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par

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Problèmes de commande optimale sur des domaines structurés et lois de commandes en boucles fermées stratifiées.

Composition du jury de thèse:

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Préface

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M.	Piernicola Bettiol	Examinateur
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From this point onwards, we adopt English as main language for the exposition. However, abstracts in French for each of the four parts of the manuscript have also been included.

This dissertation was defended in public on February the 5^{th}, 2015 at ENSTA ParisTech (Palaiseau, France) before the jury consisting of:

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Prof.	Richard Vinter	Jury's president
Prof.	Hasnaa Zidani	Advisor

Contents

Notation

$\mathbb{R}$	real numbers
$\mathbb{R} \cup \{+\infty\}$	extended real numbers
$\mathbb{N}$	positive integers
$\mathbb{B}_X(x, r)$	open ball of radius $r > 0$ centered at $x \in X$ of a metric space (X, d_X)
$\mathbb{B}_X$	open unitary ball of a normed vectorial space $(X, \lvert \cdot \rvert)$
$\mathcal{K}$	state-constraints set
$\mathcal{U}$	control space
$\mathcal{M}$	smooth manifold
Θ	target set
$\mathbb{M}_{n\times m}(\mathbb{R})$	real-valued matrices of $n \times m$
$\mathcal{L}(X, Y)$	linear continuous operators from X into Y
$\mathcal{C}^p(\mathcal{O})$	p-times continuously differentiable real-valued functions on $\mathcal{O}$
$\mathcal{AC}([a, b]; \mathbb{R}^n)$	the set of absolutely continuous functions $\gamma : [a, b] \to \mathbb{R}^n$
$L^1([a, b], \mathbb{R}^n)$	the class of equivalence of the Lebesgue integrable functions $\gamma : [a, b] \to \mathbb{R}^n$
$L^1([a, b], \mathbb{R}^n; d\mu)$	the class of equivalence of the $d\mu$-integrable functions $\gamma : [a, b] \to \mathbb{R}^n$
$\lvert \cdot \rvert$	Euclidean norm in $\mathbb{R}^N$
$\langle \cdot, \cdot \rangle$	inner dot in $\mathbb{R}^N$
$\mathbb{1}_{\mathcal{S}}(\cdot)$	indicator function of a set $\mathcal{S} \subseteq X$
$\mathrm{dist}_{\mathcal{S}}(\cdot)$	distance function to a set $\mathcal{S} \subseteq X$
$d_H(\mathcal{S}_1, \mathcal{S}_2)$	Hausdorff distance between $\mathcal{S}_1$ and $\mathcal{S}_2$
$\mathrm{proj}_{\mathcal{S}}(\cdot)$	projection over the set $\mathcal{S} \subseteq X$
$\nabla\varphi$	gradient of the function $\varphi : \mathcal{O} \to \mathbb{R}$
$\nabla^2\varphi$	Hessian matrix of the function $\varphi : \mathcal{O} \to \mathbb{R}$
$d_x\varphi$	differential of the function $\varphi : \mathcal{O} \to X$ at $x \in \mathcal{O}$
$d_x^2\varphi$	second differential of the function $\varphi : \mathcal{O} \to X$ at $x \in \mathcal{O}$

Dedicated to Marcela, my friend, partner and wife.

CHAPTER 1

Introduction

The aim of this dissertation is to study some optimal control problems of ordinary differential equations (ODEs) and feedback controls from a well-structured point of view.

We consider a parametrized dynamical system on $\mathbb{R}^N$:

$$\dot{y}(s) = f(y(s), u(s)), \quad u(s) \in \mathcal{U}, \quad \text{for a.e. } s \in (t, T), \quad y(t) = x. \tag{1.1}$$

The elements that define a control system are the following: $T \in (0, +\infty]$ is the *final horizon*, $t \in [0, T)$ is the *initial time*, $x \in \mathbb{R}^N$ is the *initial position*, $u : [t, T) \to \mathcal{U}$ is the *control function* with values in the *control space* $\mathcal{U}$ and $f : \mathbb{R}^N \times \mathcal{U} \to \mathbb{R}^N$ is the *dynamics* mapping. Depending on the issue at hand, the dynamical system can be written in a more general way, as a differential inclusion:

$$\dot{y}(s) \in F(y(s)), \quad \text{for a.e. } s \in (t, T), \quad y(t) = x. \tag{1.2}$$

The set-valued map $F : \mathbb{R}^N \rightrightarrows \mathbb{R}^N$ is still called the *dynamics* map.

