PARETO FRONTS OF THE SET OF SUSTAINABLE THRESHOLDS FOR CONSTRAINED CONTROL SYSTEMS

PEDRO GAJARDO AND CRISTOPHER HERMOSILLA

ABSTRACT. This paper is concerned with a dual object of the so-called viability kernel of a control system with mixed (state-control) constraints. This object, called the set of sustainable thresholds, describes the possible thresholds for which a given initial condition is viable. In this work, we are concerned with characterizing the weak and strong Pareto fronts of the set of sustainable thresholds and providing a practical method for computing such objects based on optimal control theory and a level-set approach. A numerical example, relying on renewable resource management, is shown to demonstrate the proposed method.

1. Introduction

This paper is concerned with the analysis of discrete-time control problems with mixed (state-control) constraints. Particular emphasis is placed on the *set of sustainable thresholds* ([3, 18, 21, 20]), which is the collection of all possible thresholds for which a given initial condition is viable or sustainable throughout time. The latter means that some control, along with its corresponding controlled trajectory, satisfies prescribed mixed constraints parametrized by thresholds. Accordingly, the set of sustainable thresholds provides a good picture of the current state of a system in terms of the thresholds that can be maintained in a sustainable way. A small set of sustainable thresholds means that the current state of the system is vulnerable in the sense that the room for maneuvering in terms of sustainability is reduced. In addition, the set of sustainable thresholds is, in a certain sense, a dual object of the so-called viability kernel ([2]). On the one hand, the set of sustainable thresholds relates to the parameters (thresholds) that make a given initial condition viable; on the other hand, the viability kernel is concerned with the initial conditions that are sustainable throughout time for a given set of thresholds.

The aim of this work is twofold. First, we characterize the weak and strong Pareto fronts of the set of sustainable thresholds and second, we provide a practical method for approximating the set of sustainable thresholds for a finite horizon setting. To accomplish these goals, we use optimal control tools and show that by using some suitable costs, one can: (i) describe the strong Pareto front of the set of sustainable thresholds by solving a finite number of optimal control problems and (ii) characterize the weak Pareto front as the zero level-set of an auxiliary value

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function. We then use this characterization and the dynamic programming principle to provide an implementable scheme for approximating the set of sustainable thresholds and its weak Pareto front.

Let us emphasize that the mixed constraint setting in this work appears in control theory in several frameworks. Two notable cases, which are the main motivations for this work, are sustainable management problems and bio-economic modeling; cf. [4, 5, 6, 9, 10, 11, 12, 13, 14, 16, 19, 23, 24, 25].

Finally, we remark that the approach that we have taken for characterizing the weak Pareto front of the set of sustainable thresholds is inspired by the level-set approach [22, 1] used for a time-continuous optimal control problem with (pure) state constraints. In addition, one of our main results (Theorem 4.1) is similar in essence to the works cited in [8, 17] for studying the admissible set of a constrained control system. The common point is that both works use a minimax problem to characterize the boundary of the object of interest; in our case, this is the set of sustainable thresholds and in theirs, it is the admissible set.

This manuscript is organized as follows. In Section 2, we present the problem, formally define the set of sustainable thresholds and introduce the basics for the rest of the paper. In Sections 3 and 4, we study the strong and weak Pareto fronts, respectively, of the set of sustainable thresholds. Finally, in Section 5, we report some numerical simulations that illustrate the scheme proposed for computing the set of sustainable thresholds.

2. Preliminaries on discrete-time control systems

Given a finite horizon $N \in \mathbb{N}$, an initial state $\xi \in \vec{X}$ and a finite sequence of controls $\vec{u} = (u_k)_{k=0}^N$, we consider the discrete-time control system:

$$(D_\xi^{\vec{u}}) \qquad x_{k+1} = F(k, x_k, u_k), \quad k \in [\![0 : N]\!], \quad x_0 = \xi.$$

In this setting, $F : [\![0 : N]\!] \times \vec{X} \times \vec{U} \to \vec{X}$ is the dynamics, $\vec{X}$ is a vector space (called the state space) and $\vec{U}$ is the control space. Here, $[\![p : q]\!]$ stands for the collection of all integers between p and q (inclusive).

We denote by $\mathbb{U}$ the collection of all possible controls; that is,

$$\mathbb{U} := \left\{ \vec{u} = (u_k)_{k=0}^N \,\middle|\, u_0, \ldots, u_N \in \vec{U} \right\} \cong \vec{U}^{N+1}.$$

A solution of the control system $(D_\xi^{\vec{u}})$ associated with a control $\vec{u} \in \mathbb{U}$ is an element of the space

$$\mathbb{X} := \left\{ \vec{x} = (x_k)_{k=0}^{N+1} \,\middle|\, x_0, \ldots, x_{N+1} \in \vec{X} \right\} \cong \vec{X}^{N+2},$$

that satisfies the initial time condition $x_0 = \xi$.

Consequently, a solution of $(D_\xi^{\vec{u}})$, which is uniquely determined by the control $\vec{u}$ and initial state ξ, is denoted in the sequel by $\vec{x}_\xi(\vec{u})$ to emphasize its dependence on the initial data of the problem (control and state).

2.1. **Constraints and sustainable thresholds.** In many real applications, the outputs and inputs of dynamical systems such as $(D_\xi^{\vec{u}})$ are restricted to a prescribed set. These constraints reflect physical or economical restrictions. In this work, we are concerned with the so-called *mixed constraints*, that is, a set of restrictions that the control $\vec{u} \in \mathbb{U}$ together with its corresponding trajectories $\vec{x}_\xi(\vec{u}) = (x_k)_{k=0}^{N+1}$

must satisfy. This context is more general than what is usually addressed in the literature, where theory has traditionally focused on pure control or pure state constraints (separately). In this paper, we are mainly concerned with constraints that can be represented as the level-set of a given constraint mapping $g : [\![0 : N]\!] \times \vec{X} \times \vec{U} \to \mathbb{R}^m$

$$g(k, x_k, u_k) \leq c, \quad \forall\ k \in [\![0 \colon N]\!]. \tag{I^c}$$

The mixed constraints are determined by a parameter $c \in \mathbb{R}^m$, which is a given vector of thresholds. The focus is on this parameter rather than on the initial conditions as in viability theory [2].

In particular, we are interested in finding, for a given initial state, all possible thresholds $c \in \mathbb{R}^m$ for which that initial condition is sustainable throughout time. The latter means that some control, along with its corresponding controlled trajectory, satisfy the mixed constraints (I^c). The collection of all such thresholds is called *set of sustainable thresholds* and is defined for a given initial condition $\xi \in \vec{X}$ as follows:

$$\mathbb{S}(\xi) := \{c \in \mathbb{R}^m \ \mid\ \exists\ \vec{u} \in \mathbb{U},\, \vec{u} \text{ and } \vec{x}_\xi(\vec{u}) \text{ satisfy } (\mathrm{I}^c)\}\,. \tag{1}$$

Remark 2.1. It is not difficult to see that for any $\xi \in \vec{X}$, its set of sustainable thresholds is always nonempty. Note as well that, if the dynamics F are jointly linear in state and control, the control space $\vec{U}$ is a convex subset of a vector space, and if the components g_i of the constraint mapping g are also jointly convex in state and control, then $\mathbb{S}(\xi)$ is a convex subset of $\mathbb{R}^m$.

