STRATIFIED DISCONTINUOUS DIFFERENTIAL EQUATIONS AND SUFFICIENT CONDITIONS FOR ROBUSTNESS

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Abstract. This paper is concerned with state-constrained discontinuous ordinary differential equations for which the corresponding vector field has a set of singularities that forms a stratification of the state domain. Existence of solutions and robustness with respect to external perturbations of the right-hand term are investigated. Moreover, notions of regularity for stratifications are discussed.

1. Introduction

One important issue in Control Theory of ODEs is the *feedback synthesis*, that is, for a control system on $\mathbb{R}^N$

$$\left\{\dot{x} = f(x,u), \quad x(t) \in \mathcal{K} \subseteq \mathbb{R}^N, \quad u(t) \in \mathcal{A} \subseteq \mathbb{R}^m \right. \tag{1}$$

construct a function $U : \mathcal{K} \to \mathcal{A}$ in such a way that all the trajectories of the vector field $x \mapsto f(x, U(x))$ belong to a certain class of curves of the control system; typically, (sub)minimizers of a given cost function.

It is a well-accepted fact that *optimal feedback laws* are in general discontinuous functions of the state. For instance, the first order necessary conditions of optimality (Pontryagin maximum principle) show that even for linear systems it is likely to occur. Moreover, for Time-Optimal problems, the Brockett Condition represents a topological obstruction for the existence of such feedbacks, and even for suboptimal strategies it remains true; see for instance the discussion in [9, 23]. Therefore, the classical theory of ODEs can not be applied to a closed-loop systems arising in this way, and so existence of solution in the sense of Carathéodory and robustness with respect to perturbations on the data are more delicate to treat.

It is also known that usually the singular set of the optimal strategies have a regular structure. This means that it can be decomposed into a locally finite family of submanifolds, in such a way that in each of these sets the feedback is smooth. Indeed, many authors have shown this fact for certain classes of controlled dynamics; see for instance [12, 7, 8, 24, 21, 3] among others.

The above mentioned contributions make suitable to assume that optimal synthesis may exhibit a stratified structure where exists a partition of the state-space

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$\mathcal{K}$ into a disjoint family of sets $\{\mathcal{M}_i\}_{i\in\mathcal{I}}$ (each of the $\mathcal{M}_i$ is called *stratum*), such that

$$U(x) = U_i(x), \quad \text{whenever } x \in \mathcal{M}_i$$

and in some of these strata, the ODE

$$\dot{x} = f(x, U_i(x))$$

admits arcs remaining on $\mathcal{M}_i$ for at least a small interval of time. In fact, it is likely to have a subfamily of strata where no trajectory of the system can stay for a set of times with positive measure. In this way, trajectories may *slide* from one stratum to another of bigger dimension.

The aim of this work is to study ODEs arising in this context. For this purpose the concept of *stratified vector fields (SVF)* is introduced. The main issues are the existence of solutions and the robustness of trajectories with respect to external perturbations on the velocity. Particular emphasis is put on the interplay between regularity conditions on the sets $\mathcal{M}_i$ and pointwise conditions on the vector field in order to ensure the existence of solutions. In particular, the case of ODEs on closed sets with empty interior is treated, and to do this the notion of *relatively wedgedness* is required.

Let us mention that the notions of SVF and relatively wedged set are already present in the literature, however their definitions are slightly different from the adopted in this work. In particular, the first one was brought to light in the notes [20] in order to prove properties about Whitney stratifications. The principal and most important difference between both definitions is that here a SVF does not need to be defined for all the strata, and in fact, as described above, it is likely that this situation occurs. On the other hand, the second concept was studied in [2] with the purpose of proving a similar existence theorem of trajectories of a stratified differential inclusion defined everywhere on the space. In that case, the definition of relatively wedged set was made only for submanifolds that are also proximal smooth sets while here the definition is more general.

The main tool for studying the stability of solutions to stratified ODEs is the *modulus outward-pointing*. The main feature of this function is that it measures the maximum size of the perturbation and it describes the class singularities allowed in order to make the system stable in the sense described earlier.

Similar works dealing with discontinuous ODEs can be found in the literature. To the best of our knowledge, the closest one is [19] where the author study the properties of a discontinuous ODE starting from an axiomatic definition of stratified solutions. Other papers focused on 3-dimensional piecewise smooth dynamical systems and switching surface are [17, 25] where a qualitative analysis is made. On the other hand, the literature on *generalized solutions* is quite large and many authors in the last decades have addressed their attention into the problem; see for instance [5, 6, 13, 14, 16, 11] and the references therein.

Further works dealing with stratified systems and Hamilton-Jacobi equation can be found in the literature; see for instance [4, 2, 22, 15].

This paper is organized as follows. Section 2 describes the model of discontinuous ODE at issue and what is understood as a solution to this. Section 3 deals with the problem of existence of solutions. In Section 4 some regularity notions over the stratification are discussed. In particular, the notions of curvature and relatively wedgedness are considered. In the last section the problem of robustness of the stratified solutions is investigated and the already referred outward-pointing modulus of a vector field is introduced. Finally, in the Appendix, some properties of this last function are discussed.

1.1. **Notations.** Throughout this paper, $\mathbb{R}$ denotes the sets of real numbers, $|\cdot|$ the Euclidean norm on $\mathbb{R}^N$, $\mathbb{B}$ the unit open ball $\{x \in \mathbb{R}^N :\ |x| < 1\}$, $\mathbb{B}(x,r) = x + r\mathbb{B}$ and $\mathbb{S}$ is the unit sphere $\{x \in \mathbb{R}^N :\ |x| = 1\}$. For a set $\mathcal{S} \subseteq \mathbb{R}^N$, $\text{int}(\mathcal{S}), \overline{\mathcal{S}}$ and $\text{bdry}(\mathcal{S})$ denote its interior, closure and boundary, respectively. If $\mathcal{S}$ is convex, $\text{ri}(\mathcal{S})$ and $\text{rbd}(\mathcal{S})$ stand for its relative interior and boundary, respectively. The Lebesgue measure of a Borel set $J \subseteq \mathbb{R}$ is indicated by $\text{meas}(J)$.

The distance function to $\mathcal{S}$ is $\text{dist}_{\mathcal{S}}(x) = \inf\{|x-y| :\ y \in \mathcal{S}\}$. If the infimum is attained, it is called the projections of x over $\mathcal{S}$ and it is denoted by $\text{proj}_{\mathcal{S}}(x)$. For two compact sets of $\mathbb{R}^N$, $\mathcal{S}_1$ and $\mathcal{S}_2$, the Hausdorff distance between them is given by

$$d_H(\mathcal{S}_1, \mathcal{S}_2) = \max\left\{ \sup_{x\in\mathcal{S}_2} \text{dist}_{\mathcal{S}_1}(x), \sup_{x\in\mathcal{S}_1} \text{dist}_{\mathcal{S}_2}(x) \right\}$$

and the distance between two cones of $\mathbb{R}^N$, C_1 and C_2, is defined as follows

$$\mathcal{D}(C_1, C_2) = d_H(C_1 \cap \mathbb{S}, C_2 \cap \mathbb{S}).$$

A sequence of cones $\{C_n\}$, is said to converge to C, another cone of $\mathbb{R}^N$, provided $\mathcal{D}(C_n, C) \to 0$ as $n \to +\infty$. It is denoted by $C_n \to C$.

For a family $\{v_1, ..., v_p\}$ of vector in $\mathbb{R}^N$, $\text{span}\{v_1, ..., v_p\}$ and $\text{cone}\{v_1, ..., v_p\}$ stand for the space and the cone generated by the family, respectively. Besides, $\mathcal{SO}(N)$ indicates the set of orthonormal matrices of dimension N whose determinant is equal to one.

Let $\mathcal{M}$ be a submanifold of $\mathbb{R}^N$ and let $x \in \mathcal{M}$. The tangent and normal spaces to $\mathcal{M}$ at x are denoted by $\mathcal{T}_{\mathcal{M}}(x)$ and $\mathcal{N}_{\mathcal{M}}(x)$.

2. Statement of the problem

Before describing the problem, some concepts need to be introduced. In particular, a precise definition of *stratification* is required and also the basic assumption over this need to be settled.

2.1. **Preliminaries.** The elementary objects necessary to define a stratification of a set in $\mathbb{R}^N$ are the embedded manifolds. The definition is recalled for sake of completeness.

Definition 2.1. A subset $\mathcal{M} \subseteq \mathbb{R}^N$ is an embedded manifold of $\mathbb{R}^N$ provided for every $\bar{x} \in \mathcal{M}$ there exists an open neighborhood $\Theta \subseteq \mathbb{R}^N$ of $\bar{x}$ such that

$$\Theta \cap \mathcal{M} = \{x \in \Theta :\ h(x) = 0\}$$

where $h : \Theta \to \mathbb{R}^{N-d}$ is a smooth function whose derivative $Dh(x)$ is surjective on Θ and $d \in \{0, \ldots N\}$. In this case, d is called the dimension of the manifold.

In this paper it is always assumed that smooth means at least of class $\mathcal{C}^2$.

Besides, when dealing with stratification it is usual to consider additional regularity assumptions in the way the strata fix together. The basic assumption, in this sense, is the so called *Whitney-(a) Condition*. The reader is referred to [20] for a further discussion about this condition.

Definition 2.2. Let $\mathcal{M}_i$ and $\mathcal{M}_j$ be two embedded manifolds of $\mathbb{R}^N$. The pair $(\mathcal{M}_i, \mathcal{M}_j)$ satisfies the Whitney-(a) condition at $x \in \mathcal{M}_i \cap \overline{\mathcal{M}_j}$ if for each sequence $\{x_n\} \subseteq \mathcal{M}_j$ with $x_n \to x$ the following holds:

$$\mathcal{T}_{\mathcal{M}_j}(x_n) \to \mathcal{T} \quad \Rightarrow \quad \mathcal{T}_{\mathcal{M}_i}(x) \subseteq \mathcal{T}. \tag{2}$$

Through this paper the following binary relation is used to set up a hierarchy on a stratification whose *strata* are exactly $\{\mathcal{M}_i\}_{i\in\mathcal{I}}$.

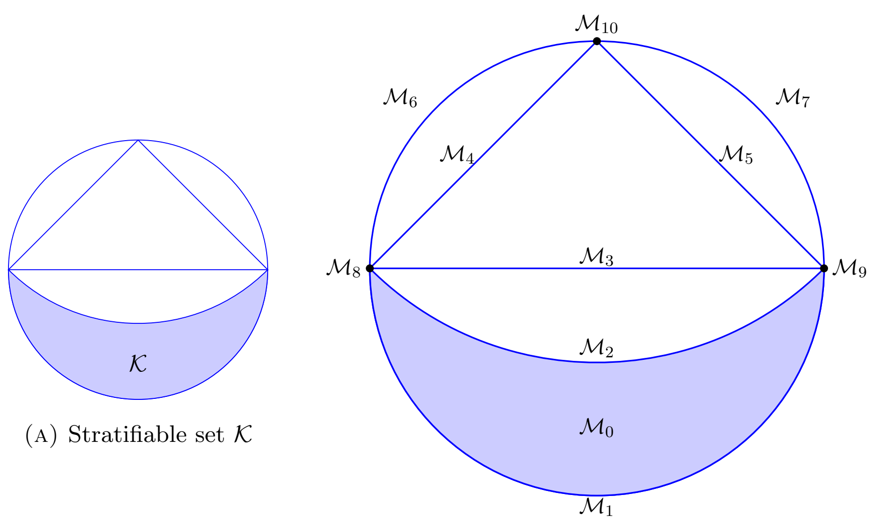
$$\forall i, j \in \mathcal{I}: \qquad i \preceq j \text{ (or } j \succeq i) \quad \Longleftrightarrow \quad \mathcal{M}_i \subseteq \overline{\mathcal{M}}_j.$$

Definition 2.3. A set $\mathcal{K} \subseteq \mathbb{R}^N$ is said to be (Whitney) stratifiable if there exists a locally finite collection $\{\mathcal{M}_i\}_{i\in\mathcal{I}}$ of embedded manifolds of $\mathbb{R}^N$ such that:

- $\mathcal{K} = \bigcup_{i\in\mathcal{I}} \mathcal{M}_i$ and $\mathcal{M}_i \cap \mathcal{M}_j = \emptyset$ when $i \neq j$,
- Whenever $\mathcal{M}_i \cap \overline{\mathcal{M}_j} \neq \emptyset$, necessarily $i \preceq j$.
- For any $i, j \in \mathcal{I}$, if $i \preceq j$ then pair $(\mathcal{M}_i, \mathcal{M}_j)$ satisfies the Whitney-(a) condition at each $x \in \mathcal{M}_i$.