Under mild hypotheses, given a *measurable* control function $u(\cdot)$, the control system (1.1) admits a *unique* solution, named the *state* of the system, which is an *absolutely continuous* function. To emphasis the reliance on control and initial data, we reserve the notation $y^u_{t,x}(\cdot)$ for such trajectory.

Notable examples of controlled vector fields are the linear systems

$$\dot{y}(s) = Ay(s) + Bu(s), \quad \text{for a.e. } s \in (t, T).$$

and the control-affine ones

$$\dot{y}(s) = f_0(y(s)) + \sum_{i=1}^{m} u_i(s) f_i(y(s)), \quad u(s) = (u_1(s), \ldots, u_m(s)) \in \mathcal{U}, \quad \text{for a.e. } s \in (t, T).$$

In many real applications, the state variable is constrained to remain in a subset $\mathcal{K} \subseteq \mathbb{R}^N$. These constraints reflect physical or economical restrictions. Consequently, we are usually concerned with controlled trajectories that verify

$$y(s) \in \mathcal{K}, \quad \forall s \in [t, T), \tag{1.3}$$

where $\mathcal{K}$ is a closed set, called the *state-constraints set*. The collection of controls which make a solution to (1.1) feasible on $\mathcal{K}$ is known as the *admissible controls* and is given by

$$\mathbb{U}^T_t(x) := \left\{ u : [t, T) \to \mathcal{U} \text{ measurable} \mid y^u_{t,x}(s) \in \mathcal{K},\ \forall s \in [t, T) \right\}, \quad \forall (t, x) \in [0, T) \times \mathcal{K}.$$

Likewise, the set of *admissible trajectories* of (1.2) is defined via

$$\mathbb{S}_t^T(x) := \left\{ y \in \mathcal{AC}([t,T];\mathbb{R}^N) \mid y \text{ satisfies (1.2) and (1.3)} \right\}, \quad \forall (t,x) \in [0,T) \times \mathcal{K}.$$

We consider a general optimal control process with fixed final horizon $T > 0$

$$\inf \left\{ \int_t^T e^{-\lambda s} \ell(y_{t,x}^u(s), u(s)) ds + e^{-\lambda T} \psi(y_{t,x}^u(T)) \;\middle|\; u \in \mathbb{U}_t^T(x) \right\}, \; \forall (t,x) \in [0,T) \times \mathcal{K}, \tag{1.4}$$

where $\lambda \geq 0$ is the *discount factor*, $\psi : \mathbb{R}^N \to \mathbb{R} \cup \{+\infty\}$ is the *final cost* and $\ell : \mathbb{R}^N \times \mathcal{U} \to \mathbb{R}$ is the *running cost*. For the purposes of the thesis, we are specially interested in the *infinite horizon problem* ($T = +\infty$ and $\psi \equiv 0$) and in the *Bolza problem* ($T < +\infty$ and $\lambda = 0$).

A noteworthy scenario of the Bolza case is the *Mayer problem* ($\ell \equiv 0$). In this situation, there is no direct dependence upon the control on the cost. In consequence, the optimization model (1.4) can be written in terms of the trajectories of the differential inclusion (1.2), i.e.

$$\inf \left\{ \psi(y(T)) \;\middle|\; y \in \mathbb{S}_t^T(x) \right\} \quad \forall (t,x) \in [0,T) \times \mathcal{K}.$$

A classical type of optimal control process is the quadratic one, which is determined via:

$$\ell(y,u) = \langle Qy, y \rangle + \langle Ru, u \rangle \quad \text{and} \quad \psi(y) = \langle Py, y \rangle, \quad \forall y \in \mathbb{R}^N, \; u \in \mathcal{U}.$$

The Value Function is the mapping that associates any initial time t and initial position x with the cost-to-go of the optimization problem (1.4). We reserve the letter $\vartheta(\cdot)$ to denote this function everywhere in the manuscript. In general this map is defined on $[0,T] \times \mathcal{K}$ and may take unbounded values, that is, $\vartheta : [0,T] \times \mathcal{K} \to \mathbb{R} \cup \{+\infty\}$. For the infinite horizon case the initial time is fixed at $t = 0$, so the functional to be minimized depends exclusively on the initial state. Consequently, with a slight abuse of notation, we write

$$\vartheta(x) := \inf \left\{ \int_0^\infty e^{-\lambda s} \ell(y_x^u(s), u(s)) ds \;\middle|\; u \in \mathbb{U}(x) := \mathbb{U}_0^\infty(x) \right\}, \quad \forall x \in \mathcal{K}.$$