In this paper, for a given threshold vector $c \in \mathbb{R}^m$, the viability kernel ([2, 9]) is given by

$$\mathbb{V}(c) := \left\{\xi \in \vec{X} \ \mid\ \exists\ \vec{u} \in \mathbb{U},\, \vec{u} \text{ and } \vec{x}_\xi(\vec{u}) \text{ satisfy } (\mathrm{I}^c)\right\}.$$

Thus, the viability kernel and the set of sustainable thresholds are related via the equivalence below, which somewhat explains the duality between these two objects: for any $\xi \in \vec{X}$ and $c \in \mathbb{R}^m$ we have

$$\xi \in \mathbb{V}(c) \quad \iff \quad c \in \mathbb{S}(\xi). \tag{2}$$

The importance of the set of sustainable thresholds lies in the trade-off between the number of restrictions $m \in \mathbb{N}$ and the dimension of the state space $\vec{X}$. For problems with several state variables, computing $\mathbb{V}(c)$ may be too expensive or impractical in terms of computational time, even if there are only a couple of restrictions; this is the so-called *curse of dimensionality in dynamic programming*. However, in the same situation (several state variables with few constraints), the computational time of estimating $\mathbb{S}(\xi)$ can be considerably lowered because, essentially, the complexity of computing $\mathbb{V}(c)$ and $\mathbb{S}(\xi)$ is the same, but the latter is an object in a lower-dimensional Euclidean space.

In some applications, we believe that the set of sustainable thresholds is visually more appealing because it contains more useful information than the viability kernel and has mathematical properties (not shared by the viability kernel) that should make its numerical computation more tractable, as we show in this work. For instance, in natural resource management problems where the state of the resources studied is represented by several variables (e.g., age-class models), the viability kernel cannot provide visual information for decision-makers who are considering constraints. This case does not occur when one has few constraints to satisfy, for instance, two: one preservation-type constraint (e.g., to maintain at least a minimal level of the resource) and one production-type constraint (e.g., to maintain at least a minimal level of harvesting or profit). In this situation, the set of sustainable thresholds can be illustrated in the plane (two dimensions associated with the two constraints) as shown in [18]. Thus, this set represents the good or bad health of the system under study in terms of sustainability, allowing visualization of the existing trade-offs between different constraints in defining sustainability. In these contexts, $\mathbb{S}(\xi)$ also provides a good picture of the current state ξ in terms of the thresholds that can be maintained in a sustainable way. A small set $\mathbb{S}(\xi)$ means that the current state ξ is vulnerable in the sense that the room for maneuvering in terms of sustainability is reduced. In Figure 1, we illustrate the set of sustainable thresholds for two different initial states ξ and ξ', where (I^c) consists of only two constraints (i.e., the threshold space is of dimension two). In this illustration, we can see that the state ξ is better than ξ' in the sense that $\mathbb{S}(\xi') \subset \mathbb{S}(\xi)$.

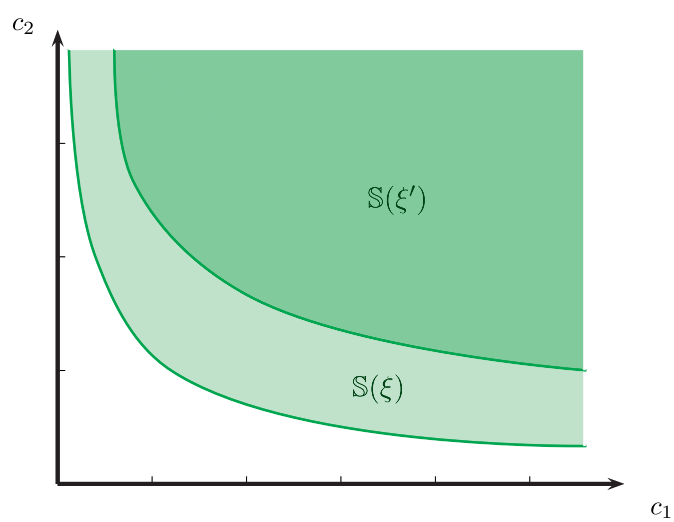

Figure 1. Sketch of the set of sustainable thresholds for two different initial states.

The dependence of the set of sustainable thresholds with respect to the initial condition, that is, the regularity of the set-valued map $\xi \mapsto \mathbb{S}(\xi)$, is not studied in this work. Nevertheless, we note that under mild assumptions, such as the standing assumptions we consider later on, this set-valued map has a closed graph. Moreover, under slightly stronger assumptions, it is also possible to show that $\xi \mapsto \mathbb{S}(\xi)$ is lower semicontinuous. However, since this set-valued map has unbounded images, properties such as upper semicontinuity are unlikely to hold. Appropriate notions of (semi)continuity for set-valued maps with unbounded images, such as sub-Lipschitz continuity, can be considered but require further developments that are beyond the scope of this work. We plan to study this more thoroughly elsewhere.

2.2. **Pareto fronts.** Because of the structure of the mixed constraint (I^c), it is clear that if $c^* \leq c$ (component-wise), then for any $\xi \in \vec{X}$, we have

$$c^* \in \mathbb{S}(\xi) \quad \Longrightarrow \quad c \in \mathbb{S}(\xi). \tag{3}$$

In other words, $\mathbb{S}(\xi) + \mathbb{R}^m_+ = \mathbb{S}(\xi)$, and therefore, the set of sustainable thresholds can be characterized by its boundary, particularly by its weak Pareto front; See Remark 2.2 below. Let us recall that a given vector $c^* \in \mathbb{R}^m$ is said to be (Pareto) dominated by another vector $c \in \mathbb{R}^m$ if $c \leq c^*$ (component-wise) and there exists $i \in [\![1 : m]\!]$ such that $c_i < c_i^*$. Additionally, c^* is said to be strongly (Pareto) dominated by c if $c < c^*$ (component-wise). With these concepts at hand, the weak and strong Pareto fronts of any set $S \subset \mathbb{R}^m$ are defined as follows:

- The strong Pareto front of S is the set of all $c^* \in S$, which are not dominated by another element of S; that is, $c^* \in S$ belongs to the strong Pareto front of S provided that
$$\forall c \in S, \quad c \leq c^* \quad \Longrightarrow \quad c = c^*.$$
- The weak Pareto front of S is the collection of all $c^* \in S$, which are not strongly dominated by another element of S. In other words, $c^* \in S$ belongs to the weak Pareto front of S provided that
$$\forall c \in S, \quad \exists i \in [\![1 : m]\!], \quad c_i^* \leq c_i.$$
The elements of the strong and weak Pareto front are called (respectively) strong and weak Pareto minimals.

Our task in this paper is to study the weak and strong Pareto fronts of the set of sustainable thresholds $\mathbb{S}(\xi)$ for a given initial condition $\xi \in \vec{X}$ in order to obtain a full description of this set. The approach we have taken is based on optimal control theory. The details are explained in the next section.