When a set $\mathcal{K}$ admits a stratification in the way described before, $\imath(x)$ will stand for the index $i \in \mathcal{I}$ such that $x \in \mathcal{M}_i$.

The next figure shows an example of a stratifiable set $\mathcal{K}$ in $\mathbb{R}^2$ and a stratification of this set. In Figure 1b, $\mathcal{M}_0$ is a stratum of full dimension ($d = 2$) (which coincides with $\text{int}(\mathcal{K})$), $\mathcal{M}_1, \dots, \mathcal{M}_7$ are the one dimensional strata and finally, $\mathcal{M}_8$, $\mathcal{M}_9$ and $\mathcal{M}_{10}$ are strata of dimension 0, that is, they are isolated points.



(A) Stratifiable set $\mathcal{K}$

(B) Stratification of $\mathcal{K}$

FIGURE 1. Example of a stratifiable set and its stratification.

Remark 1. An important class of sets that admits a stratification as described above are the polytopes of $\mathbb{R}^N$. In fact, these sets can be decomposed into a finite number of open convex polytopes of the form:

$$P = \left\{ x \in \mathbb{R}^N \;\middle|\; \begin{array}{ll} \langle \eta_k, x\rangle = \alpha_k, & k = 1, \dots, n, \\ \langle \eta_k, x\rangle < \alpha_k, & k = n+1, \dots, m \end{array} \right\}$$

where $\eta_1, \dots, \eta_m \in \mathbb{R}^N$.

Furthermore, the class of sets that admits a stratification is quite broad, it includes sub-analytic and semi-algebraic sets and also definable sets of an o-minimal structure; see for instance [26, 18].

2.2. **Stratified vector fields.** As mentioned in the introduction, it is suitable to have some strata where no curve can slide for. Whence, a selection of index where the vector fields are defined need to be involved in the definition, which is outlined in the following definition.

Definition 2.4. Let $\mathcal{K} \subseteq \mathbb{R}^N$ be a stratifiable closed set whose stratification is denoted by $\{\mathcal{M}_i\}_{i\in\mathcal{I}}$. Let $\mathcal{I}_0 \subseteq \mathcal{I}$ be a subset of index such that $\{\mathcal{M}_i\}_{i\in\mathcal{I}_0}$ is dense in $\mathcal{K}$. Then a stratified vector field (SVF) is a family of tangent vector fields $G = \{(g_i, \mathcal{M}_i)\}_{i\in\mathcal{I}_0}$ such that for each $i \in \mathcal{I}_0$

$$g_i(x) \in \mathcal{T}_{\mathcal{M}_i}(x), \quad \forall x \in \mathcal{M}_i. \tag{3}$$

A SVF is said to be regular if for each $i \in \mathcal{I}_0$, g_i is continuous on $\mathcal{M}_i$ and it can be continuously extended up to $\overline{\mathcal{M}}_i$. In addition, a SVF has linear growth if there exists a constant $c > 0$ such that

$$|g_i(x)| \leq c(1 + |x|), \quad \forall i \in \mathcal{I}_0, x \in \mathcal{M}_i. \tag{4}$$

By (3), the subset of index $\mathcal{I}_0$ is in fact the selection of strata where it is allowed to slide for. Therefore, the manifold corresponding to the index $\mathcal{I}_0$ are usually called *sliding strata* and the others, *bifurcation strata*.

Figure 2 shows a SVF defined on $\mathcal{K} = \mathbb{R}^2$. In this case, the stratification of the space consists in $\mathcal{M}_0 = \{(0,0)\}$, $\mathcal{M}_1, \dots, \mathcal{M}_4$ the positive and negative semi-axis and $\mathcal{M}_5, \dots, \mathcal{M}_8$ the quadrants of $\mathbb{R}^2$. Note that in this situation $\mathcal{I} = \{0, \dots, 8\}$ and the vector fields are defined on all the strata except on $\mathcal{M}_1$ and $\mathcal{M}_2$. Therefore, $\mathcal{I}_0 = \{0, 3, 4, 5, 6, 7, 8\}$ and, $\mathcal{M}_1$ and $\mathcal{M}_2$ are the bifurcation strata of this SVF.

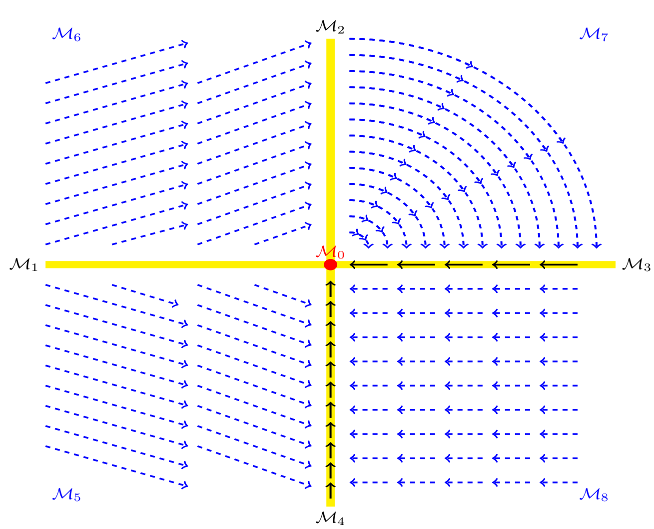


FIGURE 2. Example of stratified vector field on $\mathcal{K} = \mathbb{R}^2$

On the other hand, as G in general forms a discontinuous map with well determined singularities, it is useful to introduce some notation to identify the surrounding strata where the vector field is defined. This is denoted by

$$\mathcal{I}_0(i) := \{j \in \mathcal{I}_0 : \; j \succeq i\} \qquad \forall i \in \mathcal{I}. \tag{5}$$

2.3. **Stratified ordinary differential equations.** Once the notion of SVF is well stablished, it is possible to formalize what is the central object of investigation in this paper, namely, the discontinuous ODE engendered by this mapping.

$$(\mathbf{D}) \qquad \begin{cases} \dot{x} = g_i(x) & \text{a.e. whenever } x \in \mathcal{M}_i \\ x(0) = x_0. \end{cases}$$

For all the purposes, the equation (**D**) will be called the *stratified ODE* associated with the stratification $\{\mathcal{M}_i\}_{i\in\mathcal{I}}$ and with the dynamic $G = \{(g_i, \mathcal{M}_i)\}_{i\in\mathcal{I}_0}$. The attention is centered in Carathéodory solutions to (**D**), that is, absolutely continuous arcs satisfying

$$x(t) = x_0 + \sum_{i\in\mathcal{I}_0} \int_{J_i(t)} g_i(x(s))ds, \quad \forall t \in [0, T), \tag{6}$$

where $J_i(t) = \{s \in [0, t) : \; x(s) \in \mathcal{M}_i\}$. In particular, this implies that each $J_i(T)$ is a negligible set whenever $i \notin \mathcal{I}_0$. In order to emphasis the dependence with respect to the stratified structure of the problem, $x(\cdot)$ is called a *stratified solution* to (**D**).

Remark 2. Broadly speaking, the set of solutions defined in this way is not closed. In fact, since $\{\mathcal{M}_i\}_{i\in\mathcal{I}_0}$ is dense in $\mathcal{K}$, the set $\mathcal{I}_0(j) \neq \emptyset$ for any $j \notin \mathcal{I}_0$ and so, it may be possible to create an absolutely continuous arc that is the limit of trajectories,

$$\dot{x}_n = g_i(x_n), \quad x_n(0) = x_0^n \in \mathcal{M}_i,$$

with $x_0^n \to x_0 \in \mathcal{M}_j$ with $j \preceq i$. It could happen that the limiting trajectory lies on $\mathcal{M}_j$ for a set of times of positive measure. To avoid these situations, some types of singularities have to be ruled out; see Section 5 for more details.

Remark 3. If $x(\cdot)$ is a stratified solution to (**D**) with $G = \{(g_i, \mathcal{M}_i)\}_{i\in\mathcal{I}_0}$ having linear growth, then

$$\dot{x}(t) \leq c(1 + |x(t)|), \quad \text{a.e. } t \in [0, T).$$

and so, the Gronwall estimation holds (see [10, Proposition 4.1.4]),

$$|x(t) - x(s)| \leq (e^{c(t-s)} - 1)\,(|x(s)| + 1)\,, \quad \forall 0 \leq s < t < T.$$

Moreover, $\sup\{|x(s)| : \; s \in [0, T)\} < +\infty$.

3. Existence of solutions

Recall that a SVF is not necessarily defined everywhere on the state space. For this reason, the analysis is divided into two cases, namely, when it is defined everywhere ($\mathcal{I}_0 = \mathcal{I}$) and when it is not ($\mathcal{I}_0 \neq \mathcal{I}$). The first case in simpler and merely need the definition of SVF. The other case, is more delicate to treat and some extra hypothesis are required.

3.1. **Preliminaires.** Let us start by recalling some notions of tangent cones which are going to play a fundamental role in the forthcoming sections.

Definition 3.1. Let $\mathcal{S} \subseteq \mathbb{R}^N$ be a locally closed set and $x \in \mathcal{S}$ given.

(1) The *Bouligand cone* or *Contingent cone* to $\mathcal{S}$ at x, denoted by $\mathcal{T}_{\mathcal{S}}^B(x)$, is

$$\mathcal{T}_{\mathcal{S}}^B(x) = \left\{ v \in \mathbb{R}^N : \liminf_{t \to 0^+} \frac{\mathrm{dist}_{\mathcal{S}}(x + tv)}{t} \leq 0 \right\}.$$

(2) The *Clarke tangent cone* to $\mathcal{S}$ at x, denoted by $\mathcal{T}_{\mathcal{S}}^C(x)$, is

$$\mathcal{T}_{\mathcal{S}}^C(x) = \left\{ v \in \mathbb{R}^N : \limsup_{y \to x,\ t \to 0^+} \frac{\mathrm{dist}_{\mathcal{S}}(y + tv) - \mathrm{dist}_{\mathcal{S}}(y)}{t} \leq 0 \right\}.$$

Remark 4. Note that $\mathcal{T}_{\mathcal{S}}^C(x)$ is always convex and closed, in particular, $\mathrm{ri}\left(\mathcal{T}_{\mathcal{S}}^C(x)\right)$ and $\mathrm{rbd}\left(\mathcal{T}_{\mathcal{S}}^C(x)\right)$ are well defined. Furthermore, $\mathcal{T}_{\mathcal{S}}^C(x) \subseteq \mathcal{T}_{\mathcal{S}}^B(x)$.

The analysis of the existence of stratified solution is based in the Nagumo Theorem whose proof can be found in [1, Theorem 4.2.2] for instance.

Proposition 1 (Nagumo Theorem)**.** *Let $\mathcal{S} \subseteq \mathbb{R}^N$ be a locally compact set and consider $f : \mathcal{S} \to \mathbb{R}^N$ a continuous map. Suppose that*

$$f(x) \in \mathcal{T}_{\mathcal{S}}^B(x), \qquad \forall x \in \mathcal{S}. \tag{7}$$

Then for all $x_0 \in \mathcal{S}$ there exists $T > 0$ such that the differential equation

$$\dot{x} = f(x), \quad x(0) = x_0$$

has a solution lying in $\mathcal{S}$ on $[0, T)$.

3.2. **Case $\mathcal{I}_0 = \mathcal{I}$.** In this situation, the simpler one, the existence of local solutions is ensured by the continuity and the tangentiality of the vector fields on each stratum.