In the formulation of (1.4) we may also consider that the final horizon is not fixed, which leads to a more general class of optimal control processes. Among these, the most relevant for the exposition is the so-called *Minimum time problem* to reach a given target $\Theta \subseteq \mathbb{R}^N$

$$\inf \left\{ T \geq 0 \mid \; y \in \mathbb{S}_0^T(x) \; \text{ and } y(T) \in \Theta \right\}, \quad \forall x \in \mathcal{K}.$$

In this case we write the Value Function as $T^\Theta(\cdot)$ and name it minimum time function.

1.1 Hamilton-Jacobi-Bellman Approach

The interest in studying the optimal cost of the problem (1.4) as a function of the initial data lies in the potentiality of computing this value before solving the optimization problem. The most powerful tool for doing so is the HJB approach, which is a technique based on a functional equation known as the *Dynamic Programming Principle (DPP)*. This methodology dates from the 1950's and was first studied by Bellman and his coauthors.

This equation has different forms based on the issue at hand:

- **Infinite horizon problem:**

$$\vartheta(x) = \inf_{u\in\mathbb{U}(x)} \left\{ \int_0^\tau e^{-\lambda s}\ell(y_x^u(s),u(s))ds + e^{-\lambda\tau}\vartheta(y_x^u(\tau)) \right\}, \quad \forall x \in \mathcal{K}, \forall \tau \in [0,+\infty).$$

- **Bolza problem:**

$$\vartheta(t,x) = \inf_{u\in\mathbb{U}_t^T(x)} \left\{ \int_t^\tau \ell(s, y_{t,x}^u(s),u(s))ds + \vartheta(\tau, y_{t,x}^u(\tau)) \right\}, \quad \forall x \in \mathcal{K},\ \forall 0 \le t \le \tau \le T.$$

- **Minimum time problem:**

$$T^\Theta(x) = \inf_{y\in\mathbb{S}_0^\tau(x)} \left\{ \tau + T^\Theta(y(\tau)) \right\}, \quad \forall x \in \mathcal{K},\ \forall \tau \in [0, T^\Theta(x)].$$

The main advantage of this method is that, in essence, the Value Function is the unique mapping that verifies the DPP and therefore, the idea is to find an equivalent formulation of this optimality principle in terms of a partial differential equation called the *Hamilton-Jacobi-Bellman (HJB) equation* associated with the *Hamiltonian*

$$H(x,\zeta) := \sup\{-\langle f(x,u),\zeta\rangle - \ell(x,u) \mid u \in \mathcal{U}\}, \quad \forall x \in \mathcal{K},\ \forall \zeta \in \mathbb{R}^N.$$

For dynamical systems governed by a differential inclusion, the Hamiltonian is given by

$$H(x,\zeta) := \sup\{-\langle v,\zeta\rangle \mid v \in F(x)\} \quad \forall x \in \mathcal{K},\ \forall \zeta \in \mathbb{R}^N.$$

If there are no effective state-constraints, that is, $\mathcal{K} = \mathbb{R}^N$, and the Value Function is differentiable, then the DPP implies that $\vartheta(\cdot)$ verifies the following HJB equation:

- **Infinite horizon problem:** $\lambda\vartheta(x) + H(x,\nabla\vartheta(x)) = 0, \quad x \in \mathbb{R}^N$.
- **Bolza problem:** $-\partial_t\vartheta(t,x) + H(x,\nabla_x\vartheta(t,x)) = 0, \quad (t,x) \in (0,T)\times\mathbb{R}^N$.
- **Minimum problem:** $-1 + H(x,\nabla T^\Theta(x)) = 0, \quad x \in \mathrm{int}(\mathrm{dom}\, T) \setminus \Theta$.

