THE VIABILITY KERNEL OF DYNAMICAL SYSTEMS WITH MIXED CONSTRAINTS: A LEVEL-SET APPROACH

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Abstract. This paper deals with control systems with mixed (state-control) constraints and dynamics in discrete-time. We aim at providing a practical method for computing the viability kernel for such systems. We take a standard approach via optimal control theory, that is, we identify the viability kernel as the effective domain of the value function of a suitable optimal control problem. A dynamic programming principle can be established in order to compute that value function. However, from a practical point of view this may turn out in a slow scheme for such computation, because at each step the mixed constraints need to be verified. In this work we take a different path which has been inspired by the level-set approach studied for optimal control problems with (pure) state constraints. An algorithm is proposed for computing the viability kernel and some numerical examples are shown to demonstrate the proposed method.

Keywords. Viability kerne; Discrete-time systems; Mixed constraints; Level-set approach; Dynamic programming.

1. Introduction

This paper is concerned with the viability kernel [2] of a discrete-time control system with mixed input-output constraints. Restrictions of this type appear for example in sustainable management problems and bio-economic modeling [3, 4, 6, 16, 17, 19, 21, 24, 26, 28], where the viability approach has notably been applied to the analysis of these topics as recently reviewed in [30, 25].

Numerical methods for approximating the viability kernel have been widely investigated from different approaches; see for instance [8, 20, 29, 27, 23, 5, 14, 22] and the references therein. It is worth noting that the viability kernel can also be seen as the largest weakly invariant set of a dynamical system on a prescribed set (cf. [7]). Let us point out that most of these works deal with time-continuous model and treat input and output constraints separately.

The aim of this work is to provide a practical method for computing the viability kernel of a control system with mixed constraints on a finite horizon setting.

The problem with mixed constraints that concerns us has the same structure as in [6, 9, 18, 10, 11], and we follow as well a dynamic programming approach for computing the viability kernel, as in [9, 18, 10]. However, the novelty of this work falls on the level-set approach [23, 1] we have taken for characterizing the viability kernel. We essentially prove that the viability kernel corresponds to the zero level-set of a suitable value function associated with an optimal control problem. Let us point out that this idea has also been exploited in [23] for continuous-time systems with end-point constraints and a similar result has been obtained in [8] for state constrained systems, and later extended to a mixed constraints setting in [20].

The advantage of the level-set approach we have followed in this paper is that it avoids checking at each iteration whether the mixed constraints are satisfied or not. This is an advantage when considering practical methods for the computation of the viability kernel. As a main consequence, we obtain a method that runs as fast as dynamic programming allows, independently of the constraints imposed over the system.

The manuscript is organized as follows: In Section 2 we present the problem and set the basics for the rest of the paper. In Section 3 we present the level-set approach and in Section 4 we provide a practical method for computing the viability kernel using the level-set approach. Finally, in Section 5 we report some numerical simulations to illustrate our algorithm.

2. Preliminaries on discrete-time control systems

For a given finite horizon $N \in \mathbb{N}$, an *initial time* $n \in [\![0:N]\!]$[1], an *initial state* $\xi \in \mathbf{X}$ and a finite sequence of control $\mathbf{u} = (u_k)_{k=0}^N$, we consider the discrete-time control system

$$x_{k+1} = F(k, x_k, u_k), \quad k \in [\![n:N]\!], \quad x_n = \xi, \tag{1}$$

where $F : [\![0:N]\!] \times \mathbf{X} \times \mathbf{U} \to \mathbf{X}$ is the *dynamics*, $\mathbf{X}$ is the *state space* and $\mathbf{U}$ is the *control space*. We denote by $\mathbb{U}$ the collection of all possible controls, that is,

$$\mathbb{U} := \left\{ \mathbf{u} = (u_k)_{k=0}^N \,\middle|\, u_0, \dots, u_N \in \mathbf{U} \right\} \cong \mathbf{U}^{N+1}.$$

A solution of the control system (1) associated with a control $\mathbf{u} \in \mathbb{U}$ is an element of the space

$$\mathbb{X} := \left\{ \mathbf{x} = (x_k)_{k=0}^{N+1} \,\middle|\, x_0, \dots, x_{N+1} \in \mathbf{X} \right\} \cong \mathbf{X}^{N+2},$$

that satisfies, for some initial time $n \in [\![0:N]\!]$ and initial state $\xi \in \mathbf{X}$, the following condition

$$x_k = \xi, \quad \forall\, k = 0, \dots, n.$$

A solution of (1), which is uniquely determined by the control $\mathbf{u}$, initial time n and initial state ξ, is denoted in the sequel by $\mathbf{x}_n^\xi(\mathbf{u})$ to emphasize the dependence on the initial data of the problem (control, time and state).

2.1. Constraints and viability kernel.

In many real applications, the output and input of dynamical systems such as (1) are constrained to remain in a prescribed set. These constraints can reflect physical, economical or production restrictions (cf. [9]). In this work we are concerned with the so-called *mixed constraints*, that is, a set of restrictions that the control $\mathbf{u} \in \mathbb{U}$ along with its associated trajectories $\mathbf{x}_n^\xi(\mathbf{u}) = (x_k)_{k=0}^{N+1}$ must satisfy from some initial time $n \in [\![0:N]\!]$ onward. These constraints are represented by

$$g(k, x_k, u_k) \in \mathcal{G}, \quad \forall\, k = n, \dots, N, \tag{2}$$

where $g : [\![0:N]\!] \times \mathbf{X} \times \mathbf{U} \to \mathbf{Y}$ is the *constraints mapping* and $\mathcal{G} \subseteq \mathbf{Y}$ is the *constraints set*, which is a subset of some given space $\mathbf{Y}$.