Theorem 3.2. *Let $\mathcal{K} \subseteq \mathbb{R}^N$ be a closed stratifiable set and let $\{M_i\}_{i \in \mathcal{I}}$ be its strata. Consider a regular SVF denoted by $G = \{(g_i, \mathcal{M}_i)\}_{i \in \mathcal{I}_0}$ and suppose that $\mathcal{I}_0 = \mathcal{I}$. Then for every $x_0 \in \mathcal{K}$ there exist $T > 0$ and a stratified solution to* (**D**) *defined on the interval $[0, T)$. Moreover, if the SVF has linear growth, then $T = +\infty$.*

Proof. Let $x_0 \in \mathcal{K}$ and set $i = \imath(x_0)$. Since $i \in \mathcal{I}_0$, the following ODE is well defined

$$\dot{x} = g_i(x), \quad x(0) = x_0. \tag{$\mathbf{D_0^i}$}$$

Note that $\mathcal{M}_i$ is locally compact. Since g_i is continuous and satisfies condition (3), the Nagumo Theorem implies that ($\mathbf{D_0^i}$) has at least a solution that remains in $\mathcal{M}_i$ on an interval $[0, T)$, for $T > 0$. This is clearly a stratified solution to (**D**).

On the other hand, suppose that G has linear growth on $\mathcal{K}$. Let $x(\cdot)$ be a maximal solution (**D**) defined on $[0, T)$ and assume that $T < +\infty$. By Remark 3 there exists a constant $C = C(x_0) > 0$ such that for all $[s, t] \subseteq [0, T)$

$$|x(t) - x(s)| \leq C(e^{L(t-s)} - 1).$$

This means that for any $t_n \nearrow T$, the sequence $\{x(t_n)\}$ is a Cauchy sequence, and so the limit is well defined. Using the same inequality it is possible to see that the limits are independents of the sequences, so the limit $x(T) := \lim_{t \to T^-} x(t)$ exists and belongs to $\mathcal{K}$, which contradicts the maximality of T. So, T can not be bounded. $\square$

3.3. **Case $\mathcal{I}_0 \neq \mathcal{I}$.** In presence of bifurcation strata the existence of solutions to a stratified ODE requires additional hypothesis. For example, consider $\mathcal{K} = \mathbb{R}$, the stratification $\mathcal{M}_0 = \{0\}$, $\mathcal{M}_1 = (-\infty, 0)$ and $\mathcal{M}_2 = (0, +\infty)$, and the SVF $G = \{(\mathcal{M}_i, g_i)\}_{i\in\{1,2\}}$ with $g_1(x) = 1$ and $g_2(x) = -1$. Clearly, no solution starting from $x_0 = 0$ exists.

The problem in this case is reduced to study the existence of solution of an ODE on $\mathcal{M}$, an embedded manifold of $\mathbb{R}^N$, whose initial condition lives in $\overline{\mathcal{M}} \setminus \mathcal{M}$.

Theorem 3.3. *Let $\mathcal{K} \subseteq \mathbb{R}^N$ be a closed stratifiable set and let $\{\mathcal{M}_i\}_{i\in\mathcal{I}}$ be its strata. Consider a regular SVF denoted by $G = \{(g_i, \mathcal{M}_i)\}_{i\in\mathcal{I}_0}$ and suppose*

$$(\mathbf{H_0}) \qquad \begin{cases} \forall x \in \mathcal{K} \text{ with } \imath(x) \notin \mathcal{I}_0, \exists j \in \mathcal{I}_0(\imath(x)) \text{ and } \exists r > 0 \text{ such that:} \\ i) \;\; x + (0, r]\mathbb{B}(g_j(x), r) \cap \overline{\mathcal{M}}_j \subseteq \mathcal{M}_j. \\ ii) \; g_j(y) \in \mathcal{T}^B_{\overline{\mathcal{M}}_j}(y) \setminus \{0\}, \forall y \in \mathbb{B}(x, r) \cap \overline{\mathcal{M}}_j. \end{cases}$$

Then for every $x_0 \in \mathcal{K}$ there exist $T > 0$ and a solution to ($\mathbf{D}$) defined on $[0, T)$. Moreover, if the SVF has linear growth, then $T = +\infty$.

Proof. Let $x_0 \in \mathcal{K}$ and set $i = \imath(x_0)$. In view of Theorem 3.2, it is only necessary to consider the case $i \notin \mathcal{I}_0$. To do this, set $j \in \mathcal{I}_0(i)$ and $r > 0$ given by ($\mathbf{H_0}$).

Note that $\mathbb{B}(x_0, r) \cap \overline{\mathcal{M}}_j$ is locally compact, so by condition ($\mathbf{H_0}$) part (ii) and Nagumo Theorem, there exists a curve $x(\cdot)$ associated with the vector field g_j starting from x_0 lying in $\overline{\mathcal{M}}_j \cap \mathbb{B}(x_0, r)$ on an interval of time $[0, T)$. Moreover, $g_j(x_0) \neq 0$ and since g_j is continuous on $\overline{\mathcal{M}}_j$, by reducing T if necessary, $x(t) \in x_0 + (0, r]\mathbb{B}(g(x_0), r)$ for every $t \in [0, T)$. Therefore, by ($\mathbf{H_0}$) part (i), $x(t) \in \mathcal{M}_j$ on $[0, T)$ and so, the arc $x(\cdot)$ is a stratified solution to ($\mathbf{D}$).

Finally, if G has linear growth, the same arguments as in the proof of the previous theorem are valid, so the conclusion follows. □

Remark 5. Note that if $g_j(x) \in \operatorname{int}\left(\mathcal{T}^C_{\overline{\mathcal{M}}}(x)\right)$ in ($\mathbf{H_0}$), then conditions (i) and (ii) are automatically satisfied; see for instance [10, Section 3.6]. In this case, since $\operatorname{int}\left(\mathcal{T}^C_{\overline{\mathcal{M}}_j}(x)\right) \neq \emptyset$, $\overline{\mathcal{M}}_j$ is said to be wedged.

4. Wedgedness and regularity concepts

The main goal of this section is to provide a pointwise criterion so that Hypothesis ($\mathbf{H_0}$) holds. The principal motivation for doing this is based in the especial case when $\dim(\mathcal{M}_j) = N$; see Remark 5. The idea is to extend the notion of wedgedness to sets that are the closure of embedded manifolds of $\mathbb{R}^N$. In order to do this, some regularity concepts over those sets need to be studied.

4.1. Normal cones and embedded manifolds of $\mathbb{R}^N$.

Let us recall some notions of normal cones which are going to be of utility in the next sections.

Definition 4.1. Let $\mathcal{S} \subseteq \mathbb{R}^N$ be a locally closed set and $x \in \mathcal{S}$ given.

(1) The *Proximal normal cone* to $\mathcal{S}$ at x, denoted by $\mathcal{N}^P_{\mathcal{S}}(x)$, is the set of all $\eta \in \mathbb{R}^N$ such that
$$\sigma|x - y|^2 \geq \langle \eta, y - x\rangle \quad \forall y \in \mathcal{S},$$
for some $\sigma = \sigma(x, \eta) \geq 0$.

(2) The *Limiting normal cone* to $\mathcal{S}$ at x, denoted by $\mathcal{N}^L_{\mathcal{S}}(x)$, is given by

$$\mathcal{N}^L_{\mathcal{S}}(x) := \left\{ \lim_{n\to\infty} \eta_n : \ \exists\{x_n\} \subseteq \mathcal{S} \text{ with } x_n \to x, \exists \eta_n \in \mathcal{N}^P_{\mathcal{S}}(x_n) \right\}.$$

(3) The *Clarke normal cone* to $\mathcal{S}$ at x, denoted by $\mathcal{N}^C_{\mathcal{S}}(x)$, is exactly the convex closed hull of the $\mathcal{N}^L_{\mathcal{S}}(x)$.

Remark 6. $\mathcal{N}^P_{\mathcal{S}}(x)$ is always convex and possibly $\mathcal{N}^P_{\mathcal{S}}(x) = \{0\}$. Besides, $\mathcal{N}^L_{\mathcal{S}}(x)$ is closed, possibly non convex and always contains a nonzero vector. Additionally, its graph gr $\left(\mathcal{N}^L_{\mathcal{S}}(x)\right)$ is closed in $\mathcal{S} \times \mathbb{R}^N$, where

$$\text{gr}\left(\mathcal{N}^L_{\mathcal{S}}(x)\right) = \left\{(x,\eta) : \ \eta \in \mathcal{N}^L_{\mathcal{S}}(x)\right\}.$$

Note that by definition, the following inclusion is alway held

$$\mathcal{N}^P_{\mathcal{S}}(x) \subseteq \mathcal{N}^L_{\mathcal{S}}(x) \subseteq \mathcal{N}^C_{\mathcal{S}}(x) \quad \forall x \in \mathcal{S}.$$

Also, $\mathcal{T}^C_{\mathcal{S}}(x)$ and $\mathcal{N}^C_{\mathcal{S}}(x)$ satisfy the polar relation

$$\mathcal{T}^C_{\mathcal{S}}(x) = \left\{ v \in \mathbb{R}^N : \ \langle v, \eta\rangle \leq 0, \ \forall \eta \in \mathcal{N}^C_{\mathcal{S}}(x)\right\}.$$

As shown in the next proposition, some interesting properties come to light when studying normal and tangent cones to sets that are the closure of an embedded manifold of $\mathbb{R}^N$. Besides, something extra can be said about the structure of the Limiting and the Clarke normal cone.

Proposition 2. *Let $\mathcal{M}$ be an embedded manifold of $\mathbb{R}^N$ of dimension d then:*

(1) $\mathcal{T}^B_{\overline{\mathcal{M}}}(x) = \mathcal{T}^C_{\overline{\mathcal{M}}}(x) = \mathcal{T}_{\mathcal{M}}(x)$ for any $x \in \mathcal{M}$.

(2) $\mathcal{N}^P_{\overline{\mathcal{M}}}(x) = \mathcal{N}^C_{\overline{\mathcal{M}}}(x) = \mathcal{N}_{\mathcal{M}}(x)$ for any $x \in \mathcal{M}$. Additionally, for every $x \in \mathcal{M}$, there exists $\delta = \delta(x) > 0$ such that

$$\frac{|\eta|}{2\delta}|x-y|^2 \geq \langle \eta, y-x\rangle \qquad \forall \eta \in \mathcal{N}_{\mathcal{M}}(x), \forall y \in \overline{\mathcal{M}}.$$

(3) for any $x \in \overline{\mathcal{M}}$, $\mathcal{N}^C_{\overline{\mathcal{M}}}(x)$ contains a vectorial subspace of dimension $N-d$.

Proof. (1) This is direct consequences of [10, Proposition 2.7.4].

(2) The first part of the statement is straightforward from [10, Proposition 2.7.4]. Besides, by definition, there exists $r = r(x) > 0$ such that

$$\mathcal{N}_{\mathcal{M}}(y) = Dh(y)^T\left(\mathbb{R}^{N-d}\right), \quad \forall y \in \mathcal{M}\cap\mathbb{B}(x,r),$$

where $h : \mathbb{B}(x,r) \to \mathbb{R}^{N-d}$ is smooth and surjective on $\mathbb{B}(x,r)$. Let us fix $r_0 \in (0,r)$ and set

$$L_0 = \max_{y\in\overline{\mathbb{B}}(x,r_0)} |D^2h(y)|,$$
$$\lambda_0 = \min_{y\in\overline{\mathbb{B}}(x,r_0)} \lambda_{min}(Dh(y)Dh(y)^T).$$

Here, $\lambda_{min}(A)$ stands for the minimal eigenvalue of a square matrix A. Note that, $L_0 < +\infty$ because h is of class $\mathcal{C}^2$ and $\lambda_0 > 0$ since $Dh(y)$ is surjective on $\mathbb{B}(x,r)$. Thus, by the Mean Value Theorem

$$|Dh(x)(y-x)| \leq L_0|x-y|, \quad \forall y \in \mathcal{M}\cap\overline{\mathbb{B}}(x,r_0).$$

Let $\eta \in \mathcal{N}_{\mathcal{M}}(x)$, then there exists $\mu \in \mathbb{R}^{N-p}$ such that $\eta = Dh(x)^T\mu$ and so $\lambda_0|\mu|^2 \leq |\eta|^2$. Hence, by the previous estimation

$$\langle \eta, y-x\rangle = \langle Dh(x)^T\mu, y-x\rangle \leq \frac{L_0}{\sqrt{\lambda_0}}|\eta||y-x|^2, \quad \forall y \in \mathcal{M}\cap\overline{\mathbb{B}}(x,r_0).$$

If $y \notin \mathcal{M} \cap \overline{\mathbb{B}}(x, r_0)$, then the inequality holds trivially with $\frac{r_0}{2}$. Therefore,

$$\delta = \min\left\{\frac{r_0}{2}, \frac{\sqrt{\lambda_0}}{2L_0}\right\},$$

in case of $L_0 \neq 0$, otherwise, $\delta = \frac{r_0}{2}$. Finally, by an argument of density, the inequality holds up to $\overline{\mathcal{M}}$.