However, the Value Function is rarely differentiable, and consequently, solutions to the HJB equations need to be understood in a weak sense. The most suitable framework to deal with these equations is the *Viscosity Solutions Theory* introduced by Crandall-Lions in 1983 in [45]. This methodology is based on two semisolution concepts, namely the *viscosity supersolution* and *subsolution*, respectively. The theory provides existence and uniqueness for a much more general class of fully nonlinear partial differential equations, known as the Hamilton-Jacobi equations (not necessarily related to an optimal control problem). Classical surveys on the topic are the paper of Crandall-Ishii-Lions [44], and the manuscripts of Bardi-Capuzzo-Dolcetta [13] and Barles [14] among many others.

Initially, the theory of *viscosity solutions* was designed for continuous functions but it was extended to non-continuous frameworks. Ishii in [77] introduced a notion of solution using upper and lower semicontinuous envelope functions. For the plan of the dissertation, the most relevant definition of discontinuous viscosity solution is the one introduced by Barron-Jensen in [18] and it is known as *bilateral viscosity solution*. Using tools from Nonsmooth Analysis, Frankowska in [51] showed that bilateral solutions are intrinsically related to monotone properties of the Value Function along trajectories. This idea was already investigated by Subbotin in [124, 125] for the context of differential games, where the notions of *u-stable* and *v-stable* functions were introduced in order to characterize the Value Function of a differential game. This approach is also closely connected with the exposition of the manuscript, because in order to characterize the constrained Value Function we use the notions of *weakly decreasing* and *strongly increasing* functions along trajectories of the control system.

1.1.1 Constrained viscosity solutions

If the state-constraints are not trivial, many details need to be taken into account. The principal difference between state-constrained and unconstrained processes, lies in the structure of the admissible trajectories map $\mathbb{S}_t^T(\cdot)$. Indeed, in absence of state-constraints this multifunction is locally Lipschitz continuous in $\mathbb{R}^N$ and, by contrast, in the constrained case it may vary from point to point in a very complicated way.

This fact has three important consequences which make the study more delicate to treat.

- $\vartheta(\cdot)$ may neither be continuous nor real-valued even if the data is regular.
- The Value Function is a *constrained viscosity solution* of the HJB equation, that is,

  $$\vartheta(\cdot) \text{ is a supersolution on } \mathcal{K} \text{ and a subsolution on } \operatorname{int}(\mathcal{K}).$$

  Nevertheless, it may not be the unique function that verifies this.
- The sole information about $\vartheta(\cdot)$ on the boundary comes from the supersolution.

The first point entails technical difficulties that can be treated anyway in the setting of bilateral viscosity solutions. Nevertheless, without additional compatibility assumptions involving the dynamics and the state-constraints set, there is no known technique that allows to identify the Value Function as the unique solution, in a weak sense, of an HJB equation. This is mainly due to the lack of information on the boundary of the state-constraints. In particular, the HJB equation may have many solutions in a same class of functions which precludes a possible characterization. The works of Ishii-Koike [78] and Bokanowski-Forcadel-Zidani [23] pointed out that, in the general case, HJB equation should be completed by additional information on boundary of the state-constraints.

To deal with the aforementioned difficulties, the current literature provides principally two approaches. The first one consists in looking for conditions in order to ensure that the Value Function is uniformly continuous on its whole domain, so that the information in the interior of the state-constraints is enough to determine the Value Function on the entire domain. This approach was started by Soner in [120, 121] and then consecutively studied by several authors; we refer to [88, 38, 89, 94, 78, 95, 123, 42, 99, 52] among many others. This strategy leads to a continuous notion of constrained viscosity solution of the HJB equation.

It was shown by Soner in [120, 121], that if the Value Function is uniformly continuous on the state-constraints, it is then the unique constrained viscosity solution of the HJB equation. This was done for a fairly wide class of state-constraints sets; cf. [13, Chapter 4.5].

However, this result turned out the quest into finding sufficient conditions to assure the uniform continuity of the optimal cost map. Here is when the compatibility assumptions start playing a role. The first one that appeared in the literature is the so-called *Inward Pointing Condition (IPC)*. It was equally introduced by Soner in the context of open domains with smooth boundary but, as a matter of fact, it has been object of subsequence extension to less restrictive frameworks; we refer for instance to the works of Stern [123], Clarke-Stern [42] and more recently, Frankowska-Mazzola [52]. Under the former circumstances, if $\mathbf{n}_{\text{ext}}(x)$ is the unit exterior normal to $\mathcal{K}$ at $x \in \partial\mathcal{K}$, the condition can be stated as follows:

$$\inf_{u\in\mathcal{U}} \langle f(x,u), \mathbf{n}_{\text{ext}}(x)\rangle < 0, \quad \forall x \in \partial\mathcal{K}.$$

The IPC has as main goal to provide a *Neighboring Feasible Trajectories (NFT)* theorem, which basically says that any feasible trajectories can be approximated by a sequence of arcs which remain in the interior of the state-constraints. Under these circumstances, the NFT theorem certifies the continuity of the Value Function. From a geometrical point of view, it says that at each point of $\partial\mathcal{K}$, there exists a controlled vector field pointing into $\mathcal{K}$; see Figure 1.1 for a graphic illustration.