Remark 2.2. First, notice that strong Pareto minimals are also weak Pareto minimals. So, from (3) one can deduce that

$$\mathcal{P}^s\left(\mathbb{S}(\xi)\right) + \mathbb{R}^m_+ \subseteq \mathcal{P}^w\left(\mathbb{S}(\xi)\right) + \mathbb{R}^m_+ \subseteq \mathbb{S}(\xi), \tag{4}$$

where $\mathcal{P}^s(S)$ and $\mathcal{P}^w(S)$ denote the strong and weak Pareto fronts of a set S, respectively. Thanks to the structure of the mixed constraint (I^c) and the standing assumptions, we will be able to prove (see Theorem 3.6) that for any $c \in \mathbb{S}(\xi)$ one can find a strong Pareto minimal $c^* \in \mathbb{S}(\xi)$ such that $c^* \leq c$. This in turn shows that the inclusions in (4) are attained as equalities, and therefore, the strong (or weak) Pareto front of $\mathbb{S}(\xi)$ allows to recover the whole set of sustainable thresholds. Moreover, by the same arguments it follows that $\mathcal{P}^s(\mathbb{S}(\xi))$ is the smallest subset of $\mathbb{S}(\xi)$ that allows to recover $\mathbb{S}(\xi)$ by adding the cone $\mathbb{R}^m_+$, which means that the strong Pareto front of $\mathbb{S}(\xi)$ can be interpreted in some sense as the extreme points of $\mathbb{S}(\xi)$ whenever this set is convex.

2.3. **Standing assumptions.** In this work, we assume that the data of the dynamical system ($D_\xi^{\vec{u}}$) with mixed constraints (I^c) satisfy the following basic conditions, which we term *standing assumptions*:

- **(H1)** $F(k,\cdot,\cdot)$ is continuous on $\vec{X}\times\vec{U}$ for any $k\in[\![0\!:\!N]\!]$
- **(H2)** $g(k,\cdot,\cdot)$ is lower semicontinuous on $\vec{X}\times\vec{U}$ for $k\in[\![0\!:\!N]\!]$.
- **(H3)** $\vec{X}$ is a finite-dimensional Banach space.
- **(H4)** $\vec{U}$ is a nonempty compact metric space.

These hypotheses ensure that, for a given threshold vector $c\in\mathbb{R}^m$, the set of feasible solutions to the dynamical system ($D_\xi^{\vec{u}}$)-(I^c) is compact in $\mathbb{X}$. Since $\vec{u}\mapsto\vec{x}_\xi(\vec{u})$ is a continuous map, and $\mathbb{U}$ is a compact metric space,

$$A_\xi := \{(\vec{x},\vec{u})\in\mathbb{X}\times\mathbb{U}\mid \vec{x}=\vec{x}_\xi(\vec{u})\}$$

is a compact subset of $\mathbb{X}\times\mathbb{U}$. Furthermore, the set

$$B_c := \{(\vec{x},\vec{u})\in\mathbb{X}\times\mathbb{U}\mid g(k,x_k,u_k)\le c,,\ \forall k\in[\![0\!:\!N]\!]\}$$

is closed in $\mathbb{X}\times\mathbb{U}$. Now, since the set of admissible trajectories is exactly the projection of $A_\xi\cap B_c$ over $\vec{X}$, we conclude that set of feasible solutions to the dynamical system ($D_\xi^{\vec{u}}$) with mixed constraints (I^c) is compact (possibly empty) in $\mathbb{X}$. A similar argument shows that the set of admissible controls for the dynamical system ($D_\xi^{\vec{u}}$) with mixed constraints (I^c) is compact in $\mathbb{U}$.

3. The set of sustainable thresholds through optimal control

Let us consider a generic optimal control problem and its corresponding optimal value (viewed as a function of the threshold vector):

$$\vartheta_\xi(c) := \inf_{\vec{u}\in\mathbb{U}}\left\{\mathcal{J}(\vec{x}_\xi(\vec{u}),\vec{u})\mid \vec{u}\text{ and }\vec{x}_\xi(\vec{u})\text{ satisfy }(\mathrm{I}^c)\right\},\tag{5}$$

where the function $\mathcal{J}:\mathbb{X}\times\mathbb{U}\to\mathbb{R}$ is a generic cost that will take a closed form when appropriate.

Remark 3.1. The standing assumptions imply that $\vartheta_\xi(c)>-\infty$ for any $\xi\in\vec{X}$ provided that $\mathcal{J}$ is lower semicontinuous. This follows from the fact that the set of feasible solutions to the dynamical system ($D_\xi^{\vec{u}}$) with mixed constraints (I^c) is compact in $\mathbb{X}$. If that set is empty, then $\vartheta_\xi(c)=+\infty$; otherwise, the solution is attained and so $\vartheta_\xi(c)>-\infty$; the latter is a consequence of minimizing a lower semicontinuous map over a compact set.

Note that the set of sustainable thresholds $\mathbb{S}(\xi)$ is the collection of all thresholds c for which $\vartheta_\xi(c)$ is finite; this holds true for any finite and lower semicontinuous choice we make for the cost $\mathcal{J}$. This fact is summarized below.

Proposition 3.2. *For any $\xi\in\vec{X}$ and $c\in\mathbb{R}^m$, one has*

$$c\in\mathbb{S}(\xi)\quad\Longleftrightarrow\quad\vartheta_\xi(c)<+\infty.$$

This relation implies that if one wants to determine the set of sustainable thresholds, one may instead solve an optimization problem to check whether a given threshold is sustainable for a given initial state. Furthermore, this also suggests that, to compute the strong Pareto front of $\mathbb{S}(\xi)$, one might try to construct a suitable functional $\mathcal{J}$ from the constraint mapping g. Inspired by this idea, we provide a scheme for constructing the strong Pareto front of $\mathbb{S}(\xi)$. The main feature of this procedure is that it works from the inside in the sense that, starting from a given sustainable threshold, it provides a strong Pareto minimal.

3.1. **A characterization of the strong Pareto front.** We now show that for a given initial state $\xi \in \vec{X}$, the strong Pareto minimals of $\mathbb{S}(\xi)$ can be computed by solving a sequence of m (the dimension of the constraint space) optimal control problems. For this purpose, we will construct a scheme by considering a sequence of minimax problems obtained by setting the functional $\mathcal{J}$ to

$$\mathcal{J}^i(\vec{x}, \vec{u}) := \max_{k=0,\dots,N} g_i(k, x_k, u_k), \qquad \text{for some } i \in [\![1:m]\!]. \tag{6}$$

For the sake of exposition, let us introduce the mapping $\mathbf{c} : \mathbb{U} \to \mathbb{R}^m$ given by

$$\mathbf{c}(\vec{u}) := \left(\max_{k=0,\dots,N} g_1(k, x_k, u_k), \dots, \max_{k=0,\dots,N} g_m(k, x_k, u_k) \right), \tag{7}$$

where $\vec{u} = (u_k)_{k=0}^N$ and $\vec{x}_\xi(\vec{u}) = (x_k)_{k=0}^{N+1}$ is its corresponding trajectory.

