{\mathbf{u}\in\mathcal{U}} \operatorname*{ess-max}_{t\in[0,T]} \max_{i\in[\![1:m]\!]} \left(g_i\left(t, \mathbf{x}_\xi^{\mathbf{u}}(t), \mathbf{u}(t)\right) - c_i\right), \qquad \forall c \in \mathbb{R}^m,$$

with now $\mathcal{U}$ being the collection of measurable functions $u : [0,T] \to \mathbf{U}$. Notice that in this case, ensuring the existence of an optimal control is not as straightforward as in the discrete-time setting, specially for problems with mixed-constraints. Compactness of Trajectories is a delicate issue in that framework; see for instance [12].

Finally, while we have provided a result ensuring the Lipschitz continuity of the SST, our numerical simulations reveal situations where the SST is more robust to perturbations of the input than we could foresee. These additional sources of robustness seem intricate, but identifying them precisely is an important matter. Indeed, the sensitivity of the SST with respect to error in the input is crucial in applications where the initial conditions is often known with imprecision.

All these issues demand further analysis, and go beyond the scope of this paper. We plan to treat them independently and report them in another manuscript.

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