In particular, we are interested in finding all posible initial states such that for some control, that control along with its corresponding controlled trajectories,

[1] Given two integers $p \leq q$, we denote by $[\![p:q]\!]$ the collection of all integer between p and q

satisfy (2). The collection of all such initial conditions is the corresponding *viability kernel* ([2, 9]), which is

$$\mathbb{V}_n := \left\{ \xi \in \mathbf{X} \;\middle|\; \exists \mathbf{u} \in \mathbb{U},\ \mathbf{u} \text{ and } \mathbf{x}_n^\xi(\mathbf{u}) \text{ satisfy } (2) \right\}. \tag{3}$$

In the literature (cf. [2]), viability kernels are usually associated with infinite horizon problems. In the framework of this paper, we consider only the finite horizon case as done for instance in [9]. This is essentially motivated by the use of the dynamic programming principle (DPP for short) we present in §4.1. An extension to the infinite horizon setting should be possible under mild modifications and further assumptions.

2.1.1. *A practical case of study.* The theory we are going to develop in this paper is rather general. However, it is worth mentioning a specific type of mixed constraints that is of particular interest when considering practical applications. This is the case when $\mathbf{Y} = \mathbb{R}^p$ for some $p \in \mathbb{N}$ and, for a given thresholds vector $c = (c_1, \ldots, c_p) \in \mathbb{R}^p$, the constraints set in (2) is of the form:

$$\mathcal{G}_c := \{(v_1, \ldots, v_p) \in \mathbb{R}^p \mid v_i \leq c_i,\ \forall i = 1, \ldots, p\}. \tag{4}$$

In this situation, the constraint (2) reduces to

$$g_i(k, x_k, u_k) \leq c_i, \quad \forall\ k = n, \ldots, N; \quad \forall\ i = 1, \ldots, p,$$

where $g_i : [\![0 : N]\!] \times \mathbf{X} \times \mathbf{U} \to \mathbb{R}$. The numerical demonstrations we exhibit in §5 are instances of this case.

2.2. **Standing assumptions.** Along this work we assume that the data of the dynamical system (1) with mixed constraints (2) satisfy the following basic conditions, which are going to be referred in the sequel as *Standing Assumptions*:

(H1) $F(k, \cdot, \cdot)$ and $g(k, \cdot, \cdot)$ are continuous for $k \in [\![0 : N]\!]$.
(H2) $\mathbf{X}$ and $\mathbf{Y}$ are finite-dimensional Banach spaces.
(H3) $\mathbf{U}$ is a nonempty compact metric space.
(H4) $\mathcal{G} \subseteq \mathbf{Y}$ is a nonempty closed subset.

The hypotheses we have done so far ensure that the set of feasible solutions to the dynamical system (1)-(2) is compact in $\mathbb{X}$. Indeed, note first that $\mathbf{u} \mapsto \mathbf{x}_n^\xi(\mathbf{u})$ is a continuous map, and so, since $\mathbb{U}$ is a compact metric space, we have that

$$A_n^\xi := \left\{ (\mathbf{x}, \mathbf{u}) \in \mathbb{X} \times \mathbb{U} \mid \mathbf{x} = \mathbf{x}_n^\xi(\mathbf{u}) \right\}$$

is a compact subset of $\mathbb{X} \times \mathbb{U}$. Furthermore, the set

$$B_n^\xi := \{(\mathbf{x}, \mathbf{u}) \in \mathbb{X} \times \mathbb{U} \mid g(k, x_k, u_k) \in \mathcal{G},\ \forall k \in [\![n : N]\!]\}$$

is closed in $\mathbb{X} \times \mathbb{U}$. Now, since the set of admissible trajectories is exactly the projection of $A_n^\xi \cap B_n^\xi$ over $\mathbf{X}$ we conclude that set of feasible solutions to the dynamical system (1)-(2) is compact (possibly empty) in $\mathbb{X}$. This shows too that the set of admissible controls for this dynamical system is compact (possibly empty) in $\mathbb{U}$.

3. Optimal control and the level-set approach

The idea we follow in this paper for computing the viability kernel is based on the DPP and in the level-set approach [1]. For this reason, let us begin by considering a general optimal control problem and its corresponding value function

$$\vartheta_n(\xi) := \inf_{\mathbf{u}\in\mathbb{U}} \left\{ \mathcal{J}_n(\mathbf{x}_n^\xi(\mathbf{u}),\mathbf{u}) \;\middle|\; \mathbf{u} \text{ and } \mathbf{x}_n^\xi(\mathbf{u}) \text{ satisfy } (2) \right\} \tag{5}$$

defined for $\xi \in \mathbf{X}$, where for each $n \in [\![0:N]\!]$, the cost $\mathcal{J}_n : \mathbb{X}\times\mathbb{U} \to \mathbb{R}$ is a given function. Note that the formulation of the optimization problem (5) is general enough and it covers several problems as for example:

- Bolza-type problems:
  for a running cost $L : [\![0:N]\!] \times \mathbf{X} \times \mathbf{U} \to \mathbb{R}$ and an end-point cost $\varphi : \mathbf{X} \to \mathbb{R}$
  $$\mathcal{J}_n(\mathbf{x},\mathbf{u}) := \sum_{k=n}^{N} L(k, x_k, u_k) + \varphi(x_{N+1}).$$
- Min-max problems:
  for an instantaneous cost $\psi : [\![0:N]\!] \times \mathbf{X} \times \mathbf{U} \to \mathbb{R}$
  $$\mathcal{J}_n(\mathbf{x},\mathbf{u}) := \max_{k=n,\dots,N} \psi(k, x_k, u_k).$$

In these examples, if $L(k,\cdot,\cdot)$ and $\psi(k,\cdot,\cdot)$ (for each $k \in [\![0:N]\!]$) as well as φ are lower semicontinuous (lsc for short), then the respective cost $\mathcal{J}_n$ is lsc too. Furthermore, since the standing assumptions imply that the mapping $\mathbf{u} \mapsto \mathbf{x}_n^\xi(\mathbf{u})$ is continuous on $\mathbb{U}$, it follows that, for an initial state $\xi \in \mathbf{X}$ and initial time $n \in [\![0:N]\!]$, the mapping $\mathbf{u} \mapsto \mathcal{J}_n(\mathbf{x}_n^\xi(\mathbf{u}),\mathbf{u})$ is lsc over $\mathbb{U}$.