(3) Take $x \in \overline{\mathcal{M}}$. If $x \in \mathcal{M}$ or $d = N$, the conclusion is straightforward and the vectorial subspace coincides with the normal space and with $\{0\}$, respectively. On the other hand, suppose that $x \in \overline{\mathcal{M}} \setminus \mathcal{M}$, then there exists $\{x_n\} \subseteq \mathcal{M}$ with $x_n \to x \in \overline{\mathcal{M}}$. Whence, for each $n \in \mathbb{N}$, $\{\mathcal{N}^C_{\overline{\mathcal{M}}}(x_n)\}$ is a subspace of dimension $N-d$ and it is possible to select an orthonormal base for this subspace, denoted by $\{\eta_n^1, \ldots, \eta_n^{N-d}\}$. Without loss of generality, for every $i = 1, \ldots, N-d$, each sequence $\eta_n^i \to \eta^i$. Clearly, $\{\eta^1, \ldots, \eta^{N-d}\}$ is also an orthonormal family of vectors. Note that for any $n \in \mathbb{N}$, $\eta_n^i \in \mathcal{N}^L_{\overline{\mathcal{M}}}(x_n)$ and since this cone has closed graph on $\overline{\mathcal{M}}$, each $\eta^i \in \mathcal{N}^L_{\overline{\mathcal{M}}}(x)$. Note also that $-\eta^i \in \mathcal{N}^L_{\overline{\mathcal{M}}}(x)$. Finally, $\operatorname{span}\{\eta^1, \ldots, \eta^{N-d}\} \subseteq \mathcal{N}^C_{\overline{\mathcal{M}}}(x)$ and the result follows.

□

Note that δ in Proposition 2 is the radius of a closed ball centered at $x + \frac{\delta}{|\eta|}\eta$ which intersects $\mathcal{M}$ only at x. In this sense, it is possible to interpret this quantity as the curvature of $\mathcal{M}$.

Definition 4.2. Let $\mathcal{M}$ be an embedded manifold of $\mathbb{R}^N$.

(1) The radius of curvature of $\mathcal{M}$ at $x \in \mathcal{M}$, denoted by $\kappa(x)$, is given by

$$\kappa(x) = \sup\left\{\frac{2\langle \eta, y - x\rangle}{|y-x|^2} : \ \eta \in \mathcal{N}_{\mathcal{M}}(x) \cap \mathbb{S}, \ y \in \mathcal{M} \setminus \{x\}\right\}.$$

(2) $\mathcal{M}$ is said to have constant curvature if there is a constant $\kappa_0 \in \mathbb{R}$ such that $\kappa(x) \leq \kappa_0$ for any $x \in \mathcal{M}$.

The notion of bounded curvature can be extended to stratifiable sets in the following way.

Definition 4.3. A stratifiable subset of $\mathbb{R}^N$ is said to have bounded curvature if each its stratum has bounded curvature.

Next proposition shows the importance of the Whitney-(a) condition for the analysis. Roughly speaking, it provides a hierarchy between the Clarke normal cones of the strata.

Proposition 3. *Let $\mathcal{K}$ be stratifiable and let $i, j \in \mathcal{I}$ be two index associated with the stratification such that $i \preceq j$. Then $\mathcal{N}^C_{\overline{\mathcal{M}}_j}(x) \subseteq \mathcal{N}_{\mathcal{M}_i}(x)$ for any $x \in \mathcal{M}_i \cap \overline{\mathcal{M}}_j$.*

Proof. Note first that if $\mathcal{S}_1 \subseteq \mathcal{S}_2$ then $\mathcal{N}^P_{\mathcal{S}_2}(x) \subseteq \mathcal{N}^P_{\mathcal{S}_1}(x)$ for every $x \in \mathcal{S}_1$. Let $\eta \in \mathcal{N}^L_{\overline{\mathcal{M}}_j}(x)$, by definition there exist two sequences $\{x_n\} \subseteq \overline{\mathcal{M}}_j$ and $\{\eta_n\} \subseteq \mathbb{R}^N$ such that $\eta_n \in \mathcal{N}^P_{\overline{\mathcal{M}}_j}(x_n)$ with $x_n \to x$ and $\eta_n \to \eta$.

Recall that the stratification is locally finite, so there is only a finite number of indices $k \in \mathcal{I}$ satisfying

$$x \in \overline{\mathcal{M}}_k, \quad \mathcal{M}_i \subseteq \overline{\mathcal{M}}_k \quad \text{and} \quad \mathcal{M}_k \subseteq \overline{\mathcal{M}}_j. \tag{8}$$

Therefore, there exist a stratum $\mathcal{M}_k$ that satisfies (8) and a subsequence of $\{x_n\}$ that belong to $\mathcal{M}_k$. Without loss of generality, assume that the subsequence is the whole sequence and that $\mathcal{T}_{\mathcal{M}_k}(x_n) \to \mathcal{T}$.

By the previous remark and by Proposition 2

$$\eta_n \in \mathcal{N}^P_{\overline{\mathcal{M}_j}}(x_n) \subseteq \mathcal{N}^P_{\overline{\mathcal{M}_k}}(x_n) = \mathcal{N}_{\mathcal{M}_k}(x_n)$$

Therefore, η is orthogonal to $\mathcal{T}$ and so, by virtue of the Whitney-(a) condition applied to $(\mathcal{M}_i, \mathcal{M}_k)$, one gets that $\mathcal{T}_{\mathcal{M}_i}(x) \subseteq \mathcal{T}$ and in particular, $\eta \in \mathcal{N}_{\mathcal{M}_i}(x)$.

Finally, since $\mathcal{N}^L_{\overline{\mathcal{M}_j}}(x) \subseteq \mathcal{N}_{\mathcal{M}_i}(x)$ for any $x \in \mathcal{M}_i$, by taking convex closed hull in the last inclusion, the proof is completed. □

4.2. **Relatively wedged sets.** In Proposition 2 it was shown that $\mathcal{N}^C_{\overline{\mathcal{M}}}(x)$ can be rewritten as $\mathcal{N}^d \times \mathbb{R}^{N-d}$ (after a change of coordinate), where $\mathcal{N}^d \subseteq \mathbb{R}^d$ is convex closed cone. Note that this decomposition is not unique, in particular, the Clarke tangent cone may be contained in a affine subspace of dimension strictly lower than d. Nevertheless, if $\overline{\mathcal{M}}$ is wedged (see Remark 5) then this decomposition is unique (identifying $\mathbb{R}^N$ with $\mathbb{R}^N \times \{0\}$) and even more, it is continuous with respect to x. These facts motivate the following definition.

Definition 4.4 (Relatively wedged)**.** Let $\mathcal{M}$ be an embedded manifold of $\mathbb{R}^N$ of dimension d. We say that $\overline{\mathcal{M}}$ is relatively wedged around $x_0 \in \overline{\mathcal{M}}$ provided there exists a neighborhood Θ of x_0 and a continuous mapping $A : \overline{\mathcal{M}} \cap \Theta \to \mathcal{SO}(N)$, $x \mapsto A_x$ such that:

$$A_x\left(\mathcal{N}^C_{\overline{\mathcal{M}}}(x)\right) = \mathcal{N}^d_x \times \mathbb{R}^{N-d} \quad \text{with} \quad \mathcal{N}^d_x \text{ pointed in } \mathbb{R}^d.$$

A similar notion of *relative wedgedness* was also studied in [2] for embedded manifold of $\mathbb{R}^N$ without asking the continuity of the change of coordinates, but requiring an additional regularity property over the closure of the $\mathcal{M}$.

Remark 7. Suppose $\overline{\mathcal{M}}$ is relatively wedged around $x \in \overline{\mathcal{M}}$.

(1) If $d = N$, by taking A_x as the identity matrix of dimension N the original notion of wedgedness is recovered.
(2) The projection $\pi_x : \mathcal{N}^C_{\overline{\mathcal{M}}}(x) \to \mathcal{N}^d_x \times \{0\}^{N-d}$ is closed in the following sense: if $x_n \to x$ and $\eta_n \to \eta$ as $n \to +\infty$, where $\eta_n \in \mathcal{N}^C_{\overline{\mathcal{M}}}(x_n)$, then $\pi_{x_n}(\eta_n) \to \pi_x(\eta)$ as $n \to +\infty$.
(3) Using similar arguments as in the wedged case, it is possible to prove that $y \mapsto \mathcal{N}^C_{\overline{\mathcal{M}}}(y)$ is graph-closed at x; see [10, Proposition 3.6.7 and 3.6.8].

The importance of this property and the relation with hypothesis ($\mathbf{H_0}$) is summarized in the next proposition.

Proposition 4. *Let $\mathcal{K}$ be a stratifiable set and let $\mathcal{M}$ be one of the strata. Suppose that $\overline{\mathcal{M}}$ relatively wedged around $x \in \overline{\mathcal{M}} \setminus \mathcal{M}$.*

(1) Then $v \in \operatorname{ri}\left(\mathcal{T}^C_{\overline{\mathcal{M}}}(x)\right)$ if and only if $\exists \sigma > 0$ such that

$$\langle v, \eta\rangle \leq -\sigma|\pi_x(\eta)| \quad \forall \eta \in \mathcal{N}^C_{\overline{\mathcal{M}}}(x). \tag{9}$$

(2) Let $g : \overline{\mathcal{M}} \to \mathbb{R}^N$ continuous with $g(y) \in \mathcal{T}_{\mathcal{M}}(y)$ for every $y \in \mathcal{M}$ with $g(x) \in \operatorname{ri}\left(\mathcal{T}^C_{\overline{\mathcal{M}}}(x)\right)$, then there exists $r > 0$ such that

$$g(y) \in \mathcal{T}^C_{\overline{\mathcal{M}}}(y), \quad \forall y \in \mathbb{B}(x, r) \cap \overline{\mathcal{M}}.$$

(3) Suppose that $\mathcal{K}$ has constant curvature, then there exists $r>0$ such that

$$(x+(0,r]\mathbb{B}(v,r))\cap\overline{\mathcal{M}}\subseteq\mathcal{M}.$$

Proof. (1) For simplicity, assume that $\mathcal{N}^C_{\overline{\mathcal{M}}}(x)=\mathcal{N}^d_x\times\mathbb{R}^{N-d}$. In particular, $\mathcal{T}^C_{\overline{\mathcal{M}}}(x)=\mathcal{T}^d_x\times\{0\}^{N-d}$ and $v=(w,0)$. Since $\overline{\mathcal{M}}$ relatively wedged at x, $v\in\operatorname{ri}\left(\mathcal{T}^C_{\overline{\mathcal{M}}}(x)\right)$ if and only if $w\in\operatorname{int}(\mathcal{T}^d_x)$ relative to $\mathbb{R}^d$. However, this is equivalent to the existence of $\sigma>0$ such that

$$\langle w,\zeta\rangle\leq-\sigma|\zeta|\quad\forall\zeta\in\mathcal{N}^d_x.$$

Finally, since $\langle v,\eta\rangle=\langle w,\pi_x(\eta)\rangle$ the equivalence is stated.

(2) Reasoning by contradiction, there are two sequences $\{x_n\}\subseteq\overline{\mathcal{M}}$ and $\{\eta_n\}\in\mathbb{S}$ with $x_n\to x$ and $\eta_n\in\mathcal{N}^C_{\overline{\mathcal{M}}}(x_n)$ such that $\langle g(x_n),\eta_n\rangle>0$. Since $g(y)\in\mathcal{T}_{\mathcal{M}}(y)$ for every $y\in\mathcal{M}$,

$$\langle g(y),\eta\rangle=0,\quad\forall y\in\overline{\mathcal{M}},\eta\in\mathcal{N}^C_{\overline{\mathcal{M}}}(y)\text{ such that }\pi_y(\eta)\in\{0\}^d\times\mathbb{R}^{N-d}.$$

Thus, it is always possible to pick η_n such that $|\eta_n|=|\pi_{x_n}(\eta_n)|=1$

On the other hand, by Remark 7, $y\mapsto\mathcal{N}^C_{\overline{\mathcal{M}}}(y)$ is closed-graph at x, so without loss of generality, $\eta_n\to\eta\in\mathcal{N}^C_{\overline{\mathcal{M}}}(x)\cap\mathbb{S}$ and, as g is continuous, $\langle g(x),\eta\rangle\geq0$. However, this contradicts the first part of this proposition, so the proof is completed.