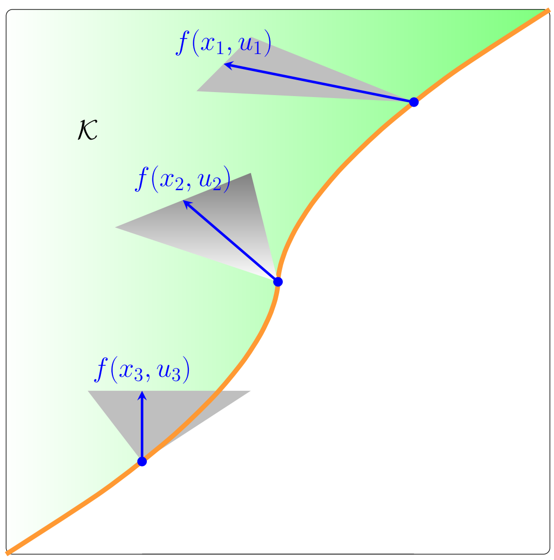


Figure 1.1: An example of Inward Pointing Condition.

The preceding condition is merely a sufficient requirement for ensuring the continuity of the Value Function, nevertheless, it is not difficult to construct an instance in which the IPC fails at a single point and the Value Function is not more than lower semicontinuous; see [13, Example 5.3 page 276]. Furthermore, the IPC is not a generic property and may fail even for very simple cases as the following situation shows.

Example 1.1.1. *Consider a mechanical system governed by a second order equation for which the velocity and the position are bounded:*

$$\ddot{y}(s) = \varphi(y(s), \dot{y}(s), u(s)), \;\; u(s) \in \mathcal{U}, \;\; a.e. \;\; s \in [0,T), \quad y(s) \in [a,b], \;\; \dot{y}(s) \in [c,d], \;\; \forall s \in [0,T).$$

By using the transformation $y_1 = y$ and $y_2 = \dot{y}$, the system can be rewritten as:

$$\begin{pmatrix} \dot{y}_1(s) \\ \dot{y}_2(s) \end{pmatrix} = f(y_1(s), y_2(s), u(s)) := \begin{pmatrix} y_2(s) \\ \varphi(y_1(s), y_2(s), u(s)) \end{pmatrix}, \ (y_1(s), y_2(s)) \in \mathcal{K}_0 = [a,b] \times [c,d].$$

In particular,

$$\langle f(x_1, x_2, u), \mathbf{n}_{ext}(a, x_2) \rangle = -x_2, \quad \forall x_2 \in (c,d), \forall u \in \mathcal{U}.$$

Notice that this quantity does not depend on the control nor in the initial dynamics φ but only on the sign of x_2, and so, for some values of x_2 the dynamics will point into $\mathcal{K}_0$ and for others it will point into $\mathbb{R}^2 \setminus \mathcal{K}_0$. A similar analysis can be done for the boundary points in $\{b\} \times (c,d)$.

The second approach mentioned earlier assumes that the Value Function may not be continuous, and seeks for conditions with a view to guarantee that the informations coming from the interior of the state-constraints reach the boundary. This methodology was introduced by Frankowska-Vinter in [54] using Nonsmooth Analysis techniques, and then it was extended to more general situations by Frankowska and his coauthors [53, 52].

In [54] the authors have shown, for convex-valued dynamics, that the Value Function is the unique lower semicontinuous solution of the HJB equation (in the constrained bilateral sense). However, to do so, the authors have to assume a compatibility assumption called the *Outward Pointing Condition (OPC)*; this kind of assumptions were already considered by Blanc in [21] for exit-time problems. This requirement for an open domain with smooth boundary can be written as

$$\sup_{u \in \mathcal{U}} \langle f(x,u), \mathbf{n}_{\text{ext}}(x) \rangle > 0, \quad \forall x \in \partial\mathcal{K}.$$

The techniques used in [54, 53, 52] heavily rely upon an NFT theorem as well.