Remark 3.1. *The hypotheses we have done so far ensure that $\vartheta_n(\xi) > -\infty$ for any $\xi \in \mathbf{X}$ provided $\mathcal{J}_n$ is lsc. This follows from the fact that the set of feasible solutions to the dynamical system (1)-(2) is compact in $\mathbb{X}$. If that set is empty, then $\vartheta_n(\xi) = +\infty$, otherwise a solution is attained and $\vartheta_n(\xi) > -\infty$; the latter being a result of minimizing an lsc map on a compact set.*

Note that the viability kernel $\mathbb{V}_n$ agrees with the domain of the value function ϑ_n, and this holds true for any finite choice we make for the cost $\mathcal{J}_n$; here we use the convention $\inf\{\emptyset\} = +\infty$. We summarize this below.

Proposition 3.1. *For any $n \in [\![0:N]\!]$, $\xi \in \mathbf{X}$ and $\mathcal{G} \subseteq \mathbf{Y}$ one has*

$$\xi \in \mathbb{V}_n \quad \Longleftrightarrow \quad \vartheta_n(\xi) < +\infty.$$

This implies that if one wants to determine the viability kernel, one may consider to solve an optimization problem instead, and set the viability kernel as the domain of the value function of that optimization problem.

A DPP can be stated for the value function ϑ_n (cf. [9]), which can be used for computing such function and its domain. However, from a practical point of view two major issues arise: (i) the value function is likely to have infinite values (otherwise the viability kernel is the whole space) and (ii) the mixed constraints need to be verified at each step of computation, a fact that may considerably increase the computational time. This is also the case when pure state constraints are considered; cf. [29]. We take instead a different path, which has been inspired by the level-set approach studied in [1] for time-continuous optimal control problems with (pure) state constraints.

3.1. **Level-set approach.** Let us consider the following auxiliary optimal control problem and its corresponding value function

$$\omega_n(\xi, z) := \inf_{\mathbf{u}\in\mathbb{U}} \max\left\{\mathcal{J}_n(\mathbf{x}_n^\xi(\mathbf{u}),\mathbf{u}) - z, R_n(\mathbf{x}_n^\xi(\mathbf{u}),\mathbf{u})\right\} \tag{6}$$

defined for $\xi \in \mathbf{X}$ and $z \in \mathbb{R}$, where $R_n : \mathbb{X} \times \mathbb{U} \to \mathbb{R}$ is a function that satisfies

$$R_n(\mathbf{x},\mathbf{u}) \le 0 \;\Leftrightarrow\; g(k, x_k, u_k) \in \mathcal{G} \quad \forall k \in [\![n\!:\!N]\!]. \tag{7}$$

Note that (6) corresponds to an optimal control problem with only dynamical and input constraints (no mixed or state constraints). Also, for any $\mathbf{u} \in \mathbb{U}$ it follows that

$$R_n(\mathbf{x}_n^\xi(\mathbf{u}),\mathbf{u}) \le 0 \quad\Longleftrightarrow\quad \mathbf{u} \text{ and } \mathbf{x}_n^\xi(\mathbf{u}) \text{ satisfy (2)}.$$

Thus, the function R_n characterizes the constraints (2) of the optimal control problem via its zero level-set.

Remark 3.2. *A simple way to construct the function R_n from the initial data of the problem is got by taking*

$$R_n(\mathbf{x},\mathbf{u}) := \max_{k=n,\dots,N} \Phi(k, x_k, u_k), \tag{8}$$

where $\Phi : [\![0\!:\!N]\!] \times \mathbf{X} \times \mathbf{U} \to \mathbb{R}$ is a function such that

$$g(k, x, u) \in \mathcal{G} \quad\Longleftrightarrow\quad \Phi(k, x, u) \le 0.$$

A suitable choice for Φ could be the distance function

$$\Phi(k, x, u) = \operatorname{dist}\left(g(k, x, u), \mathcal{G}\right).$$

Moreover, for the case exposed in §2.1.1, that is, when for a given thresholds vector $c \in \mathbb{R}^p$ the set of constraints is given by (4), one may also consider the analogous with the signed distance function:

$$\Phi^c(k, x, u) = \max_{i=1,\dots,p} \left\{g_i(k, x, u) - c_i\right\}. \tag{9}$$

In this case, the functions $\Phi^c(k,\cdot,\cdot)$ are continuous for each $k \in [\![0 : N]\!]$. Consequently, R_n defined by (8) with Φ^c is also continuous.

Under mild assumptions, value functions having the structure of (6) attain the infimum at some control. Note that in the following statement, the function R_n does not necessarily satisfy (7). For other results concerning existence of optimal trajectories for discrete-time systems we refer to [15].