(3) By contradiction, suppose there exist some sequence $\{t_n\}\subseteq(0,+\infty)$ and $\{v_n\}\subseteq\mathbb{R}^N$ with $t_n\to0$ and $v_n\to v$ such that $y_n:=x+t_nv_n\in\overline{\mathcal{M}}\setminus\mathcal{M}$. Since $\mathcal{K}$ is stratifiable, without loss of generality, there exists a stratum $\mathcal{M}_i\subseteq\overline{\mathcal{M}}$ with $x\in\overline{\mathcal{M}}_i$ such that $\{y_n\}\subseteq\mathcal{M}_i$. Take $\eta_n\in\mathcal{N}^L_{\overline{\mathcal{M}}}(y_n)$ with $|\eta_n|=|\pi_{y_n}(\eta_n)|=1$. By passing to a subsequence if necessary, $\eta_n\to\eta$ with $\eta\in\mathcal{N}^C_{\overline{\mathcal{M}}}(x)$ and $|\pi_x(\eta)|=1$

By Proposition 3, $\eta_n\in\mathcal{N}_{\mathcal{M}_i}(y_n)$ and since $\mathcal{M}_i$ has constant curvature, there exists $\kappa_0>0$ such that

$$\frac{\kappa_0}{2}|y_n-y|^2\geq\langle\eta_n,y-y_n\rangle\qquad\forall y\in\overline{\mathcal{M}}_i,\forall n\in\mathbb{N}.$$

Evaluating at $y=x$

$$\langle v_n,\eta_n\rangle\geq-\frac{\kappa_0t_n}{2}|v_n|^2\qquad\forall n\in\mathbb{N}.$$

Letting $n\to+\infty$

$$\langle v,\eta\rangle\geq0.$$

However, $|\pi_x(\eta)|=1$ which contradicts (9). So the conclusion follows. □

In view of this last proposition, if $\mathcal{K}$ has bounded curvature as in Definition 4.3 and

$$(\mathbf{H}_0^\sharp)\qquad\begin{cases}\forall x\in\mathcal{K}\text{ with }\imath(x)\notin\mathcal{I}_0,\exists j\in\mathcal{I}_0(\imath(x))\text{ such that:}\\ i)\ \ \overline{\mathcal{M}}_j\text{ is relatively wedged.}\\ ii)\ g_j(x_0)\in\operatorname{ri}\left(\mathcal{T}^C_{\overline{\mathcal{M}}_j}(x_0)\right).\end{cases}$$

holds then ($\mathbf{H_0}$) also holds and so Theorem 3.3.

Theorem 4.5. *Let $\mathcal{K} \subseteq \mathbb{R}^N$ be a closed stratifiable set that has bounded curvature. Consider a regular SVF denoted by $G = \{(g_i, \mathcal{M}_i)\}_{i\in\mathcal{I}_0}$ and suppose that ($\mathbf{H}_0^\sharp$) holds. Then for every $x_0 \in \mathcal{K}$ there exist $T > 0$ and a solution to (**D**) defined on $[0, T)$. Moreover, if the SVF has linear growth, then $T = +\infty$.*

5. Necessary condition for Robustness

This final section has as purpose to study the robustness of a stratified ODE under external perturbations. The principal issue here is to show conditions in order to ensure that the corresponding solutions to that type of discontinuous ODE are still stratified solutions of the original system, even if the velocities are slightly perturbed.

Hereinafter, the stratification associated with the set $\mathcal{K}$ is supposed to be relatively wedged, that is, the closure of each stratum $\mathcal{M}_i$ is relatively wedged in the sense of Definition 4.4.

5.1. **Robustness with respect to external perturbations.** Let $\mathcal{S}$ be a closed set and $g : \mathcal{S} \to \mathbb{R}^N$ a given vector field. Let $x_0 \in \mathcal{S}$ and consider $\{x_0^n\} \subseteq \mathcal{S}$ and $\{\xi_n\} \subseteq L^1([0, T])$ such that $x_0^n \to x_0$ and $\xi_n \to 0$ in $L^1([0, T])$. Suppose that for each $n \in \mathbb{N}$, there exists $x_n : [0, T] \to \mathbb{R}^N$ a Carathéodory solution to the perturbed system:

$$\begin{cases} \dot{x} = g(x) + \xi_n & \text{a.e. on } (0, T), \\ x(t) \in \mathcal{S}, & \text{for any } t \in [0, T], \\ x(0) = x_0^n. \end{cases} \tag{10}$$

Let $x(\cdot)$ be an accumulation point of $\{x_n(\cdot)\}$ in the topology of the uniform convergence on $[0, T]$ and suppose that this is a solution to

$$\begin{cases} \dot{x} = g(x) & \text{on } (0, T), \\ x(t) \in \mathcal{S}, & \text{for any } t \in [0, T], \\ x(0) = x_0. \end{cases} \tag{11}$$

In this case, the map g is said to be *robust with respect to external perturbations*. It is not difficult to see that if g is continuous (which is not in general the case of a SVF), then the property holds. However, thank to tameness of the singular set of a stratified dynamics, gathering the continuity on each strata, it is possible to state such property by ruling some types singularities out.

5.2. **Outward-pointing modulus.** In order to study the way how the continuous part of a SVF interact between each other, the *outward-pointing modulus* of a vector field is introduced.

Definition 5.1. Let $\mathcal{M}$ be an embedded manifold of $\mathbb{R}^N$ and $g : \overline{\mathcal{M}} \to \mathbb{R}^N$. Suppose that $\overline{\mathcal{M}}$ is relatively wedged. For any $x \in \overline{\mathcal{M}} \setminus \mathcal{M}$, the outward-pointing modulus of g at x is given by:

$$\alpha_g(x) = \max\left\{\langle g(x), \eta\rangle : \ \eta \in \mathcal{N}^C_{\overline{\mathcal{M}}}(x) \text{ s.t. } |\eta| = |\pi_x(\eta)| = 1\right\}.$$

The condition $|\pi_x(\eta)| = 1$ is important in the definition, because if this is omitted, one would get $\alpha_g(x) = 0$ whenever $\dim(\mathcal{M}) < N$ and $g(x) \in \mathcal{T}^C_{\overline{\mathcal{M}}}(x)$, whereas, if $\dim(\mathcal{M}) = N$ it can be strictly negative. Indeed, $\alpha_g(x) < 0$ if and only if $g(x) \in \operatorname{int}\left(\mathcal{T}^C_{\overline{\mathcal{M}}}(x)\right)$.

The main characteristic of this function is that provides a generalization to lower-dimension embedded manifolds of the previous remark, where the role of the interior is played by the relative interior of the tangent cone of Clarke. This is recapitulated in the next proposition.

Proposition 5. *Let $\mathcal{M}$ be an embedded manifold of $\mathbb{R}^N$ for which $\overline{\mathcal{M}}$ is relatively wedged around $x \in \overline{\mathcal{M}} \setminus \mathcal{M}$ and let $g : \overline{\mathcal{M}} \to \mathbb{R}^N$ be a vector field. Then*

$$\alpha_g(x) < 0 \quad \text{if and only if} \quad g(x) \in \operatorname{ri}\left(\mathcal{T}^C_{\overline{\mathcal{M}}}(x)\right).$$

In addition, if g is continuous at x and satisfies $g(y) \in \mathcal{T}_{\mathcal{M}}(y)$, $\forall y \in \mathcal{M}$, then

1. *$\alpha_g(x) = 0$ if and only if $g(x) \in \operatorname{rbd}\left(\mathcal{T}^C_{\overline{\mathcal{M}}}(x)\right)$.*
2. *$\alpha_g(x) > 0$ if and only if $g(x) \notin \mathcal{T}^C_{\overline{\mathcal{M}}}(x)$.*

Proof. By Proposition 4, $g(x) \in \operatorname{ri}\left(\mathcal{T}^C_{\overline{\mathcal{M}}}(x)\right)$ if and only if $\exists \sigma > 0$ such that

$$\langle g(x), \eta \rangle \leq -\sigma |\pi_x(\eta)| \quad \forall \eta \in \mathcal{N}^C_{\overline{\mathcal{M}}}(x).$$

So, first equivalence comes directly from the definition of the modulus. Moreover, by the previous part and the polar relation between $\mathcal{T}^C_{\overline{\mathcal{M}}}(\cdot)$ and $\mathcal{N}^C_{\overline{\mathcal{M}}}(\cdot)$,

$$\begin{aligned} &\text{if } g(x) \in \operatorname{rbd}\left(\mathcal{T}^C_{\overline{\mathcal{M}}}(x)\right) \quad \text{then} \quad \alpha_g(x) = 0,\\ &\text{if } g(x) \notin \mathcal{T}^C_{\overline{\mathcal{M}}}(x) \qquad \text{then} \quad \alpha_g(x) \geq 0.\\ &\text{if } \alpha_g(x) > 0 \qquad \text{then} \quad g(x) \notin \mathcal{T}^C_{\overline{\mathcal{M}}}(x). \end{aligned}$$

Note that if $g(x) \notin \mathcal{T}^C_{\overline{\mathcal{M}}}(x)$ and $\alpha_g(x) = 0$, then

$$\langle g(x), \eta \rangle \leq 0, \quad \forall \eta \in \mathcal{N}^C_{\overline{\mathcal{M}}}(x) \text{ s.t. } |\pi_x(\eta)| \neq 0.$$

On the other hand, since $\langle g(y), \eta \rangle = 0$ for any $\eta \in \mathcal{N}_{\mathcal{M}}(y)$ if $g(y) \in \mathcal{T}_{\mathcal{M}}(y)$, if in addition g is continuous on $\overline{\mathcal{M}}$, one gets

$$\langle g(x), \eta \rangle = 0, \quad \eta \in \mathcal{N}^C_{\overline{\mathcal{M}}}(x) \text{ s.t. } |\pi_x(\eta)| = 0.$$

Hence,

$$\langle g(x), \eta \rangle \leq 0, \quad \eta \in \mathcal{N}^C_{\overline{\mathcal{M}}}(x),$$

meaning that $g(x) \in \mathcal{T}^C_{\overline{\mathcal{M}}}(x)$ which is not possible. So, if $g(x) \notin \mathcal{T}^C_{\overline{\mathcal{M}}}(x)$ then necessarily $\alpha_g(x) > 0$ and the proof is completed. □

5.3. **The externally perturbed model.** Recall that the main characteristic of the ODE models presented in this work is that there may exist some strata where no trajectory can slide for. However, in presence of external perturbations this feature may not be held because for any $i \notin \mathcal{I}_0$ and $j \in \mathcal{I}_0(i)$, it is possible to construct a continuous perturbation $\xi : \mathcal{M}_i \to \mathbb{R}^N$ such that

$$g_j(x) + \xi(x) \in \mathcal{T}_{\mathcal{M}_i}(x), \quad \forall x \in \mathcal{M}_i.$$

Whence, trajectories of an externally perturbed model may sliding for $\mathcal{M}_i$, so an equation written as (**D**) may not have sense.

To avoid this problem, one may replace the initial stratified dynamics by a set-valued map which take into account all the *significative* directions of a SVF, namely

$$G^E(x) = \bigcup_{i \in \mathcal{I}_0(\imath(x))} \{g_i(x)\} \cap \mathcal{T}^B_{\mathcal{M}_i}(x).$$

Remark 8. The idea of using a set-valued map corresponding to the essential directions has already been explored in others related works for studying Hamilton-Jacobi equation with discontinuous data; see for instance [2, 22].