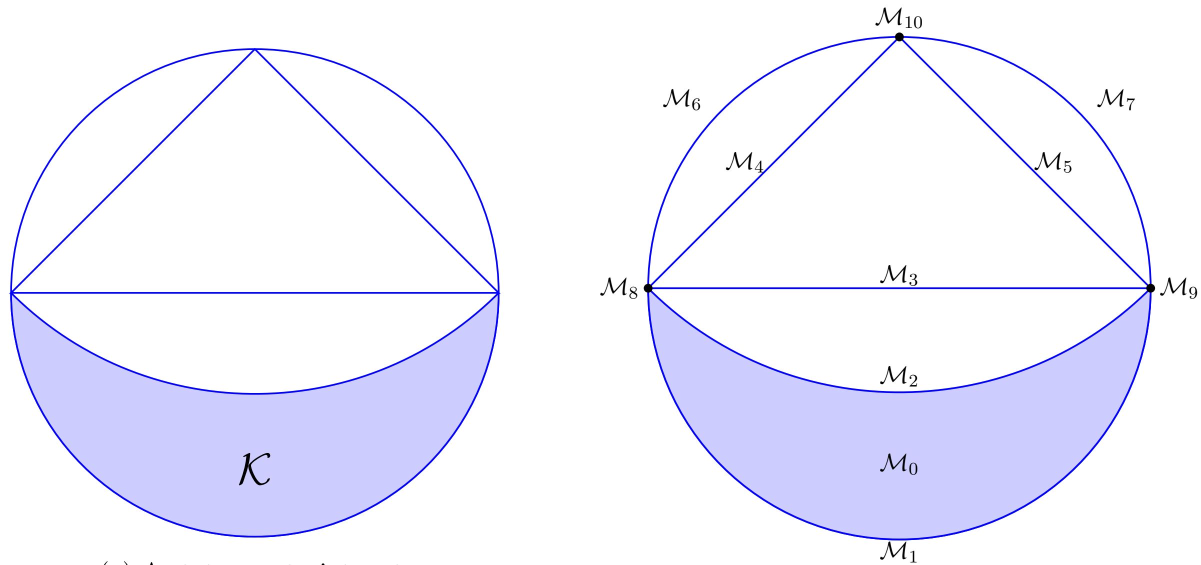
On the other hand, since the OPC has a similar nature than the IPC, it is not difficult to see that in Example 1.1.1, where we have exhibited that the IPC fails, the OPC also fails. Additionally, the OPC can be interpreted as an IPC for the backward dynamics and so it is not a generic property either.

The preceding discussion yields to wonder if there are other alternative compatibility assumptions, different from the pointing conditions, that allow to identify the Value Function as the single generalized solution of an HJB equation.

> **Question:** *Is it possible to characterize the Value Function of an optimal control problem in the presence of state-constraints with new compatibility assumptions or further structural requirements?*

To the best of our knowledge, very little has been said about this question in the current literature. This fact has motivated part of the work we have developed for this manuscript. We have addressed the contributions of this thesis regarding this question in Chapters 4, 5 and 6. In these expositions we study optimal control problems with well-structured state-constraints and dynamics. In each situation we provide a theorem (in the lower semicontinuous context) which allows to identify the Value Function of the corresponding problem as the unique solution of an HJB systems of inequalities. This has been accomplished without making use of any pointing condition, but other type of assumptions, well-suited for the structure of the problems.

The main feature of the theory we have exposed in Chapter 4 is that the set of state-constraints admits a stratified structure, meaning that it can be decomposed into a locally finite



(a) A state-constraints set.

(b) A particular stratification of $\mathcal{K}$.