Remark 3.3. Let us note that the images of the mapping $\mathbf{c} : \mathbb{U} \to \mathbb{R}^m$ introduced above are actually sustainable thresholds. Indeed, let $\vec{u} \in \mathbb{U}$ be a control and $\vec{x}_\xi(\vec{u}) = (x_k^i)_{k=0}^{N+1}$ be the corresponding trajectory. Then, in particular, $\mathbf{c}(\vec{u}) \in \mathbb{S}(\xi)$ because, by definition, we have

$$g(k, x_k, u_k) \leq \mathbf{c}(\vec{u}) \quad \text{(component-wise)}, \ \forall k \in [\![0:N]\!].$$

Definition 3.4. Given $\xi \in \vec{X}$, we say that a set-valued mapping $\mathcal{P}_\xi : [\![1:m]\!] \times \mathbb{S}(\xi) \rightrightarrows \mathbb{R}^m$ is a Pareto operator provided that

$$\mathcal{P}_\xi(i, c) = \{\mathbf{c}(\vec{u}^i) \mid \vec{u}^i \in \mathbb{U} \text{ and } \ \vartheta_\xi^i(c) = \mathcal{J}^i(\vec{x}_\xi(\vec{u}^i), \vec{u}^i)\},$$

where

$$\vartheta_\xi^i(c) := \inf_{\vec{u} \in \mathbb{U}} \left\{ \mathcal{J}^i(\vec{x}_\xi(\vec{u}), \vec{u}) \middle| \quad \vec{u} \text{ and } \vec{x}_\xi(\vec{u}) \text{ satisfy } (\mathrm{I}^c) \ \right\}. \tag{8}$$

Note that in Definition 3.4, $\mathcal{J}^i$ and $\mathbf{c}$ are the mappings given by (6) and (7), respectively. Let us now see that a Pareto operator has nonempty values, and moreover, its images are sustainable thresholds.

Proposition 3.5. *For any $\xi \in \vec{X}$, a Pareto set-valued operator $\mathcal{P}_\xi : [\![1:m]\!] \times \mathbb{S}(\xi) \rightrightarrows \mathbb{R}^m$ has nonempty values and*

$$\mathcal{P}_\xi(i, c) \subset \mathbb{S}(\xi), \qquad \forall i \in [\![1:m]\!], \ c \in \mathbb{S}(\xi).$$

Proof. To see that a Pareto operator has nonempty values, it is enough to see that $\vartheta_\xi^i(c) < +\infty$ because the existence of at least one optimal control is guaranteed by the standing assumptions under these circumstances (Remark 3.1). For a given $c \in \mathbb{S}(\xi)$, there is by definition $\vec{u} \in \mathbb{U}$ such that $\vec{u}$ together with $\vec{x}_\xi(\vec{u})$ satisfy (I^c). In particular, the optimal control problem associated with $\vartheta_\xi^i(c)$ is feasible, which means its value is finite and attained at some control $\vec{u}^i \in \mathbb{U}$.

Finally, the fact that $\mathcal{P}_\xi(i, c) \subset \mathbb{S}(\xi)$ follows from Remark 3.3. □ □

With the notion of the Pareto operator at hand, we are now ready to introduce a scheme for finding strong Pareto minimals of the set of sustainable thresholds as claimed above. In particular, the following theorem implies that for any $c \in \mathbb{S}(\xi)$ one can find a strong Pareto minimal $c^* \in \mathbb{S}(\xi)$ such that $c^* \leq c$ (component-wise) as claimed in Remark 2.2.

Theorem 3.6. *For any initial condition $\xi \in \vec{X}$, sustainable threshold $c^0 \in \mathbb{S}(\xi)$ and permutation $\sigma : [\![1:m]\!] \to [\![1:m]\!]$, consider the sequence $c^1, \dots, c^m$ generated inductively by a Pareto operator $\mathcal{P}_\xi : [\![1:m]\!] \times \mathbb{S}(\xi) \rightrightarrows \mathbb{S}(\xi)$ as follows:*

$$c^i \in \mathcal{P}_\xi(\sigma(i), c^{i-1}), \quad i \in [\![1:m]\!].$$

Then, c^m belongs to the strong Pareto front of $\mathbb{S}(\xi)$ with

$$c^m \leq c^{m-1} \leq \ldots \leq c^1 \leq c^0 \quad \textit{(component-wise)}$$

and

$$\vartheta_\xi^{\sigma(i)}(c^{i-1}) = c^j_{\sigma(i)}, \qquad \forall i \in [\![1:m]\!],\ j \in [\![i:m]\!].$$

Here, $\vartheta^i_\xi(\cdot)$ is given by (8). In particular,

$$c^m = \left(\vartheta^1_\xi\left(c^{\sigma(1)-1}\right), \dots, \vartheta^m_\xi\left(c^{\sigma(m)-1}\right)\right).$$

Proof. For the sake of simplicity, let us consider only the case where the permutation σ is the identity; that is, $\sigma(i) = i$ for any $i \in [\![1:m]\!]$. For more general permutations, the proof is analogous (it is sufficient to redefine the constraint mapping g by changing the order of the components).

Note that the sequence $c^1, \dots, c^m$ is well-defined. Indeed, this is a straightforward consequence of the induction principle and Proposition 3.5. In particular, we have $c^1, \dots, c^m \in \mathbb{S}(\xi)$.

Let us continue by showing that $c^i \leq c^{i-1}$ for any $i \in [\![1:m]\!]$. Given that a control $\vec{u}^i$ in the definition of $\mathcal{P}_\xi(i, c^{i-1})$ (see Definition 3.4) is, in particular, a feasible control for problem $\vartheta^i_\xi(c^{i-1})$; we have

$$c^i_j = \max_{k=0,\dots,N} g_j(k, x^i_k, u^i_k) \leq c^{i-1}_j, \qquad \forall j \in [\![1:m]\!].$$

Therefore, we have $c^i \leq c^{i-1}$ (component-wise) for any $i \in [\![1:m]\!]$.

Let us now prove that c^m is a strong Pareto minimal. Let $c \in \mathbb{S}(\xi)$ be such that $c \leq c^m$. Assume for the sake of contradiction that $c \neq c^m$. Let $i \in \{1, \dots, m\}$ be an index such that $c_i < c^m_i$. In particular, we have

$$c_i < c^m_i \leq c^i_i = \vartheta^i_\xi(c^{i-1}).$$

However, since $c \in \mathbb{S}(\xi)$, there is a control $\vec{u} = (u_k)_{k=0}^N \in \mathbb{U}$ such that $\vec{u}$ and $\vec{x}_\xi(\vec{u}) = (x_k)_{k=0}^{N+1}$ satisfy

$$g(k, x_k, u_k) \leq c \leq c^m \leq c^{i-1}, \qquad \forall k \in [\![0:N]\!].$$

This means that $\vec{u}$ is feasible for the optimal control problem associated with $\vartheta^i_\xi(c^{i-1})$, and thus by definition

$$\vartheta^i_\xi(c^{i-1}) \leq \max_{k=0,\dots,N} g_i(k, x_k, u_k) \leq c_i;$$

this leads to a contradiction. Therefore, $c = c^m$, and consequently, c^m is a strong Pareto minimal.