Proposition 3.2. *If $\mathcal{J}_n$ and R_n are lsc for $n \in [\![0\!:\!N]\!]$, then for any $\xi \in \mathbf{X}$ and $z \in \mathbb{R}$, there is an optimal control $\mathbf{u} \in \mathbb{U}$ so that the infimum defined in (6) is attained. In other words, there is $\mathbf{u} \in \mathbb{U}$ so that*

$$\omega_n(\xi, z) = \max\left\{\mathcal{J}_n(\mathbf{x}_n^\xi(\mathbf{u}),\mathbf{u}) - z, R_n(\mathbf{x}_n^\xi(\mathbf{u}),\mathbf{u})\right\}.$$

Proof. It is straightforward recalling that $\mathbb{U} \cong \mathbf{U}^{N+1}$ is a nonempty compact metric space, and the functional defined from $\mathbb{U}$ into $\mathbb{R}$ given by

$$\mathbf{u} \mapsto \max\left\{\mathcal{J}_n(\mathbf{x}_n^\xi(\mathbf{u}),\mathbf{u}) - z, R_n(\mathbf{x}_n^\xi(\mathbf{u}),\mathbf{u})\right\}$$

is lsc for $n \in [\![0 : N]\!]$, $\xi \in \mathbf{X}$ and $z \in \mathbb{R}$ fixed (it is the maximum of two lsc functions). □

3.2. **Properties of the auxiliary value function.** We now focus on some useful properties of the auxiliary value function ω_n defined by (6), that will allow us to use this function for solving the original optimal control problem (5), and determine its domain, which a posteriori will provide a practical way for computing the viability kernel of the controlled system (1)-(2).

For the remainder of this section $n \in [\![0\!:\!N]\!]$ and $\xi \in \mathbf{X}$ are fixed, and we assume, without further mentioning, that the cost $\mathcal{J}_n$ and the penalization function R_n satisfy

(H5) $\mathcal{J}_n$ and R_n are lsc for $n \in [\![0\!:\!N]\!]$.

We begin by showing a monotonicity property of the auxiliary value function.

Proposition 3.3. *The function $z \mapsto \omega_n(\xi, z)$ given by (6) is non-increasing.*

Proof. Let $z_1 \leq z_2$ and $\mathbf{u}_1 \in \mathbb{U}$ be an optimal control for $\omega_n(\xi, z_1)$ given by Proposition 3.2. Since $\max\{a, c\} \leq \max\{b, c\}$ if $a \leq b$, we have that

$$\max\left\{\mathcal{J}_n(\mathbf{x}_n^\xi(\mathbf{u}_1), \mathbf{u}_1) - z_2, R_n(\mathbf{x}_n^\xi(\mathbf{u}_1), \mathbf{u}_1)\right\} \leq \omega_n(\xi, z_1).$$

Since the left handside is greater than or equal to $\omega_n(\xi, z_2)$, the conclusion follows. □

Let us set now some further notation: for given $\lambda \in \mathbb{R}$, we define

$$\mathbb{U}_n(\xi, \lambda) := \left\{\mathbf{u} \in \mathbb{U} \;\mid\; R_n(\mathbf{x}_n^\xi(\mathbf{u}), \mathbf{u}) \leq \lambda\right\}$$

and

$$v_n(\xi, \lambda) := \inf_{\mathbf{u} \in \mathbb{U}_n(\xi,\lambda)} \left\{\mathcal{J}_n(\mathbf{x}_n^\xi(\mathbf{u}), \mathbf{u}) - \lambda\right\}. \tag{10}$$

Notice that the set $\mathbb{U}_n(\xi, \lambda) \subseteq \mathbb{U}$ is compact (eventually empty). Hence, one has that $v_n(\xi, \lambda) \in \mathbb{R}$ if and only if $\mathbb{U}_n(\xi, \lambda) \neq \emptyset$. Since we are considering a function R_n satisfying (7), it follows that

$$\mathbf{u} \in \mathbb{U}_n(\xi, 0) \quad \Leftrightarrow \quad \mathbf{u} \;\text{ and }\; \mathbf{x}_n^\xi(\mathbf{u}) \text{ satisfy (2)}.$$

Therefore $v_n(\xi, 0) = \vartheta_n(\xi)$. Moreover, from the definition of the viability kernel, one has

$$\mathbb{U}_n(\xi, 0) \neq \emptyset \;\Leftrightarrow\; \xi \in \mathbb{V}_n \;\Longleftrightarrow\; \vartheta_n(\xi) < +\infty. \tag{11}$$

The following proposition links the level-set of the function $\omega_n(\xi, \cdot)$ with the value $v_n(\xi, \lambda)$ defined in (10).

Proposition 3.4. *For any $\lambda \in \mathbb{R}$, one has*

$$\{z \in \mathbb{R} \;\mid\; \omega_n(\xi, z) \leq \lambda\} \subseteq [v_n(\xi, \lambda), +\infty).$$

Proof. Take $z \in \mathbb{R}$ such that $\omega_n(\xi, z) \leq \lambda$. Let $\bar{\mathbf{u}} \in \mathbb{U}$ be an optimal control for the value function $\omega_n(\xi, z)$, which is provided by Proposition 3.2. Then, on the one hand

$$R_n(\mathbf{x}_n^\xi(\bar{\mathbf{u}}), \bar{\mathbf{u}}) \leq \omega_n(\xi, z) \leq \lambda,$$

which implies that $\bar{\mathbf{u}} \in \mathbb{U}_n(\xi, \lambda)$. Since $\mathbb{U}_n(\xi, \lambda) \neq \emptyset$ we get that $v_n(\xi, \lambda)$ is finite. On the other hand, we have

$$\mathcal{J}_n(\mathbf{x}_n^\xi(\bar{\mathbf{u}}), \bar{\mathbf{u}}) - z \leq \omega_n(\xi, z) \leq \lambda.$$

Hence,

$$v_n(\xi, \lambda) = \inf_{\mathbf{u} \in \mathbb{U}_n(\xi,\lambda)} \mathcal{J}_n(\mathbf{x}_n^\xi(\mathbf{u}), \mathbf{u}) - \lambda \leq \mathcal{J}_n(\mathbf{x}_n^\xi(\bar{\mathbf{u}}), \bar{\mathbf{u}}) - \lambda \leq z.$$

□

We complement the preceding proposition with a series of equivalent properties.