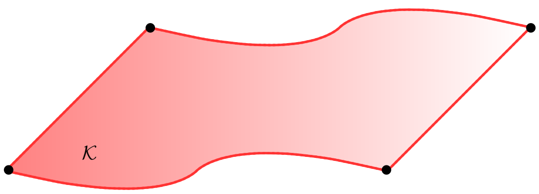
Figure 1.2: An example of stratifiable state-constraints set on $\mathbb{R}^2$.

family of embedded manifolds of $\mathbb{R}^N$ or *strata*. This framework allows us to treat broader classes of state-constraints than the NFT approach because we do not need to impose the condition $\overline{\operatorname{int}\mathcal{K}} = \mathcal{K}$ which is essential for the NFT theorems. In Figure 1.2a we show an example of a state-constraints that can be covered by our work. Indeed, a possible stratification has been illustrated in Figure 1.2b. Here $\mathcal{M}_0 = \operatorname{int}(\mathcal{K})$, $\mathcal{M}_1, \ldots, \mathcal{M}_7$ are bounded curves and $\mathcal{M}_8$, $\mathcal{M}_9$ and $\mathcal{M}_{10}$ are single points. If we consider the infinite horizon problem with $\mathcal{K}$ as in Figure 1.2a and the stratification given in Figure 1.2b, the theory we propose yields to claim that the Value Function is the unique lower semicontinuous function with superlinear growth that verifies the following systems of inequalities in the viscosity sense:

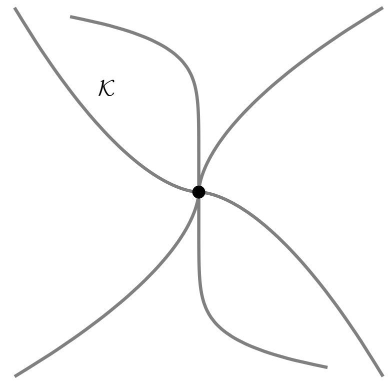
$$\begin{aligned}
\lambda\vartheta(x) + H(x, \nabla\vartheta(x)) &\geq, 0 && x \in \mathcal{K},\\
\lambda\vartheta(x) + H(x, \nabla\vartheta(x)) &\leq 0, && x \in \operatorname{int}\mathcal{K},\\
\lambda\vartheta(x) + H_i(x, \nabla\vartheta_i(x)) &\leq 0, && x \in \mathcal{M}_i,\ i = 1, \ldots, 7,\\
\lambda\vartheta(x) - \min_{u\in\mathcal{U}}\{\ell(x,u) \mid f(x,u) = 0\} &\leq 0, && x \in \mathcal{M}_i,\ i = 8, \ldots, 10,
\end{aligned}$$

where $H_i : \mathcal{M}_i \times \mathbb{R}^N \to \mathbb{R} \cup \{+\infty\}$ is a Hamiltonian that is either $-\infty$ or locally Lipschitz, and ϑ_i is the restriction of ϑ to $\mathcal{M}_i$. Notice that the first two inequalities lead to the usual notion of constrained bilateral viscosity solution; see for instance [54, 123, 42, 131]. Thus, the contribution of the dissertation complements the standard constrained HJB equations.

Furthermore, the collection of stratifiable sets is quite vast and includes closed manifolds with or without boundary and semilinear, semialgebraic and finitely subanalytic sets. Among these, we might also count in the topological networks as its extension to larger dimensions. The latter remark motivates the development reported in Chapter 9 where we extend some results of Chapter 4 to a discontinuous dynamical setting. Indeed, in a purely network context, the dynamics and cost might differ from branch to branch which yield to a discontinuous dynamical systems that does not fit completely in the framework of Chapter 4, and so, it deserves a specialized treatment. This study is developed in detail in Chapter 9, where we deal with standard topological networks and generalized d-dimensional networks, that is, the situation in which the junctions and branches are embedded manifolds of dimension $d-1$ and d, respectively.



(a) The closure of an embedded manifold with piecewise smooth boundary.



(b) A topological network.

Figure 1.3: Some further examples of stratifiable state-constraints sets.

The compatibility assumptions in Chapter 4 are written in terms of the stratification and correspond to, first a Lipschitz property of the dynamics restrained to each stratum, and second, to a local controllability condition over the strata where a chattering phenomenon may occur; in Figure 1.2a the controllability assumption only matters on $\mathcal{M}_1$ and $\mathcal{M}_2$, this is because we need to control the possible *chattering* curves between $\mathcal{M}_0$ and, $\mathcal{M}_1$ or $\mathcal{M}_2$.