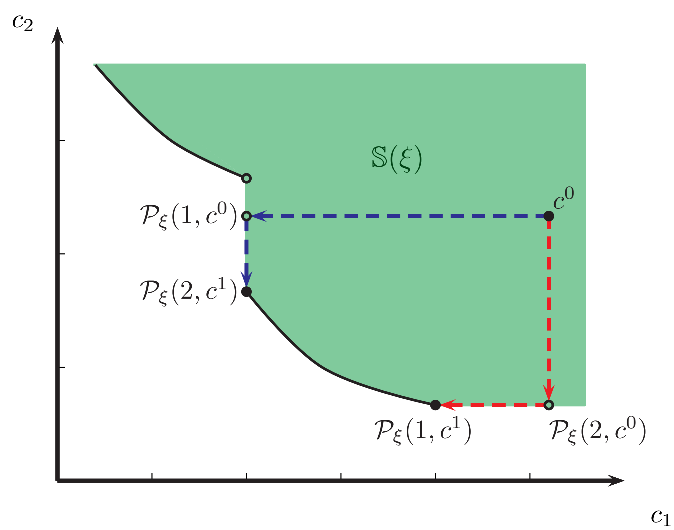

FIGURE 2. Sketch of sequence of thresholds generated by Theorem 3.6 for a problem with $m = 2$. The black line indicates the strong Pareto front of the example. Here, $\mathcal{P}_\xi(1, c^1)$ and $\mathcal{P}_\xi(2, c^1)$ are the two strong Pareto minimals found by the scheme using the two possible permutations on $[\![1:2]\!]$ starting from $c^0 \in \mathbb{S}(\xi)$.

Finally, we have by definition that $c^i_i = \vartheta^i_\xi(c^{i-1})$ for any $i \in [\![1:m]\!]$. Thus, let $i \in [\![1:m-1]\!]$ and $j \in [\![i+1:m]\!]$. Then, since $c^j \le c^i \le c^{i-1}$, we have that $\vec{u}^j$, the optimal control given in Definition 3.4, and its corresponding optimal trajectory are feasible for the optimal control problem associated with $\vartheta^i_\xi(c^{i-1})$. In particular, we must have

$$\vartheta^i_\xi(c^{i-1}) = \max_{k=0,\dots,N} g_i(k, x^i_k, u^i_k) \le \max_{k=0,\dots,N} g_i(k, x^j_k, u^j_k) = c^j_i \le c^i_i = \vartheta^i_\xi(c^{i-1}).$$

This completes the proof of the theorem. □ □

To give an idea of what the sequence $c^1, \dots, c^m$ generated by the preceding theorem looks like, we describe a situation with a threshold space of dimension $m = 2$ in Figure 2.

Remark 3.7. In Theorem 3.6, it is not difficult to see that if for an initial condition $\xi \in \vec{X}$, the thresholds vector $c^0 \in \mathbb{S}(\xi)$ already belongs to the strong Pareto front of $\mathbb{S}(\xi)$, then by definition of strong Pareto minimals, the sequence of thresholds $c^1, \dots, c^m$ generated by the proposed method is equal to c^0.

4. The weak Pareto front

Let us now focus on the weak Pareto front of the set of sustainable thresholds. As with the strong Pareto front, we will present a method for computing the weak Pareto front by means of optimal control tools. However, in this case, we will consider a method that identifies elements in the set from the outside; that is, we will construct a weak Pareto minimal from threshold vectors that are not sustainable

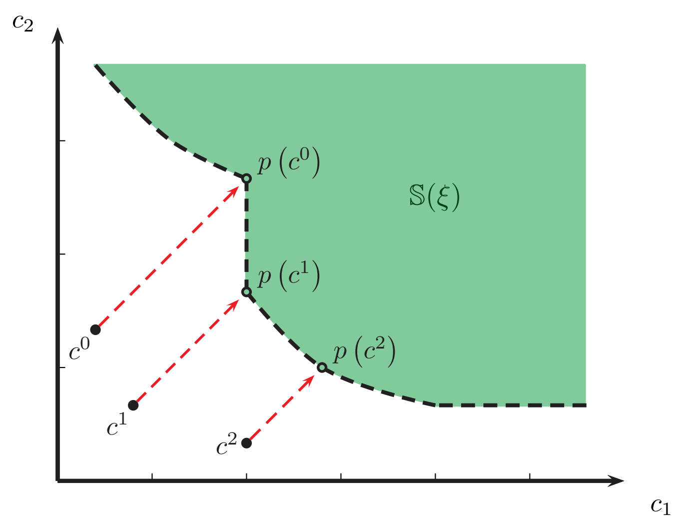

FIGURE 3. Sketch of Theorem 4.1 for a problem with $m = 2$ on the construction of weak Pareto minimals from unsustainable thresholds c^0, c^1 and c^2. The dashed black line indicates the weak Pareto front of the example.

for the given initial condition. In particular, this means that the optimal control problems we are considering do not require forcing the mixed constraint and thus are unconstrained problems. This helps somewhat in reducing the computational time.

To begin, we introduce the optimal control problem

$$\omega_\xi\left(c\right) = \min_{\vec{u}\in\mathbb{U}} \left\{ \max_{k=0,\ldots N} \Phi^c(k, x_k, u_k) \mid \vec{x}_\xi(\vec{u}) = (x_k)_{k=0}^{N+1} \right\}$$

where $c \in \mathbb{R}^m$ is a given threshold vector and $\Phi^c : [\![0\!:\!N]\!] \times \vec{X} \times \vec{U} \to \mathbb{R}$ is given by

$$\Phi^c(k, x, u) = \max_{i=1,\ldots,m} g_i(k, x, u) - c_i.$$

Note that, since $\mathbb{U}$ is (nonempty) compact and the functional to be minimized in the definition of $\omega_\xi\left(c\right)$ is lower semicontinuous, the use of the minimum instead of the infimum in $\omega_\xi\left(c\right)$ is justified, which means that the optimal value $\omega_\xi\left(c\right)$ is attained at some optimal control $\vec{u} \in \mathbb{U}$. Moreover, this implies that for any $\xi \in \vec{X}$, we have

$$c \in \mathbb{S}(\xi) \iff \omega_\xi\left(c\right) \le 0. \tag{9}$$

This equivalence shows the strong link between the level-set of the value function $\omega_\xi\left(c\right)$ and the set $\mathbb{S}(\xi)$. Furthermore, there is a rather straightforward way to construct a point in the weak Pareto front of $\mathbb{S}(\xi)$ from any given *unsustainable* threshold $c \in \mathbb{R}^m$ through the value function $\omega_\xi\left(c\right)$. In this context, unsustainable means that $\omega_\xi\left(c\right) > 0$. We describe this situation in Figure 3.

Theorem 4.1. *Let $c^*, \bar{c} \in \mathbb{R}^m$. We then have the following:*

(1) *c^* is a weak Pareto minimal of $\mathbb{S}(\xi)$ if and only if $\omega_\xi(c^*) = 0$.*
(2) *If $\omega_\xi(\bar{c}) > 0$, then*

$$p(\bar{c}) := \bar{c} + \omega_\xi(\bar{c})\,\vec{1}$$

belongs to the weak Pareto front of $\mathbb{S}(\xi)$, where $\vec{1} = (1, 1, \ldots, 1) \in \mathbb{R}^m$.