In this case, the perturbed model corresponds to

$$\begin{cases} \dot{x} \in G^E(x) + \sigma(x)\xi & \text{a.e. } t \in (0,T), \\ x(t) \in \mathcal{K}, & \text{for every } t \in [0,T], \end{cases} \tag{12}$$

where $\sigma : \mathcal{K} \to [0, +\infty)$ is continuous, $\xi : [0,T] \to \mathbb{R}^N$ is measurable.

However, the initial model is still of concern and so, it would be interesting to know when a Carathéodory solution to (12) is also a solution in the stratified sense, as much as in the unperturbed ODE (**D**). It turns out that a sort of maximality condition over the choice of the index $\mathcal{I}_0$ is required. To state the hypothesis, let us introduce some notation, set $\alpha_j(x) = \alpha_{g_j}(x)$ for any $x \in \mathcal{K}$ and $j \in \mathcal{I}_0(x)$.

$$(\mathbf{H_1}) \qquad \begin{cases} \text{For any } i \in \mathcal{I} \text{ and } j \in \mathcal{I}_0(i): \\ i) \quad \text{The sign of } \alpha_j(\cdot) \text{ is constant all along } \mathcal{M}_i. \\ ii) \quad \text{If } i \notin \mathcal{I}_0,\ \alpha_j(x) \neq 0,\ \forall x \in \mathcal{M}_i. \\ iii) \text{ If } i \in \mathcal{I}_0 \text{ and } \alpha_j(x) = 0 \text{ with } g_i(x) \neq 0,\ \forall x \in \mathcal{M}_i, \\ \qquad \text{then } g_j(x) = g_i(x)\ \forall x \in \mathcal{M}_i. \end{cases}$$

Remark 9. The first two points in ($\mathbf{H_1}$) can be weakened by requiring, for example, that the first one holds only locally and that the second is replaced by a condition that ensures that no $j \in \mathcal{I}_0(i)$ will satisfy $g_j(x) \in \mathcal{T}_{\mathcal{M}_i}(x)$.

However, the third point seems to be an essential assumption. Indeed, if it does not hold taking a suite of initial condition living in $\mathcal{M}_j$ but converging to $x_0 \in \mathcal{M}_i \subseteq \overline{\mathcal{M}}_j$. Whence, the limit of the those trajectories dwells in $\mathcal{M}_i$, because of the continuity of g_j and also because $g_j(x) \in \mathcal{T}_{\mathcal{M}_i}(x)$ whenever $x \in \mathcal{M}_i$. If $g_i \neq g_j$, then the limiting arc can not be a stratified trajectory. So in this sense, this hypothesis is a necessary condition for robustness.

Note that in particular, the set of singularities enlisted in [3] for planar systems with affine control systems as well as the set of singularities found in the construction of the feedback in [12] satisfy this condition. Furthermore, this also implies that, if $g_i(x) \neq 0$, then

$$G^E(x) \cap \mathcal{T}_{\mathcal{M}_i}(x) = \begin{cases} \{g_i(x)\} & \text{if } i \in \mathcal{I}_0 \\ \emptyset & \text{if } i \notin \mathcal{I}_0, \end{cases} \quad \forall x \in \mathcal{M}_i. \tag{13}$$

Besides, an upper bound for the size of the perturbation is also essential. Before giving this bound, let us introduce more notation.

$$\begin{aligned} \mathcal{I}_i^+(x) &:= \{j \in \mathcal{I}_0(\imath(x)) :\ \alpha_j > 0 \text{ all along } \mathcal{M}_i\}, \\ \mathcal{I}_i^-(x) &:= \{j \in \mathcal{I}_0(\imath(x)) :\ \alpha_j < 0 \text{ all along } \mathcal{M}_i\}, \\ \mathcal{I}_i^0(x) &:= \{j \in \mathcal{I}_0(\imath(x)) :\ \alpha_j = 0 \text{ all along } \mathcal{M}_i\}. \end{aligned}$$

Thus, the upper bound for the size of the perturbations is given by

$$\sigma(x) \leq \frac{1}{2} \min \left\{ \min_{j \in \mathcal{I}_i^+(x)} \alpha_j(x), \min_{j \in \mathcal{I}_i^-(x)} -\alpha_j(x) \right\}, \quad \forall x \in \mathcal{M}_i. \tag{14}$$

Proposition 6. *Let $\mathcal{K} \subseteq \mathbb{R}^N$ be a closed stratifiable set and let $\{\mathcal{M}_i\}_{i\in\mathcal{I}}$ be its strata. Consider a regular SVF denoted by $G = \{(g_i, \mathcal{M}_i)\}_{i\in\mathcal{I}_0}$. Assume that ($\mathbf{H_1}$) holds. Let $\xi : [0,T] \to \overline{\mathbb{B}}$ and $\sigma : \mathcal{K} \to [0,+\infty)$ be two given measurable functions. Suppose $\sigma(\cdot)$ satisfying (14) and $x : [0,T] \to \mathcal{K}$ is a Carathéodory solution to (12). Set $J_i = \{t \in [0,T] :\ x(t) \in \mathcal{M}_i\}$, then*

i) $\forall i \notin \mathcal{I}_0$, $\text{meas}(J_i) = 0$,
ii) $\forall i \in \mathcal{I}_0$, $\dot{x} = g_i(x) + \sigma(x)\xi$ *a.e. on* $J_i \cap \{t \in [0,T] :\ g_i(x(t)) \neq 0\}$.

Proof. let $i \in \mathcal{I}$ such that $\text{meas}(J_i) > 0$, then by the Lebesgue Differentiation Theorem there exists $\tilde{J}_i \subseteq J_i$ measurable with $\text{meas}(\tilde{J}_i) = \text{meas}(J_i)$ such that

$$\lim_{\varepsilon\to 0^+} \frac{1}{\varepsilon}\int_t^{t+\varepsilon} \dot{x}(s)ds = \dot{x}(t) \in G^E(x(t)) + \sigma(x(t))\xi(t), \quad \forall t \in \tilde{J}_i.$$

Assume that the set $\tilde{J}_i$ does not contain isolated points.

On the other hand, since the stratification is locally finite, for any $t \in \tilde{J}_i$ there exist $j \succeq i$ with $j \in \mathcal{I}_0$, a sequence $\{t_n\} \subseteq \tilde{J}_i \setminus \{t\}$ with $t_n \to t$ such that

$$\frac{x(t_n) - x(t)}{t_n - t} \to \dot{x}(t) = g_j(x(t)) + \sigma(x(t))\xi(t).$$

Since, $x(t_n) \in \mathcal{M}_i$ then $\dot{x}(t) \in \mathcal{T}^B_{\mathcal{M}_i}(x(t)) = \mathcal{T}_{\mathcal{M}_i}(x(t))$.

i) Suppose $i \notin \mathcal{I}_0$ and let $t \in \tilde{J}_i$. By Proposition 3,

$$\langle \dot{x}(t), \eta\rangle = 0, \quad \forall \eta \in \mathcal{N}^C_{\mathcal{M}_j}(x(t)),\ \forall t \in \tilde{J}_i.$$

In addition, for any $\eta \in \mathcal{N}^C_{\mathcal{M}_j}(x(t))$ with $|\eta| = |\pi_x(\eta)| = 1$ one has

$$\langle g_j(x(t)), \eta\rangle - \sigma(x(t)) \leq \langle \dot{x}(t), \eta\rangle \leq \alpha_j(x(t)) + \sigma(x(t)).$$

In particular, if $j \in \mathcal{I}_i^+(x(t))$, the leftthand side gives a contradiction with (14). If $j \in \mathcal{I}_i^-(x(t))$, the same occurs with righthand. Finally, by ($\mathbf{H_1}$), $\mathcal{I}_i^0(x(t)) = \emptyset$. Hence, $\text{meas}(J_i) = 0$ otherwise a contradiction is obtained.

ii) By (13), $G^E(x(t)) = \{g_i(x(t))\}$ for any $t \in \tilde{J}_i$. So the conclusion follows.

□

Notably, if ($\mathbf{H_1}$) holds and the perturbation are small enough, the perturbed stratified systems

$$(\mathbf{D}^\sigma) \qquad \begin{cases} \dot{x} = g_i(x) + \sigma(x)\xi & \text{a.e. whenever } x \in \mathcal{M}_i \\ x(t) \in \mathcal{K}, \quad t > 0 \end{cases}$$

makes sense, and in fact, any solution to this model is a Caratheodory solution to (12). Solutions to this equation have the following form.

$$x(t) = x_0 + \sum_{i\in I_0} \int_{J_i(t)} g_i(x(s))ds + \int_0^t \sigma(x(s))\xi(s)ds, \quad \forall t \in [0,T],$$

where $J_i(t) = \{s \in [0,t] :\ x(s) \in \mathcal{M}_i\}$.

5.4. **Main result.** The main result and principal motivation to write this paper can be summarized in the following theorem.

Theorem 5.2. *Let $\mathcal{K} \subseteq \mathbb{R}^N$ be a closed stratifiable set and let $\{\mathcal{M}_i\}_{i\in\mathcal{I}}$ be its strata. Consider a SVF denoted by $G = \{(g_i, \mathcal{M}_i)\}_{i\in\mathcal{I}_0}$. Let $\{\xi_n : [0,T] \to \overline{\mathbb{B}}\}$ be a sequence of measurable functions with $\xi_n \to 0$ in $L^1([0,T])$ and consider $\sigma : \mathcal{K} \to [0,+\infty)$ a given continuous satisfying (14). Suppose that ($\mathbf{H_1}$) holds and in addition*

$$(\mathbf{H_2}) \qquad \begin{cases} \forall i \in \mathcal{I}_0, \ \text{if } i \preceq j \text{ then } j \in \mathcal{I}_0 \text{ and } \alpha_j(x) \geq 0, \ \forall x \in \mathcal{M}_i. \\ \forall i \notin \mathcal{I}_0, \ \text{if } j \in \mathcal{I}_0(j) \text{ then } \alpha_j(x) < 0, \ \forall x \in \mathcal{M}_i. \end{cases}$$

Let $x_n(\cdot)$ be a stratified solution to

$$(15) \qquad \begin{cases} \dot{x} = g_i(x) + \sigma(x)\xi_n & \text{a.e. whenever } x \in \mathcal{M}_i \\ x(t) \in \mathcal{K}, & \text{for any } t \in [0,T] \\ x(0) = x_0^n \in \mathcal{K}. \end{cases}$$

that converges uniformly to $x : [0,T] \to \mathcal{K}$, then x is a stratified solution to ($\mathbf{D}$).

To prove this theorem, some previous lemmas are required.

Lemma 5.3. *Let $\mathcal{K} \subseteq \mathbb{R}^N$ be a closed stratifiable set and let $\{\mathcal{M}_i\}_{i\in\mathcal{I}}$ be its strata. Consider a SVF denoted by $G = \{(g_i, \mathcal{M}_i)\}_{i\in\mathcal{I}_0}$. Let $\xi : [0,T] \to \mathbb{B}$ be a measurable map and $\sigma : \mathcal{K} \to [0,+\infty)$ a continuous function that satisfies (14). Let $x(\cdot)$ be a stratified solution to ($\mathbf{D}^\sigma$) defined on $[0,T]$. Let $i \in \mathcal{I}$ and $j \in \mathcal{I}_0$ with $i \succeq j$ and* $\dim(\mathcal{M}_i) + 1 = \dim(M_j)$. *Assume that*

$$(16) \qquad \alpha_j(x) > 0 \quad \forall x \in \mathcal{M}_i.$$

Suppose that $x(t) \in \mathcal{M}_i \cup \mathcal{M}_j$ for every $t \in [0,T]$. Then, for any $\bar{x} \in \mathcal{M}_i$ there exists $\rho_i > 0$ such that if $x(0) \in \mathcal{M}_j \cap \mathbb{B}(\bar{x}, \rho_i)$ with $dist_{\mathcal{M}_i}(x(0)) \leq \rho$ for some $\rho \in (0, \rho_i)$ it is possible to find $\tau = \tau(\rho) \in (0,T]$ such that $x(\tau) \in \mathcal{M}_i$. Furthermore, $\tau(\rho) \to 0$ as $\rho \to 0$ and if $i \in \mathcal{I}_0$ then $x(t) \in \mathcal{M}_i$ on $[\tau, T]$.