Proposition 3.5. *For any $\lambda \in \mathbb{R}$, the following properties are equivalent:*

(a) *There exists $z \in \mathbb{R}$ such that $\omega_n(\xi, z) \leq \lambda$.*
(b) $\inf \left\{ R_n(\mathbf{x}_n^\xi(\mathbf{u}), \mathbf{u}) \mid \mathbf{u} \in \mathbb{U} \right\} \leq \lambda$.
(c) $\mathbb{U}_n(\xi, \lambda) \neq \emptyset$.
(d) $v_n(\xi, \lambda) < +\infty$.
(e) $v_n(\xi, \lambda) < +\infty$ *and* $\omega_n(\xi, z) \leq \lambda$ *for* $z \geq v_n(\xi, \lambda)$.
(f) $v_n(\xi, \lambda) < +\infty$ *and* $\omega_n(\xi, v_n(\xi, \lambda)) \leq \lambda$.

Proof. The implication $(a) \Rightarrow (b)$ is due to the fact that

$$\inf_{\mathbf{u} \in \mathbb{U}} R_n(\mathbf{x}_n^\xi(\mathbf{u}), \mathbf{u}) \leq \omega_n(\xi, z), \qquad \forall z \in \mathbb{R}.$$

Note that the infimum in (b) is attained, proving thus $(b) \Rightarrow (c)$. By the same arguments, one has $(c) \Leftrightarrow (d)$. Obviously one has $(e) \Rightarrow (f)$ and $(f) \Rightarrow (a)$, so, in order to finish the proof, we will show $(d) \Rightarrow (e)$.

Assume $v_n(\xi, \lambda) < +\infty$ and take $z \geq v_n(\xi, \lambda)$. Since $\mathbb{U}_n(\xi, \lambda) \neq \emptyset$, there is $\mathbf{u} \in \mathbb{U}_n(\xi, \lambda)$ so that one has $R_n(\mathbf{x}_n^\xi(\mathbf{u}), \mathbf{u}) \leq \lambda$ and

$$\mathcal{J}_n(\mathbf{x}_n^\xi(\mathbf{u}), \mathbf{u}) - \lambda = v_n(\xi, \lambda) \leq z.$$

Therefore

$$\omega_n(\xi, z) \leq \max\{\mathcal{J}_n(\mathbf{x}_n^\xi(\mathbf{u}), \mathbf{u}) - z, R_n(\mathbf{x}_n^\xi(\mathbf{u}), \mathbf{u})\} \leq \lambda,$$

which proves (e) and completes the proof. □

From propositions 3.4 and 3.5 we deduce the following result, which is a key property of our approach.

Proposition 3.6. *For any $\lambda \in \mathbb{R}$, one has*

$$\{z \in \mathbb{R} \ \mid \ \omega_n(\xi, z) \leq \lambda\} = [v_n(\xi, \lambda), +\infty).$$

In particular

$$\{z \in \mathbb{R} \ \mid \ \omega_n(\xi, z) \leq 0\} = [\vartheta_n(\xi), +\infty). \tag{12}$$

Proof. The inclusion " $\subseteq$ " was obtained in Proposition 3.4. The reverse inclusion is direct from Proposition 3.5. Indeed, if we take $z \in \mathbb{R}$ such that $z \geq v_n(\xi, \lambda)$, then $v_n(\xi, \lambda) < +\infty$ and therefore $\omega_n(\xi, z) \leq \lambda$, because of $(d) \Rightarrow (e)$ in Proposition 3.5.

Finally, the equality $v_n(\xi, 0) = \vartheta_n(\xi)$ proves (12). □

The preceding result implies that in order to compute the viability kernel of the dynamical system (1)-(2), one could first compute the value function ω_n, and then set the viability kernel as its zero level-set. In other words:

$$\mathbb{V}_n = \{\xi \in \mathbf{X} \mid \exists z \in \mathbb{R} \text{ such that } \omega_n(\xi, z) \leq 0\}. \tag{13}$$

Let us mention that the equality (12) was obtained in [1] for time-continuous optimal control problems with pure state constraints.

4. A method for computing the viability kernel

In this part of the paper we show how to use the level-set approach for computing the viability kernel based on the relation (13). Note that the viability kernel is independent of the choice done for the cost $\mathcal{J}_n$. Thus, to fix ideas, for any $(\mathbf{x}, \mathbf{u}) \in \mathbb{X} \times \mathbb{U}$ we set from now on

$$\mathcal{J}_n(\mathbf{x}, \mathbf{u}) = 0 \text{ and } R_n(\mathbf{x}, \mathbf{u}) = \max_{k=n,\dots,N} \Phi(k, x_k, u_k), \tag{14}$$

where $\Phi : [\![0\!:\!N]\!] \times \mathbf{X} \times \mathbf{U} \to \mathbb{R}$ is a given lsc function such that

$$g(k,x,u) \in \mathcal{G} \quad \Longleftrightarrow \quad \Phi(k,x,u) \leq 0. \tag{15}$$

Remark 4.1. *As we mention above, the viability kernel does not depend on the choice of $\mathcal{J}_n$ or R_n. In the remainder of the paper, we introduce a method for computing the viability kernel for $\mathcal{J}_n$ and R_n given by (14). However, we would like to remark that the proposed approach (with possibly other $\mathcal{J}_n$ or R_n) could allow to establish other methods for computing the viability kernel and provide at the same time additional information about this set, as for instance, its topological boundary.*