Proof. Let us set for any $c \in \mathbb{R}^m$

$$R^c(\vec{x}, \vec{u}) := \max_{k=0,\ldots,N} \max_{i=1,\ldots,m} \left[g_i(k, x_k, u_k) - c_i\right], \quad \forall \vec{x} \in \mathbb{X},\ \vec{u} \in \mathbb{U}. \tag{10}$$

In particular, it follows that

$$\omega_\xi(c) = \inf_{\vec{u}\in\mathbb{U}} R^c(\vec{x}_\xi(\vec{u}), \vec{u}).$$

(1) Let us assume first that $\omega_\xi(c^*) = 0$; then by (9), we obtain $c^* \in \mathbb{S}(\xi)$. To see that c^* is a weak Pareto minimal; suppose for the sake of contradiction that there exists $c \in \mathbb{S}(\xi)$ with $c < c^*$. We define

$$\delta := \min_{i=1,\ldots,m} \{c_i^* - c_i\} > 0.$$

Since $c \in \mathbb{S}(\xi)$, there exists $\vec{u} = (u_k)_{k=0}^N \in \mathbb{U}$ such that

$$g_i(k, x_k, u_k) \le c_i, \qquad \forall i \in [\![1:m]\!],\ k \in [\![0:N]\!],$$

where $\vec{x}_\xi(\vec{u}) = (x_k)_{k=0}^{N+1}$. This implies that

$$R^{c^*}(\vec{x}_\xi(\vec{u}), \vec{u}) \le \max_{i=1,\ldots,m} c_i - c_i^* = -\delta < \omega_\xi(c^*),$$

which is a contradiction. Therefore, c^* is a weak Pareto minimal of $\mathbb{S}(\xi)$.

On the other hand, assume now that c^* is a weak Pareto minimal of $\mathbb{S}(\xi)$. In particular, $c^* \in \mathbb{S}(\xi)$, so (9) yields $\omega_\xi(c^*) \le 0$. Let $\vec{u} \in \mathbb{U}$ be an optimal control for $\omega_\xi(c^*)$; that is, $R^{c^*}(\vec{x}_\xi(\vec{u}), \vec{u}) = \omega_\xi(c^*)$. Suppose for a contradiction that $R^{c^*}(\vec{x}_\xi(\vec{u}), \vec{u}) < 0$. Then

$$c := c^* + \frac{1}{2} R^{c^*}(\vec{x}_\xi(\vec{u}), \vec{u})\vec{1} < c^*.$$

However, $c \in \mathbb{S}(\xi)$ because we also have

$$\omega_\xi(c) \le R^c(\vec{x}_\xi(\vec{u}), \vec{u}) \le R^{c^*}(\vec{x}_\xi(\vec{u}), \vec{u}) - \frac{1}{2} R^{c^*}(\vec{x}_\xi(\vec{u}), \vec{u}) = \frac{1}{2} R^{c^*}(\vec{x}_\xi(\vec{u}), \vec{u}) \le 0.$$

This leads to a contradiction, so the proof of 1 is complete.

(2) It is straightforward to see that $\omega_\xi(p(\bar{c})) = \omega_\xi(\bar{c}) - \omega_\xi(\bar{c}) = 0$. Thus, in light of the first statement of Theorem 4.1, the conclusion follows.

□ □

Remark 4.2. Notice that the first part of Theorem 4.1 provides a characterization of the weak Pareto minimals of $\mathbb{S}(\xi)$. In addition, one can see that if $\omega_\xi(c) < 0$, then c belongs to the interior of $\mathbb{S}(\xi)$. This comes directly from the proof of Theorem 4.1 and the fact that $c + \frac{1}{2}\omega_\xi(c)\,\vec{1} \in \mathbb{S}(\xi)$ implies

$$c + \frac{1}{2}\omega_\xi(c)\,[-1, 1]^m \subset \mathbb{S}(\xi).$$

4.1. **Dynamic programming principle.** To compute the optimal value $\omega_\xi(c)$, we use the dynamic programming principle. This method leads to an implementable way to compute the optimal value $\omega_\xi(c)$ and, consequently, to a practical way to compute the set of sustainable thresholds and its weak Pareto front.

For the sake of exposition, for any $n \in [\![0:N]\!]$, we write

$$V_n^c(\xi) := \min_{\vec{u}\in\mathbb{U}} \left\{ \max_{k=n,\dots,N} \Phi^c(k,x_k,u_k) \mid x_{k+1} = F(k,x_k,u_k),\ k \in [\![n:N]\!],\ x_n = \xi \right\}.$$

In particular, for any $\xi \in \vec{X}$, we have

$$V_0^c(\xi) = \omega_\xi(c) \quad \text{and} \quad V_N^c(\xi) = \min_{u\in\vec{U}} \Phi^c(N,\xi,u).$$

To obtain $\omega_\xi(c)$, we use the dynamic programming principle for computing $V_0^c(\xi)$ from the sequence of value functions $V_1^c(\cdot),\dots,V_N^c(\cdot)$. In this setting, the dynamic programming principle for $(V_n^c)_{n=0}^N$ is as follows:

Proposition 4.3. *For any $n \in [\![0:N-1]\!]$, $c \in \mathbb{R}^m$ and $\xi \in \vec{X}$, we have*

$$V_n^c(\xi) = \min_{u\in\vec{U}} \max\left\{ V_{n+1}^c\left(F(n,\xi,u)\right), \Phi^c(n,\xi,u)\right\}. \tag{11}$$

Proof. For any given $n \in [\![0:N-1]\!]$, let

$$W_n^c(\xi) := \inf_{u\in\vec{U}} \max\left\{ V_{n+1}^c\left(F(n,\xi,u)\right), \Phi^c(n,\xi,u)\right\}.$$

That is, $W_n^c(\xi)$ is the right-hand side of (11) with the infimum instead of the minimum. Since $\vec{U}$ is compact and Φ^c is lower semicontinuous, it is not difficult to see that $V_N^c(\cdot)$ is lower semicontinuous. Consequently, by induction,

$$u \mapsto \max\left\{ V_{n+1}^c\left(F(n,\xi,u)\right), \Phi^c(n,\xi,u)\right\}$$

is also lower semicontinuous, which means the infimum in W_n is always attained for any $n \in [\![0:N-1]\!]$.

Similarly, as in (10), we set

$$R_n^c(\vec{x},\vec{u}) := \max_{k=n,\dots,N} \Phi^c(k,x_k,u_k), \quad \forall \vec{x}\in\mathbb{X},\ \vec{u}\in\mathbb{U},$$

as the functional to be minimized in the definition of V_n^c. It is then clear that

$$R_n^c(\vec{x},\vec{u}) = \max\left\{ R_{n+1}^c(\vec{x},\vec{u}), \Phi^c(n,x_n,u_n)\right\}.$$

If $\vec{u} = (u_k)_{k=0}^N \in \mathbb{U}$ and $\vec{x} = (x_k)_{k=0}^{N+1}$ are such that $x_{k+1} = F(k,x_k,u_k)$ for any $k \in [\![n:N]\!]$ with $x_n = \xi$, then, since $x_{n+1} = F(n,\xi,u_n)$, we have

$$R_{n+1}^c(\vec{x},\vec{u}) \geq V_{n+1}^c\left(F(n,\xi,u_n)\right).$$

From here, we readily obtain $V_n^c(\xi) \geq W_n^c(\xi)$.