Proof. By Proposition 3, $\mathcal{N}^C_{\overline{\mathcal{M}_j}}(x) \subseteq \mathcal{N}_{\mathcal{M}_i}(x)$ for any $x \in \mathcal{M}_i$. Besides, for any $\bar{x} \in \mathcal{M}_i$, there exists $r > 0$ such that for every $x \in \mathcal{M}_i \cap \mathbb{B}(\bar{x}, r)$, there is an orthonormal family of vectors $\{\eta_1(x), \dots, \eta_p(x)\}$ such that

$$\mathcal{N}_{\mathcal{M}_i}(x) = \text{span}\{\eta_1(x), \dots, \eta_p(x)\}.$$

Moreover, the maps $x \mapsto \eta_n(x)$ is continuous for every $n = 1, \dots, p$.

On the other hand, since $\overline{\mathcal{M}}_j$ is relatively wedged, $\mathcal{N}^C_{\overline{\mathcal{M}_j}}(x) = \mathcal{N}_1(x) \oplus \mathcal{N}_2(x)$ with $\mathcal{N}_2(x)$ a vector subspace of dimension q. Whence, as $\dim(\mathcal{M}_i) + 1 = \dim(M_j)$, $p = q + 1$ and without loss of generality,

$$\mathcal{N}_1(x) = \text{cone}\{\eta_1(x)\} \quad \text{and} \quad \mathcal{N}_2(x) = \text{span}\{\eta_2(x), \dots, \eta_p(x)\}.$$

Taking all this into account, one gets the following formula

$$\alpha_j(x) = \langle g_j(x), \eta_1(x) \rangle, \qquad x \in \mathcal{M}_i \cap \mathbb{B}(\bar{x}, r).$$

Moreover, it is always possible to assume that $\eta_1(x) = \sum_{n=1}^p \lambda_n \nabla h_n(x)$ with $\lambda_n \geq 0$ and $h_n(y) < 0$ for every $y \in \mathcal{M}_j \cap \mathbb{B}(\bar{x}, r)$.

Let $c_r > 0$ be an upper bound for $|g_j(x)| + |\sigma(x)|$ on $\mathbb{B}(\bar{x}, r)$ and reducing r if necessary, it is assumed that each ∇h_n is Lipschitz of modulus $L_n^r > 0$ on $\overline{\mathbb{B}}(\bar{x}, r)$

and by continuity of g_j and σ

$$\max\left\{|g_j(x)-g_j(\bar{x})|, |\sigma(x)-\sigma(\bar{x})|, \frac{r}{2L^r c_r}\right\} \leq \frac{1}{12}\alpha_j(\bar{x}), \quad \forall x \in \mathbb{B}(\bar{x}, r). \tag{17}$$

Let $\rho > 0$ to be fixed and suppose $x(0) \in \mathcal{M}_j \cap \mathbb{B}(\bar{x}, \rho)$. Set $h(t) = \sum_{n=1}^{p} \lambda_n h_n(x(t))$ and note that $h(0) < 0$. Let

$$T_r = \sup\{t \in [0,T] : h(t) < 0 \text{ and } x(t) \in \mathbb{B}(\bar{x}, r)\}.$$

Hence for a.e. $t \in (0, T_r)$ one gets

$$\begin{aligned}
\dot{h}(t) &= \sum_{n=1}^{p} \lambda_n \langle \nabla h_n(x(t)), \dot{x}(t)\rangle \\
&= \sum_{n=1}^{p} \lambda_n \langle \nabla h_n(x(t)), g_j(x(t))\rangle + \lambda_n \langle \nabla h_n(x(t)), \sigma(x(t))\xi(t)\rangle \\
&\geq \left\langle \sum_{n=1}^{p} \lambda_n \nabla h_n(x(t)), g_j(x(t)) \right\rangle - \sigma(x(t)) \left| \sum_{n=1}^{p} \lambda_n \nabla h_n(x(t)) \right|
\end{aligned}$$

Note that

$$\begin{aligned}
\left\langle \sum_{n=1}^{p} \lambda_n \nabla h_n(x(t)), g_j(x(t)) \right\rangle &= \alpha_j(\bar{x}) + \left\langle \sum_{n=1}^{p} \lambda_n \left(\nabla h_n(x(t)) - \nabla h_n(\bar{x})\right), g_j(x(t)) \right\rangle \\
&\quad \left\langle \sum_{n=1}^{p} \lambda_n \nabla h_n(\bar{x}), g_j(x(t)) - g_j(\bar{x}) \right\rangle \\
&\geq \alpha_j(\bar{x}) - L^r|x(t)-\bar{x}||g_j(x(t)| - |g_j(x(t)) - g_j(\bar{x})|
\end{aligned}$$

and

$$\left| \sum_{n=1}^{p} \lambda_n \nabla h_n(x(t)) \right| \leq 1 + L^r|x(t)-\bar{x}|.$$

Whence

$$\dot{h}(t) \geq \alpha_j(\bar{x}) - \sigma(\bar{x}) - L^r c_r|x(t)-\bar{x}| - |g_j(x(t)) - g_j(\bar{x})| - |\sigma(x(t)) - \sigma(\bar{x})| \geq \frac{1}{4}\alpha_j(\bar{x}).$$

Where this last inequality comes from (14) and (17). Therefore, $t \mapsto h(t)$ is strictly increasing and

$$h(t) \geq h(0) + \frac{1}{4}\alpha_j(\bar{x})t, \quad \forall t \in [0, T_r].$$

Hence, $T_r \leq -4h(0)\frac{1}{\alpha_j(\bar{x})}$ and since each h_n is locally Lipschitz, there exists a constant $\ell_r > 0$ such that

$$\left| \sum_{n=1}^{p} \lambda_n h_n(x) \right| \leq \ell_r \operatorname{dist}_{\mathcal{M}_i}(x) \leq \ell_r \rho, \quad \forall x \in \mathcal{M}_j \cap \mathbb{B}(\bar{x}, r),$$

and so, $T_r \leq \frac{4\ell_r \rho}{\alpha_j(\bar{x})}$. Nevertheless, since $x(\cdot)$ is continuous with essentially bounded derivatives, the time needed to escape from the $\mathbb{B}(\bar{x}, r)$, $\inf\{t \in [0,T] : x(t) \in \mathcal{M}_j \text{ with } |x(t)-\bar{x}| = r\}$ should increase as long as ρ goes to 0. Thus, there exists $\rho_i > 0$ such that

$$\inf\{t \in [0,T] : x(t) \in \mathcal{M}_j \text{ with } |x(t)-\bar{x}| = r\} > T_r$$

whenever $|x(0)-\bar{x}| \leq \rho \leq \rho_i$, and so $x(\tau) \in \mathcal{M}_i$ whenever $\rho \leq \rho_i$ with $\tau = T_r$.

Assume now that $i \in \mathcal{I}_0$ and $\tau < T$, for sake of simplicity, hereinafter $\tau = 0$. Suppose there exists $t \in (0, T]$ such that $x(t) \in \mathcal{M}_j$. Since $\mathcal{M}_j$ is relatively open on $\mathcal{M}_i \cup \mathcal{M}_j$, there exists $t_0 \in [0, t)$ such that $x(t_0) \in \mathcal{M}_i$ and $x(s) \in \mathcal{M}_j$ for every $s \in (t_0, t]$. To simply the notation, t_0 is set to be 0 and $\bar{x} = x(0)$. Let $\varepsilon \in \left(0, \frac{1}{2}\alpha_j(\bar{x})\right)$, then since σ and g_j are continuos, there exists $\delta > 0$ such that

$$|x-\bar{x}| < \delta \quad \Rightarrow \max\left\{|\sigma(x)-\sigma(\bar{x})|, |g_j(x)-g_j(\bar{x})|\right\} \leq \frac{\varepsilon}{3}.$$

By reducing t if necessary, $x(t) \in \mathbb{B}(\bar{x}, \delta)$ and

$$\max_{s\in[0,t]} \sigma(x(s)) \leq \sigma(\bar{x}) + \frac{\varepsilon}{3} \tag{18}$$

$$\max_{s\in[0,t]} |g_j(x(s)) - g_j(\bar{x})| \leq \frac{\varepsilon}{3} \tag{19}$$

Note that

$$0 > h(t) = \left\langle \sum_{n=1}^{p} \lambda_n \nabla h_n(\bar{x}), x(t)-\bar{x} \right\rangle + o(|x(t)-\bar{x}|^2)$$

So, by reducing again t if necessary,

$$\left\langle \frac{x(t)-\bar{x}}{t}, \eta_1(\bar{x}) \right\rangle \leq \frac{1}{3}\varepsilon. \tag{20}$$

On the other hand,

$$\begin{aligned}\left\langle \frac{x(t)-\bar{x}}{t}, \eta_1(\bar{x}) \right\rangle &= \frac{1}{t}\int_0^t \langle g_j(x(s)) + \sigma(x(s))\xi(s), \eta_1(\bar{x})\rangle ds \\ &\geq \alpha_j(\bar{x}) - \frac{1}{t}\int_0^t (|g_j(x(s)) - g_j(\bar{x})| + \sigma(x(s)))ds \\ &\geq \alpha_j(\bar{x}) - \sigma(\bar{x}) - \frac{2}{3}\varepsilon,\end{aligned}$$

where the last inequality is given by (18) and (19). Therefore, by 14 and (20)

$$\varepsilon \geq \alpha_j(\bar{x}) - \frac{1}{2}\alpha_j(\bar{x}) = \frac{1}{2}\alpha_j(\bar{x}).$$

Which contradicts the choice of ε, so such t_0 can not exist. □

Remark 10. The fact that $\dim(\mathcal{M}_i) + 1 = \dim(M_j)$ is crucial for the proof of the first part of the previous lemma. Indeed, if $\dim(\mathcal{M}_i) + 1 < \dim(M_j)$ the same conclusion of Lemma 5.3 does not hold, for example, consider the stratification $\mathcal{M}_0 = \{(0,0)\}$, $\mathcal{M}_1 = \{\min\{x+y, y\} < 0\}$, $\mathcal{M}_2 = \{\min\{x+y, y\} > 0\}$, $\mathcal{M}_3 = \{x+y=0,\ x<0\}$ and $\mathcal{M}_4 = \{y=0,\ x>0\}$ with $g_1 = (1,0)$, $g_2 = (0,-1)$, $g_3 = (1,-1)$ and $g_4 = (1,0)$.

It is not difficult to see that the couples $(\mathcal{M}_3, \mathcal{M}_1)$, $(\mathcal{M}_3, \mathcal{M}_2)$, $(\mathcal{M}_3, \mathcal{M}_2)$ verifies the result state in the lemma. Note also that $\alpha_1(0,0) = \frac{1}{\sqrt{2}} > 0$. However, no trajectory of the perturbed system starting from (x_0, y_0) of norm arbitrarily small but with $x_0 > 0$ and $y_0 < 0$ will reach $\mathcal{M}_0$.

A similar lemma can be stated when the sign of the outward-pointing modulus is negative. In this case, there is no restriction over the dimensions of the strata.

Lemma 5.4. *Let $\mathcal{K} \subseteq \mathbb{R}^N$ be a closed stratifiable set and let $\{\mathcal{M}_i\}_{i\in\mathcal{I}}$ be its strata. Consider a SVF denoted by $G = \{(g_i, \mathcal{M}_i)\}_{i\in\mathcal{I}_0}$. Let $\xi : [0,T] \to \mathbb{B}$ be a measurable map and $\sigma : \mathcal{K} \to [0,+\infty)$ a continuous function that satisfies (14). Let $x(\cdot)$ be a stratified solution to ($\mathbf{D}^\sigma$) defined on $[0,T]$. Let $i \in \mathcal{I}$ and $j \in \mathcal{I}_0$ with $i \succeq j$. Assume that*

$$\alpha_j(x) < 0 \quad \forall x \in \mathcal{M}_i. \tag{21}$$

Suppose that $x(t) \in \mathcal{M}_i \cup \mathcal{M}_j$ for every $t \in [0,T]$ with $x(0) \in \mathcal{M}_i$. Let

$$\tau = \inf\{t \in [0,T] : x(t) \notin \mathcal{M}_i\}.$$

If $\tau < T$ then $x(t) \in \mathcal{M}_j$ on $(\tau, T]$.