It follows from (14) that the functional R_n is lsc for any $n \in [\![0:N]\!]$, and so, by Proposition 3.2, the infimum in (6) is attained and the auxiliary value function becomes a minimax problem

$$\omega_n(\xi, z) = \min_{\mathbf{u}\in\mathbb{U},\ \mathbf{x}_n^\xi(\mathbf{u})=(x_k)_{k=0}^{N+1}} \max\left\{-z, \max_{k=n,\dots N} \Phi(k, x_k, u_k)\right\}.$$

In order to compute the auxiliary value function ω_n we use the DPP method. Note that this technique when applied to the function ω_n needs to be solved in $\dim(\mathbf{X}) + 1$ variables, because of the extra z variable. Nonetheless, because of the monotone character of the auxiliary value function with respect to that variable (Proposition 3.3), it is enough to compute ω_n only for $z = 0$ in order to recover the viability kernel. Consequently, the main result of this paper and fundamental property we use for computing the viability kernel of a system with mixed constraints is the following.

Theorem 4.1. *For any $n \in [\![0\!:\!N]\!]$ we have that*

$$\mathbb{V}_n = \{\xi \in \mathbf{X} \mid \omega_n(\xi, 0) \leq 0\}. \tag{16}$$

Proof. By Proposition 3.3 we have that

$$\omega_n(\xi, 0) \leq \omega_n(\xi, z), \qquad \forall z \geq 0.$$

On the other hand, by Proposition 3.6, we have that

$$\inf\{z \in \mathbb{R} \ \mid \ \omega_n(\xi, z) \leq 0\} = \vartheta_n(\xi) = \begin{cases} 0 & \text{if } \xi \in \mathbb{V}_n, \\ +\infty & \text{otherwise.} \end{cases}$$

The latter being a consequence of the choice of $\mathcal{J}_n$. The conclusion follows then from the characterization (13) of the viability kernel. □

Remark 4.2. *In the light of (16), observe that the viability kernel $\mathbb{V}_n$ is a closed set because $\omega_n(\cdot, 0)$ is a lsc function. In addition, $\mathbb{V}_n$ is a convex set provided that $\omega_n(\cdot, 0)$ is a quasiconvex function, which holds when the dynamics $F(k, \cdot, \cdot)$ are linear, the control space $\mathbf{U}$ is convex, and the functions $\Phi(k, \cdot, \cdot)$ in (15) are quasiconvex.*

Remark 4.3. *From the definition of the function $\omega_n(\cdot,0)$, observe that $\omega_n(\xi,0)$ is nonnegative for all $\xi \in \mathbf{X}$. Therefore, from (16) we deduce that $\xi \in \mathbb{V}_n$ if and only if $\omega_n(\xi,0)=0$. Hence, the value $\lambda = \omega_n(\xi,0)$ can be interpreted as a distance of ξ to $\mathbb{V}_n$.*

4.1. **Dynamic programming principle.** Theorem 4.1 implies that in order to compute the viability kernel one needs to calculate the value function $\omega_n(\cdot,0)$. For sake of exposition, let us denote by

$$\omega_n^0(\xi) := \omega_n(\xi,0), \qquad \forall \xi \in \mathbf{X}.$$

Note that by definition we have that

$$\omega_n^0(\xi) = \min_{\mathbf{u}\in\mathbb{U}} \left\{ \max_{k=0,\dots,N} \Phi^+(k,x_k,u_k) \mid \mathbf{x}_n^\xi(\mathbf{u}) = (x_k)_{k=0}^{N+1} \right\}$$

where $\Phi^+ := \max\{0,\Phi\}$. In particular

$$\omega_N^0(\xi) = \min_{u\in\mathbf{U}} \Phi^+(N,\xi,u).$$

The DPP for $(\omega_n^0)_{n=0}^N$ reads as follows:

Proposition 4.1. *Let $\Phi : [\![0\!:\!N]\!] \times \mathbf{X}\times\mathbf{U} \to \mathbb{R}$ be as above. For any $n \in [\![0\!:\!N-1]\!]$ and $\xi \in \mathbf{X}$ one has*

$$\omega_n^0(\xi) = \min_{u\in\mathbf{U}} \max\left\{\omega_{n+1}^0\left(F(n,\xi,u)\right), \Phi^+(n,\xi,u)\right\}. \tag{17}$$

Proof. For any $n \in [\![0\!:\!N-1]\!]$ given, let

$$W_n(\xi) := \inf_{u\in\mathbf{U}} \max\left\{\omega_{n+1}^0\left(F(n,\xi,u)\right), \Phi^+(n,\xi,u)\right\}$$

that is, the righthand side of (17) with infimum instead of minimum. Since $\mathbf{U}$ is compact and Φ is lsc, it is not difficult to see that ω_N^0 is lsc. Consequently, by induction we get that

$$u \mapsto \max\left\{\omega_{n+1}^0\left(F(n,\xi,u)\right), \Phi^+(n,\xi,u)\right\}$$

is also lsc, and then the infimum in W_n is always attained for any $n \in [\![0\!:\!N-1]\!]$.

Recall that the functional to be minimized in the definition of $\omega_n(\xi,z)$ is

$$\varphi_n\left(\mathbf{x},\mathbf{u},z\right) = \max_{k=n,\dots,N} \Phi^+(k,x_k,u_k).$$

It is then clear that

$$\varphi_n\left(\mathbf{x},\mathbf{u},z\right) = \max\left\{\varphi_{n+1}\left(\mathbf{x},\mathbf{u},z\right), \Phi^+(n,x_n,u_n)\right\}.$$

If $\mathbf{u} = (u_k)_{k=0}^N \in \mathbb{U}$ and $\mathbf{x} = \mathbf{x}_n^\xi(\mathbf{u}) = (x_k)_{k=0}^{N+1}$ is the corresponding trajectory, then recalling that $x_n = \xi$ and $x_{n+1} = F(n,\xi,u_n)$, we have that

$$\varphi_{n+1}\left(\mathbf{x}_n^\xi(\mathbf{u}),\mathbf{u}\right) \geq \omega_{n+1}^0\left(F(n,\xi,u_n)\right).$$

From here, we readily get that $\omega_n^0(\xi) \geq W_n(\xi)$.