For the other inequality, consider an arbitrary $u \in \vec{U}$ and let $\vec{u} = (u_k)_{k=0}^N \in \mathbb{U}$ be an optimal control for the optimization problem related to $V_{n+1}^c\left(F(n,\xi,u)\right)$. Let $\tilde{\vec{u}} = (\tilde{u}_k)_{k=0}^N$ be such that $\tilde{u}_n = u$ and $\tilde{u}_k = u_k$ otherwise. Let $\tilde{\vec{x}} = (\tilde{x}_k)_{k=0}^{N+1}$ be such that $\tilde{x}_{k+1} = F(k,\tilde{x}_k,\tilde{u}_k)$ for any $k \in [\![n:N]\!]$ with $\tilde{x}_n = \xi$.

Note that $\tilde{\vec{u}}$ is an admissible control for the problem associated with $V_{n+1}^c\left(F(n,\xi,u)\right)$ as well as for $V_n^c(\xi)$. Consequently,

$$V_n^c(\xi) \leq R_n^c\left(\tilde{\vec{x}},\tilde{\vec{u}}\right) = \max\left\{ R_{n+1}^c\left(\tilde{\vec{x}},\tilde{\vec{u}}\right), \Phi^c(n,\xi,u)\right\}.$$

Finally, since $V^c_{n+1}(F(n,\xi,u)) = R^c_{n+1}\left(\tilde{\vec{x}},\tilde{\vec{u}}\right)$, we can complete the proof by taking the infimum over $u \in \vec{U}$. □ □

4.2. **A scheme for computing the weak Pareto front.** To summarize, by combining Theorem 4.1 and Proposition 4.3, we obtain a practical method (Algorithm 1) to compute the weak Pareto front of the set of sustainable thresholds associated with a control system with mixed constraints.

For implementing this algorithm, it is necessary to define two meshes $\vec{X}_h \subset \vec{X}$ and $S_h \subseteq \mathbb{R}^m$ of size $0 < h \ll 1$ as computational domains (state and thresholds). Then for any $n \in [\![0\!:\!N]\!]$, the function $V^c_n(\cdot)$ in Proposition 4.3 has to be computed for every $\xi' \in \vec{X}_h$ reachable in n steps from ξ, and for all $c \in S_h$, a procedure that could be too expensive, which is not surprising because the method is based in the dynamic programming principle. Nevertheless, from Theorem 4.1 and Proposition 4.3, the method introduced in Algorithm 1 will not need a large mesh S_h, as it is explained in the example showed in Section 5.

Algorithm 1: Computing the weak Pareto front

Input: $\xi \in \vec{X}$, $N \in \mathbb{N}$, $F : [\![0\!:\!N]\!] \times \vec{X} \times \vec{U} \to \vec{X}$, $g : [\![0\!:\!N]\!] \times \vec{X} \times \vec{U} \to \mathbb{R}^m$
Let $\vec{X}_h \subset \vec{X}$ and $S_h \subseteq \mathbb{R}^m$ be two meshes of size $0 < h \ll 1$ for computational domains (state and thresholds).
For $n \in [\![0\!:\!N]\!]$ let $\vec{X}^n_h \subset \vec{X}_h$ be the set of points in $\vec{X}_h$ reachable from ξ in n steps.
Let $\mathbb{S}$ and $\mathcal{P}_w$ two empty arrays.
for $c_i \in S_h$ **do**
  for $\xi' \in \vec{X}^N_h$ **do**
    Compute $V^{c_i}_N(\xi') = \min_{u\in\vec{U}} \Phi^{c_i}(N,\xi',u)$.
Set $n = N-1$.
while $n \geq 0$ **do**
  for $c_i \in S_h$ **do**
    for $\xi' \in \vec{X}^n_h$ **do**
      Compute $V^{c_i}_n(\xi') = \min_{u\in\vec{U}} \max\left\{V^{c_i}_{n+1}(F(n,\xi',u)), \Phi^{c_i}(n,\xi',u)\right\}$.
  Set $n = n-1$.
for $c_i \in S_h$ **do**
  Save $c_i + V^{c_i}_0(\xi)\vec{1}$ in $\mathcal{P}_w$
return $\mathcal{P}_w$ *and* $\mathbb{S} = \mathcal{P}_w + \mathbb{R}^m_+$

5. Simulations

In this section, we illustrate the computation of the set of sustainable thresholds $\mathbb{S}(\xi)$ by computing the weak Pareto front for one example based on renewable resource management inspired by [7]. In this example, the stock of a renewable resource in period k is represented by $x_k \geq 0$, and its dynamics with harvesting (or catch) u_k are described by

$$x_{k+1} = F(x_k, u_k) = f(x_k) - u_k$$

where f stands for the renewable function of the stock.

For the above control system, suppose that a regulatory agency has the social objective of ensuring both current stock and catch. Reformulated from a viability viewpoint, the problem relates to sustaining both stock and catch through the thresholds $x^{\lim}$ and $h^{\lim}$ as follows:

$$\left\{ \begin{array}{l} x_{k+1} = f(x_k) - u_k, \\ x_0 = \xi \ \text{ given (the current state of the resource)} \\ x_k \geq x^{\lim} \\ u_k \geq h^{\lim}. \end{array} \right. \tag{12}$$

The computation of the set $\mathbb{S}(\xi)$ corresponds to the identification of viable thresholds $x^{\lim}$ and $h^{\lim}$ with respect to current state ξ.

We will see that the maximal sustainable yield (MSY) level is a tipping point in the determination of the sustainable thresholds. To introduce this concept, we denote by $\sigma(x)$ the harvest level obtained for the equilibrium biomass level x; that is,

$$\sigma(x) = f(x) - x.$$

Therefore, the MSY is the catch level at equilibrium for which this quantity is maximized; that is,

$$\text{MSY} = \max_{x \geq 0} \sigma(x).$$

We study this very simple example, because we can analytically compute the set of sustainable thresholds when the horizon is infinity, and then we are able to compare this analytical expression with the results given by our method by computing the weak Pareto front (see Section 4) of this set in the finite horizon case. For this purpose, it is convenient to consider the Beverton-Holt population dynamics

$$f(x) = (1+r)x\left(1 + \frac{r}{K}x\right)^{-1} \tag{13}$$

where the intrinsic growth r and carrying capacity K are positive parameters. For this Beverton-Holt growth function (13), the MSY biomass level x_{MSY} at equilibrium is given by

$$x_{\text{MSY}} = \frac{K}{1+\sqrt{1+r}}.$$

When the horizon is infinity ($N = +\infty$), the viability kernel has been calculated analytically in [9]. It is not difficult to see that equivalence (2) also holds in the case of infinite horizon, and therefore, denoting by $\mathbb{S}_\infty(\xi)$ the set of sustainable thresholds when $N = +\infty$, one can compute analytically the set $\mathbb{S}_\infty(\xi)$ from the viability kernel studied in [9]. Indeed, it is given by

$$\mathbb{S}_\infty(\xi) = \{(x^{\lim}, h^{\lim}) \mid x^{\lim} \leq \min\{x_0, K\}; \quad h^{\lim} \leq \sigma(x^{\lim})\}.$$

Since $\mathbb{S}(\xi)$ approaches to $\mathbb{S}_\infty(\xi)$ when $N \to \infty$, the objective of this example is to show how accurate is $\mathbb{S}(\xi)$ -computed by our method for N large enough- with respect to $\mathbb{S}_\infty(\xi)$.