Proof. Let $\bar{x} \in \mathcal{M}_i$ and $r > 0$ as in the previous lemma such that $x(0) \in \mathbb{B}(\bar{x}, r)$. Without loss of generality, $\tau = 0$ and let $\tau_1 > 0$ be the maximal time such that $x(t) \in \mathcal{M}_j \cap \mathbb{B}(\bar{x}, r)$ for every $t \in (0, \tau_1)$. By contradiction, suppose that $\tau_1 < T$

For any $\eta \in \mathcal{N}^C_{\mathcal{M}}(x)$ there exists a representation as

$$\eta = \sum_{n=1}^{p} \lambda_n \nabla h_n(\bar{x}).$$

Setting $h(t) = \sum_{n=1}^{p} \lambda_n h_n(x(t))$ and using the similar estimations as in Lemma 5.3, one can show that, $\dot{h}(t) < \frac{1}{4}\alpha_j(\bar{x})$ for almost all $t \in (0, \tau_1)$, and so, $t \mapsto h(t)$ is strictly decreasing. Note that $h(0) = 0$ for any η, so $x(\tau_1) \in \mathcal{M}_j$ with $|x(\tau_1) - \bar{x}| = r$ and the conclusion follows easily. □

5.5. **Proof of Theorem 5.2.**

Proof (Thm. 5.2). By ($\mathbf{H_1}$), the sets of index $\mathcal{I}_i^+(x)$, $\mathcal{I}_i^-(x)$ and $\mathcal{I}_i^0(x)$ are independent of x, so for sake of notation the dependence with respect to x is suppressed. Note that the vector field $g(x) = g_l(x)$ whenever $x \in \mathcal{M}_l$ is continuous on

$$\mathcal{S}_i = \bigcup\{\mathcal{M}_j : \ j \in \mathcal{I}_i^0\}, \quad \text{whenever } i \in \mathcal{I}_0.$$

Note also that $\mathcal{S}_i$ is locally closed around each $x \in \mathcal{M}_i$.

To prove the statement of the theorem it is enough to show that for some $\tau > 0$, $x|_{[0,\tau]}$ is a stratified solution. For this purpose, let $i = \imath(x_0)$ and $i_n = \imath(x_0^n)$, where $x_0 = x(0)$. Since the stratification is locally finite, $\{i_n\}$ is compact and so, for simplicity, it is assumed that $i_n = j \in \mathcal{I}$ for any $n \in \mathbb{N}$ with $i \preceq j$.

Suppose that $i \in \mathcal{I}_0$, then by ($\mathbf{H_2}$), $j \in \mathcal{I}_0$ as well. Let $R > 0$ such that $\mathcal{M}_i$ is the stratum of lower dimension on $\mathcal{K} \cap \overline{\mathbb{B}}(x_0, R)$ and let

$$\tau_n := \inf\left\{t \in [0,T] : \ x_n(t) \notin \mathcal{K} \cap \overline{\mathbb{B}}(x_0, R)\right\}.$$

Note that, since the set of velocities associated with (15) can be bounded uniformly with respect to n on $\mathcal{K} \cap \overline{\mathbb{B}}(x_0, R)$, there exists $\tau > 0$ such that $\tau_n > \tau$ for any $n \in \mathbb{N}$. Recall that $x_0^n \to x_0$, so it can be as close as wanted of $\mathcal{S}_i$. Moreover, it is not difficult to see that by Lemma 5.3 and ($\mathbf{H_2}$) any arc $x_n(\cdot)$ reaches $\mathcal{S}_i$ within time t_n, and even more, the sequence $\{t_n\}$ converges to zero as long as n goes to infinity. So, without loss of generality, $j \in \mathcal{I}_i^0$. Once again, by Lemma 5.3 and ($\mathbf{H_2}$), no trajectory of (15) can pass to a stratum of bigger dimension as long as the outward pointing modulus related to this stratum has positive sign. Therefore,

$x_n(t) \in \mathcal{S}_i$ for any $t \in [0, \tau]$ and any $n \in \mathbb{N}$. Recall that g is continuous on each $\mathcal{S}_i$, so the conclusion follows by passing into the limit in

$$x_n(t) = x_0^n + \int_0^t [g(x_n) + \sigma(x_n)\xi_n]\, ds.$$

Assume that $i \notin \mathcal{I}_0$. Suppose first that $j \in \mathcal{I}_0$ and then, using the same arguments as in the previous part applied to j instead of i, it can be shown that there exists $\tau_n > 0$ such that $x_n(t) \in \mathcal{S}_j$ for any $t \in [0, \tau_n]$ and any $n \in \mathbb{N}$. Now, by (**H₂**), $\alpha_j(x_0) < 0$ then by Lemma 5.4, τ_n can be uniformly bounded from below for a positive number, so it is possible to pass to the limit and get the conclusion.

Now consider the case $j \notin \mathcal{I}_0$, by (**H₂**), $\mathcal{I}_0(j) = \mathcal{I}_j^-$ and $\mathcal{I}_0(i) = \mathcal{I}_i^-$, and by Lemma 5.3, each trajectory $x_n(\cdot)$ can only dwell in strata whose indices belong to $\mathcal{I}_j^-$. So, by Lemma 5.4 there exists $k \in \mathcal{I}_j^-$ such that $x_n(t) \in \mathcal{M}_k$ in a maximal interval $(t_n, \tau_n]$ with $\tau_n > 0$. Since by (**H₂**), $k \in \mathcal{I}_0$ then by Lemma 5.3 then the sequence $\{\tau_n\}$ is uniformly bounded from below by a positive number $\tau > 0$ and using the same argument as before one can pass to the limit and get the desired result. So the proof is completed.

□

Appendix A. Properties of the outward-pointing modulus

This appendix has as a principal aim to discuss *how continuous* can be the outward-pointing modulus of a vector field. Thanks to the relatively wedgedness of the closure of the manifold, it can be shown that this is upper semicontinuous. Continuity along a submanifold contained in the frontier of the manifold is also achieved.

Proposition 7. *Let $\mathcal{M}$ be an embedded manifold of $\mathbb{R}^N$ with $\overline{\mathcal{M}}$ relatively wedged and consider $g : \overline{\mathcal{M}} \to \mathbb{R}^N$ continuous on $\overline{\mathcal{M}} \setminus \mathcal{M}$, then:*

(1) α_g upper semi-continuous on $\overline{\mathcal{M}} \setminus \mathcal{M}$.
(2) Let $\mathcal{M}_b$ be another embedded manifold of $\mathbb{R}^N$ such that $\mathcal{M}_b \subseteq \overline{\mathcal{M}} \setminus \mathcal{M}$.
   (a) Suppose $\mathcal{N}^C_{\mathcal{M}}(\cdot)$ is lower semicontinuous restricted to $\mathcal{M}_b$, then α_g is continuous restricted to $\mathcal{M}_b$.
   (b) Suppose g and $\mathcal{N}^C_{\mathcal{M}}(\cdot)$, both restricted to $\mathcal{M}_b$, are locally Lipschitz, then α_g is locally Lipschitz on $\mathcal{M}_b$.

Recall that a set-valued map $\Upsilon : \mathbb{R}^N \rightrightarrows \mathbb{R}^N$ is said to be lower semicontinuous at x if for any $x \in \Upsilon(x)$ and $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\forall y \in \operatorname{dom} \Upsilon \cap \mathbb{B}(x, \delta) \quad \Rightarrow \quad v \in \Upsilon(y) + \mathbb{B}(0, \varepsilon).$$

Moreover, if each $\Upsilon(x)$ is a closed cone, then it is said locally Lipschitz if for every $r > 0$ there exists $L_r > 0$ such that

$$\mathcal{D}(\Upsilon(x), \Upsilon(y)) \leq L_r |x - y| \qquad \forall x, y \in \operatorname{dom} \Upsilon \cap \mathbb{B}(0, r).$$

Proof. For sake of simplicity assume that $\mathcal{M}$ is a $N-$dimensional submanifold of $\mathbb{R}^N$, otherwise it is enough to replace the condition $|\eta| = 1$ by $|\eta| = |\pi_x(\eta)| = 1$.

(1) Let $x \in \overline{\mathcal{M}} \setminus \mathcal{M}$ and take $\{x_n\} \subseteq \overline{\mathcal{M}} \setminus \mathcal{M}$ such that $x_n \to x$. Then, by compactness, there exists $\eta_n \in \mathcal{N}^C_{\mathcal{M}}(x_n)$ with $|\eta_n| = 1$ such that

$$\alpha_g(x_n) = \langle g(x_n), \eta_n \rangle.$$

Without lose of generality, assume that $\eta_n \to \eta$, with $|\eta| = 1$. We know that the multifunction $x \mapsto \mathcal{N}^C_{\mathcal{M}}(x)$ has closed graph on $\overline{\mathcal{M}}$. So $\eta \in \mathcal{N}^C_{\mathcal{M}}(x)$. Thus,

$$\lim_{n\to\infty} \alpha_g(x_n) = \langle g(x), \eta\rangle \leq \alpha_g(x).$$

(2) It is enough to show that $\alpha_g(\cdot)$ restricted to $\mathcal{M}_b$ is lower semi-continuous. So, let $x \in \mathcal{M}_b$ and $\{x_n\} \subseteq \mathcal{M}_b$ such that $x_n \to x$. Since, $\mathcal{N}^C_{\mathcal{M}}(\cdot)$ is lsc restricted to $\mathcal{M}_b$, for any $\eta \in \mathcal{N}^C_{\mathcal{M}}(x)$, we can assume that, passing to a subsequence if necessary, there exists a sequence $\eta_n \in \mathcal{N}^C_{\mathcal{M}}(x_n)$ such that $\eta_n \to \eta$. Let $\bar\eta \in \mathcal{N}^C_{\mathcal{M}}(x)$ satisfying $\alpha_g(x) = \langle g(x), \bar\eta\rangle$, then $\forall n \in \mathbb{N}$

$$\alpha_g(x_n) \geq \langle g(x_n), \eta_n\rangle \geq \langle g(x_n), \bar\eta\rangle - |g(x_n)||\eta_n - \bar\eta|.$$

So, taking the liminf in the last inequality proof is completed.

(3) Let $x_0 \in \mathcal{M}_b$ and $r > 0$, consider $x, y \in \mathbb{B}(x_0, r) \cap \mathcal{M}_b$. Let L_r be the Lipschitz constant of $g(\cdot)$ and $\mathcal{N}^C_{\mathcal{M}}(\cdot)$ on $\mathbb{B}(x_0, r)\cap\mathcal{M}_b$. Consider too $C_r > 0$ an upper bound for the norm of g on $\mathbb{B}(x_0, r) \cap \mathcal{M}_b$. By compactness, take $\eta_x \in \mathcal{N}^C_{\mathcal{M}}(x) \cap \mathbb{S}$ such that $\alpha_g(x) = \langle g(x), \eta_x\rangle$, then $\forall \eta_y \in \mathcal{N}^C_{\mathcal{M}}(y) \cap \mathbb{S}$

$$\begin{aligned}\alpha_g(x) - \alpha_g(y) &\leq \langle g(x), \eta_x\rangle - \langle g(y), \eta_y\rangle,\\ &\leq \langle g(x) - g(y), \eta_x\rangle + \langle g(y), \eta_x - \eta_y\rangle,\\ &\leq L_r|x - y| + C_r|\eta_x - \eta_y|.\end{aligned}$$

Therefore, taking the infimum over all the vectors of $\mathcal{N}^C_{\mathcal{M}}(y) \cap \mathbb{S}$

$$\alpha_g(x) - \alpha_g(y) \leq L_r|x - y| + C_r \mathrm{dist}_{\mathcal{N}^C_{\mathcal{M}}(y)\cap\mathbb{S}}(\eta_x),$$

but,

$$\mathrm{dist}_{\mathcal{N}^C_{\mathcal{M}}(y)\cap\mathbb{S}}(\eta_x) \leq \mathcal{D}(\mathcal{N}^C_{\mathcal{M}}(x), \mathcal{N}^C_{\mathcal{M}}(y)) \leq L_r|x - y|.$$

Finally, since it is possible to change the roles between x and y, there exists a constant $L_r(\alpha_g) > 0$ such that

$$|\alpha_g(x) - \alpha_g(y)| \leq L_r(\alpha_g)|x - y| \qquad \forall x, y \in \mathbb{B}(x_0, r) \cap \mathcal{M}_b.$$

□

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