For the other inequality, consider $\mathbf{u} = (u_k)_{k=0}^N \in \mathbf{U}$ an optimal control for the optimal control problem related to $\omega_{n+1}^0\left(F(n,\xi,u)\right)$. Let $\tilde{\mathbf{u}} = (\tilde{u}_k)_{k=0}^N$ be defined via $\tilde{u}_n = u$ and $\tilde{u}_k = u_k$ otherwise, for some $u \in \mathbf{U}$. Note that $\tilde{\mathbf{u}}$ is an admissible control for the problem associated with $\omega_{n+1}^0\left(F(n,\xi,u)\right)$ as well as for $\omega_n^0(\xi)$. Consequently,

$$\omega_n^0(\xi) \leq \varphi_n\left(\mathbf{x}_n^\xi(\tilde{\mathbf{u}}),\tilde{\mathbf{u}}\right) = \max\left\{\varphi_{n+1}\left(\mathbf{x}_n^\xi(\tilde{\mathbf{u}}),\tilde{\mathbf{u}}\right), \Phi^+(n,\xi,u)\right\}.$$

Finally, since $\omega_{n+1}^0(F(n,\xi,u)) = \varphi_{n+1}\left(\mathbf{x}_n^\xi(\tilde{\mathbf{u}}),\tilde{\mathbf{u}}\right)$ taking infimum over $u \in \mathbf{U}$ the proof is complete. $\square$

Remark 4.4. *It is not unexpected to obtain a DPP for the function ω_n^0 because it is well-known that the viability kernel satisfies the backward DPP (cf.* [9, Proposition 4.2]*). The novelty of the algorithm proposed below is that the DPP is obtained for a function that takes values in $\mathbb{R}_+$ and not in $\{0, +\infty\}$ as in* [9] *or values in $\{0, 1\}$ as in* [12]*. Allowing to have more than two values for the involved functions in the algorithm provides a deeper insight into the situation of the initial condition ξ, as pointed out in Remark 4.3. Indeed, the value $\omega_n^0(\xi) > 0$ gives a quantitative idea of how far an initial condition ξ is from the viability kernel. For instance, if the set of constraints $\mathcal{G}$ is given by $\mathcal{G}_c$ (see (4)) for a given vector $c = (c_1, \ldots, c_p) \in \mathbb{R}^p$, and the penalization function is given by $R_n(\mathbf{x}, \mathbf{u}) := \max_{k=n,\ldots,N} \Phi^c(k, x_k, u_k)$ with $\Phi^c(k, x_k, u_k)$ defined as in (9), one can see straightforwardly that if $\omega_n^0(\xi) = \lambda > 0$ (i.e., $\xi \notin \mathbb{V}_n$), then ξ belongs to the viability kernel associated to the (less demanding) set of constraints $\mathcal{G}_{c+\lambda\vec{1}}$, where $\vec{1} = (1, 1, \ldots, 1) \in \mathbb{R}^p$.*

4.2. **A scheme for computing the viability kernel.** To sum up, by combining Theorem 4.1 and Proposition 4.1, we obtain a practical method for computing the viability kernel of a system with mixed constraints, which has been summarized in Algorithm 1.

Algorithm 1: Computing the viability kernel

Input: $N \in \mathbb{N}$, $F : [\![0\!:\!N]\!] \times \mathbf{X} \times \mathbf{U} \to \mathbf{X}$, $g : [\![0\!:\!N]\!] \times \mathbf{X} \times \mathbf{U} \to \mathbf{Y}$, $\mathcal{G} \subseteq \mathbf{Y}$, $\Phi : [\![0\!:\!N]\!] \times \mathbf{X} \times \mathbf{U} \to \mathbf{Y}$ satisfying (15)

```
Let X_h ⊆ X be a mesh of size 0 < h ≪ 1 of a computational domain.
Let V_N, V_{N-1}, ..., V_0 be empty arrays.
for ξ_i ∈ X_h do
    Compute ω_N^0(ξ_i) = min_{u∈U} Φ^+(N, ξ_i, u).
    if ω_N^0(ξ_i)=0 then
        Save ξ_i in V_N.
Set X_h = V_N and n = N − 1.
while X_h ≠ ∅ and n ≥ 0 do
    for ξ_i ∈ X_h do
        Compute ω_n^0(ξ_i) = min_{u∈U} max {ω_{n+1}^0 (F(n, ξ_i, u)), Φ^+(n, ξ_i, u)}.
        if ω_n^0(ξ_i)=0 then
            Save ξ_i in V_n.
    Set X_h = V_n and n − 1 ← n.
return V_N, V_{N-1}, ..., V_0
```

5. Simulations

In this section we illustrate our approach by computing the viability kernel for two examples in which this set can be obtained analytically. These examples are taken from [13], where the authors study the following prey-predator controlled dynamical system:

$$\left\{ \begin{aligned} x_{k+1} &= x_k F_x(x_k, y_k, u_k^x) \\ y_{k+1} &= y_k F_y(x_k, y_k, u_k^y). \end{aligned} \right. \tag{18}$$

Here x_k (prey) and y_k (predator) represent the biomass of two species at period (typically years) $k \geq n$ (n the initial time). The controls $u_k^x,\ u_k^y \geq 0$ comprise the harvesting effort for each species, being the catches $u_k^x x_k$ and $u_k^y y_k$ measured in biomass. The functions F_x and F_y represent the species growth factors.