In [15] this example is also analyzed for the same purposes; that is, the authors compare the analytical solution of $\mathbb{S}_\infty(\xi)$ with the results of a method that they introduce.

Remark 5.1. Note that the constraints $x_k \geq x^{\lim}$ and $u_k \geq h^{\lim}$ in (12) are given in the opposite sense of the constraints established in (I^c) for the general formulation. Instead of reformulating these constraints as $-x_k \leq -x^{\lim}$ and $-u_k \leq -h^{\lim}$ to fit the general formulation, we deal directly with the original constraints; therefore, the definition of $\mathbb{S}(\xi)$ changes slightly. Instead of $\mathbb{S}(\xi) + \mathbb{R}^m_+ = \mathbb{S}(\xi)$, dealing with the original constraints will result in $\mathbb{S}(\xi) - \mathbb{R}^m_+ = \mathbb{S}(\xi)$. On the other hand, for the problem studied in this example, we have no interest in the negative thresholds $x^{\lim}$ and $h^{\lim}$. For this reason, in Figure 4, we only depict the sustainable thresholds in the positive orthant.

Figure 4 displays both the set of sustainable thresholds $\mathbb{S}_\infty(\xi)$ in the infinite horizon case, computed analytically (first row), and the numerical approximation of $\mathbb{S}(\xi)$ when $N = 20$ (second row) for initial conditions ξ in the following cases: (i) $0 < \xi < x_{\mathrm{MSY}}$ (first column); (ii) $x_{\mathrm{MSY}} < \xi < K$ (second column); and (iii) $K < \xi$ (third column). This procedure was conducted by computing the weak Pareto front of $\mathbb{S}(\xi)$ and then using the equality $\mathbb{S}(\xi) - \mathbb{R}^m_+ = \mathbb{S}(\xi)$ to obtain the set $\mathbb{S}(\xi)$.

In the positive orthant of $\mathbb{R}^2$, we consider the mesh

$$S_h = \{(jh, c_2^{\max}) \mid j = 0, 1, \ldots, N_h\} \cup \{(c_1^{\max}, jh) \mid j = 0, 1, \ldots, N_h\}, \tag{14}$$

with $0 < h \ll 1$ as the size of the mesh, $N_h \in \mathbb{N}$, and $c_1^{\max}$, $c_2^{\max} > 0$ large enough. For each vector $\bar{c}$ in this mesh, we compute $\omega_\xi(\bar{c})$. Taking $c_1^{\max}$ and $c_2^{\max} > 0$ sufficiently large ensures that vectors $\bar{c}$ in the mesh are not in $\mathbb{S}(\xi)$ (see Remark 5.1). Hence, we obtain $\omega_\xi(\bar{c}) < 0$[1]. Therefore, from Theorem 4.1, we find that $p(\bar{c}) := \bar{c} + \omega_\xi(\bar{c})\,\vec{1}$ is in the weak Pareto front for all $\bar{c}$ in the mesh. Thus, we obtain the weak Pareto front of $\mathbb{S}(\xi)$ and, a fortiori, the entire set $\mathbb{S}(\xi)$.

For obtaining Figure 4, the mesh S_h in Algorithm 1 given by (14), we have taken $N_h = 80$ (i.e, 160 vectors of thresholds). The same number of points was considered in the mesh $\vec{X}_h$, the discretization of the state space. The algorithm was implemented in Python (Jupyter notebook) and the CPU time for the horizon $N = 20$ was 240 seconds.

[1]Following Theorem 4.1, for vectors $\bar{c} \notin \mathbb{S}(\xi)$, one has $\omega_\xi(\bar{c}) > 0$. In this example, we are considering constraints in the opposite sense (see Remark 5.1), so we obtain the same result with the opposite sign for the function $\omega_\xi(\cdot)$.

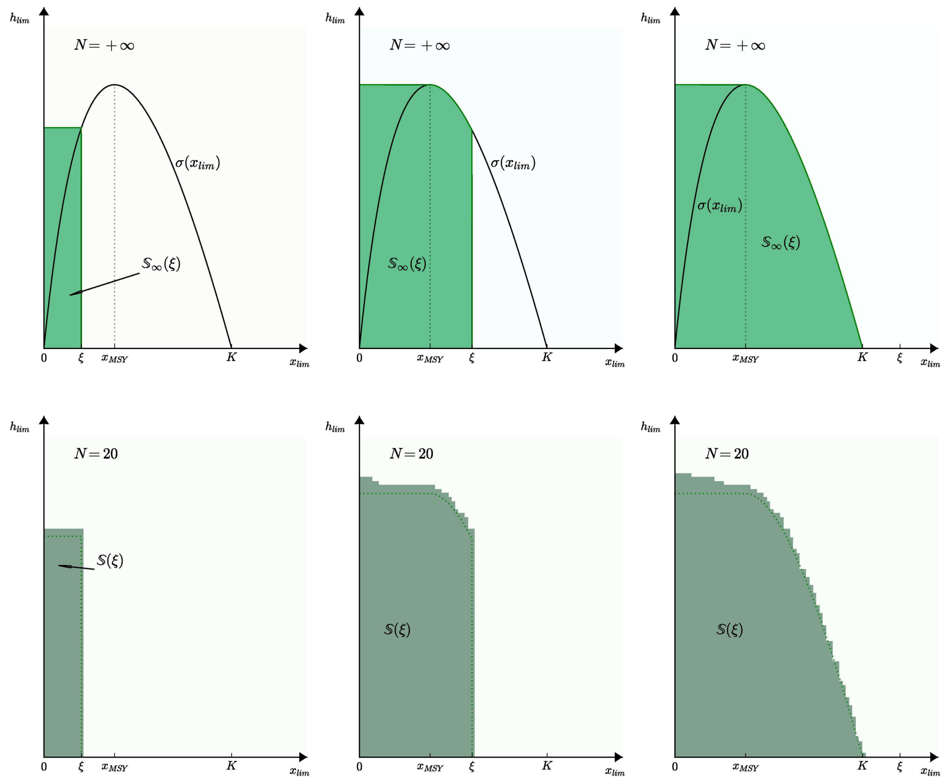

FIGURE 4. Set of sustainable thresholds $\mathbb{S}_\infty(\xi)$ (first row) and their approximations given by $\mathbb{S}(\xi)$ (second row) with $N = 20$ for different initial conditions ξ: first column: $\xi < x_{\mathrm{MSY}}$; second column: $x_{\mathrm{MSY}} < \xi < K$; third column: $\xi > K$. The parameters for the stock dynamics are set to $r = 1.75$ and $K = 50$.

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