Given thresholds $x_{min},\ y_{min} \geq 0$ for minimal biomass levels, and $C_{min}^x,\ C_{min}^y \geq 0$ for minimal catch levels, the goal is to compute the viability kernel associated with the dynamics (18) and the following constraints:

$$x_k \geq x_{min}, \quad y_k \geq y_{min}, \quad u_k^x x_k \geq C_{min}^x, \quad u_k^y y_k \geq C_{min}^y.$$

Whenever the functions $F_x(x, y, \cdot)$ and $F_y(x, y, \cdot)$ are continuous, decreasing, and $F_x(x, y, u^x)$ and $F_y(x, y, u^y)$ are non positive for controls u^x, u^y sufficiently large, then (see [13, Proposition 2]), the viability kernel is

$$\mathbb{V}_n = \left\{ (x, y) \;\middle|\; \begin{array}{ll} x \geq x_{min}, & xF_x(x, y, C_{min}^x/x) \geq x_{min} \\ y \geq y_{min}, & yF_y(x, y, C_{min}^y/y) \geq y_{min} \end{array} \right\}, \tag{19}$$

for an initial time $n \in [\![0 : N-1]\!]$.

In Figure 1 we show the viability kernels obtained with our method, for two examples of dynamics F_x and F_y, for a finite horizon $N = 3$ and an initial time $n = 0$. The examples considered, taken from [13], are:

- Example 1: $\begin{cases} F_x(x, y, u^x) = Rx - \alpha xy - u^x x \\ F_y(x, y, u^y) = Ly + \beta xy - u^y y \end{cases}$;
- Example 2: $\begin{cases} F_x(x, y, u^x) = Rx - \frac{R}{\kappa} x^2 - \alpha xy - u^x x \\ F_y(x, y, u^y) = Ly + \beta xy - u^y y. \end{cases}$

The values of the positive parameters R, L, α, β, and κ are also borrowed from [13], where they are obtained from data of the anchovy-hake (prey, predator respectively) Peruvian fisheries. The thresholds considered are: $x_{min} = 7 \times 10^6$ tons, $y_{min} = 2 \times 10^5$ tons, $C_{min}^x = 3,779,300$ tons, and $C_{min}^y = 39,760$ tons.

In Figure 1 we also depict (with a continuous black line) the boundary of the viability kernel $\mathbb{V}_n$ given by (19) in order to check the accuracy of our method. The viability kernels are not exactly the same, this is due to the fact that one of the key assumptions in [13] for computing these sets analytically, is to consider unbounded controls u^x and u^y. For our approach and of course for simulation purposes, we have to consider bounded controls, which is anyway a more realistic hypothesis, because negative levels of biomasses (obtained for very large controls) make no sense.

6. Conclusion

In this paper we have introduced the level-set approach for discrete-time control systems with mixed constraints in a rather general framework. This approach allows us to characterize the viability kernel as the zero level-set of an auxiliary value function with only dynamical and input constraints, and thus, to implement a practical scheme for computing the viability kernel. The method has been illustrated with two examples where the viability kernel can be obtained analytically, showing then the pertinence of our algorithm. One of the main advantage of the algorithm proposed is that it does not spend time checking if the restrictions are satisfied at each iteration. Finally, let us emphasize that the method we propose also provides a notion of sensitivity when computing the viability kernel. Indeed, recall that the value $\omega_n^0(\xi)$ gives a quantitative idea of how far an initial condition

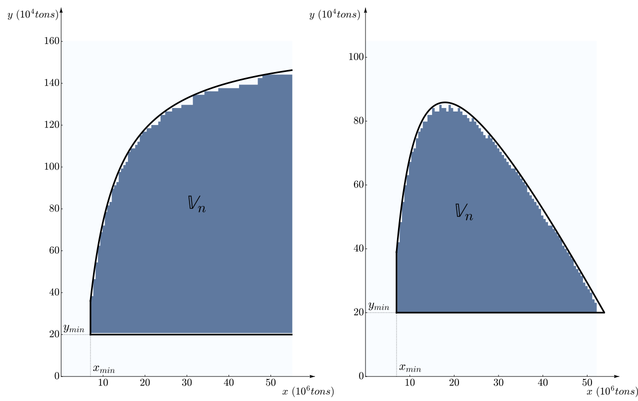

FIGURE 1. Viability kernels obtained in [13] for examples 1 (left) and 2 (right). Model parameters are: (Example 1) $R = 1.98$, $L = 0.955$, $\alpha = 1.220 \times 10^{-6}$ (tons)$^{-1}$, $\beta = 4.15 \times 10^{-8}$ (tons)$^{-1}$; (Example 2) $R = 2.25$, $L = 0.945$, $\kappa = 67,113 \times 10^3$ tons, $\alpha = 1.220 \times 10^{-6}$ (tons)$^{-1}$, $\beta = 4.845 \times 10^{-8}$ (tons)$^{-1}$. The continuous black lines delimit the real viability kernels, obtained with the analytical expression (19).

ξ is from the viability kernel. This can be seen more explicitly for the structure mentioned in §2.1.1. Indeed, if $\lambda = \omega_n^0(\xi) > 0$, then, although ξ is not viable for the constraints set $\mathcal{G}_c$ given by (4), it is for another set of constraints represented by the mixed constraints

$$g_i(k, x_k, u_k) \leq c_i + \lambda, \quad \forall\, k = n, \ldots, N; \;\; \forall\, i = 1, \ldots, p.$$

This is an advantage when considering other approaches based on the DPP with value functions that take only two possible values, as discussed in Remark 4.3 and 4.4.

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