Chapter 5 and Chapter 6 are dedicated to study convex state-constraints. In the first case we investigate problems with linear-like dynamics. We obtain two NFT theorems taking advantage only of the structure of the state-constraints and the dynamics. The Accessibility Lemma of Convex Analysis turns out to be crucial for the analysis. One of the main result of this chapter (Theorem 5.2.1) implies in particular that if the controlled vector field is linear, the Value Function $\vartheta(\cdot)$ of the infinite horizon problem is the only lower semicontinuous function, which is a viscosity supersolution on $\mathcal{K}$ and a subsolution on $\mathrm{ri}(\mathcal{K})$ of

$$\lambda\vartheta(x) + H(x, \nabla\vartheta(x)) = 0.$$

In Chapter 6 we study the state-constraints in the light of a class of penalization maps known as the *Legendre functions*. This approach allows us to associate the state-constrained problem with an unconstrained one, and so, use the already known theory of HJB equations with unrestricted state-space on the auxiliary optimal process and afterwards, transport the result to the original problem by mean of a suitable change of coordinates. In this framework, we are able to prove (Theorem 6.2.1) that, for dynamics having an absorbing property at the boundary, the Mayer Value Function is the unique uniformly continuous function on $\mathrm{ri}(\mathcal{K})$ that is a viscosity solution of

$$-\partial_t\vartheta(t,x) + H(x, \nabla_x\vartheta(t,x)) = 0, \quad \forall (t,x) \in (0,T) \times \mathrm{ri}(\mathcal{K}).$$

The fundamental tool needed for the analysis is the *Legendre change of coordinates*. This object together with its principal features are detailed Section 2.3.1.

We finally mention that there are some alternative methodologies to aboard the HJB approach. For instance, in [4], Altarovici-Bokanowski-Zidani have shown that, under fairly general assumptions, it is always possible to compute $\vartheta(\cdot)$ via an auxiliary problem without state-constraints. That article is devoted to study the epigraph of the Value Function via an exact penalization technique, which leads to a constructive way for determining the optimal cost mapping and to its numerical approximation.

1.2 Optimal Feedback Laws

The ultimate goal of optimal processes, as any optimization model, is to find at least one minimizer. To be more accurate, we seek to determine for any $x \in \mathcal{K}$ and $t \in [0,T)$ a measurable control $u^*_{t,x} \in \mathbb{U}^T_t(x)$ and the respective optimal trajectory of the dynamical system $y^*_{t,x} \in \mathbb{S}^T_t(x)$, which minimize the cost involved in the problem. If this synthesis procedure can be done for any (t,x) that belongs to a subset $\Omega \subseteq [0,+\infty)\times\mathcal{K}$, we can thereupon construct a map $U : \Omega \to \mathcal{U}$ via the optimality condition

$$U(s, y^*_{t,x}(s)) = u^*_{t,x}(s), \quad \text{for a.e. } s \in [t,T],\ \forall (t,x) \in \Omega.$$

This function is called an *optimal feedback* for the control system and the methodology is usually referred as the *feedback synthesis.*

Example 1.2.1. *Consider the soft landing problem. This is a minimum time problem that can be modeled as follows,*

$$\min T \quad s.t.\ \dot{y}(s) = \begin{pmatrix} y_2(s) \\ u(s) \end{pmatrix},\ u(s) \in [-1,1] \text{ for a.e. } s \in [0,T], \quad y(0) = x,\ y(T) = (0,0).$$

The optimal synthesis (see Figure 1.4) does not depend upon the time but only on the state, and it is given by

$$U(x) = \begin{cases} 1 & x \in \{2x_1 < -\,sign(x_2)x_2^2\} \cup \{2x_1 = x_2^2, x_2 < 0\}, \\ -1 & x \in \{2x_1 > -\,sign(x_2)x_2^2\} \cup \{2x_1 = -x_2^2, x_2 > 0\}, \\ 0 & x = (0,0), \end{cases} \quad x \in \mathbb{R}^2.$$

One way to reckon with the feedback synthesis is through the so-called *Pontryagin's maximum principle.* This technique is an evolution of the *Lagrange's multipliers rule*, well-suited for optimal control problems, which delves for necessary conditions of optimality. Consequently, it allows to single out a family of *candidates* to local solutions which may or not be minimizers. There is a huge literature addressed to necessary conditions; see for instance the classical monographs of Pontryagin-Boltayanski-Gramgrelidze-Mischenko [103], Lee-Markus [85], some more recent books, Vinter [131], Schäettler-Ledzewicz [118] and Clarke [40], and the survey of Hartl-Sethi-Vickson [66].

Remark 1.2.1. *Several authors have address their into conditions that ensure the optimality of the candidates to solution obtained by means of the Pontryagin's maximum principle. One*

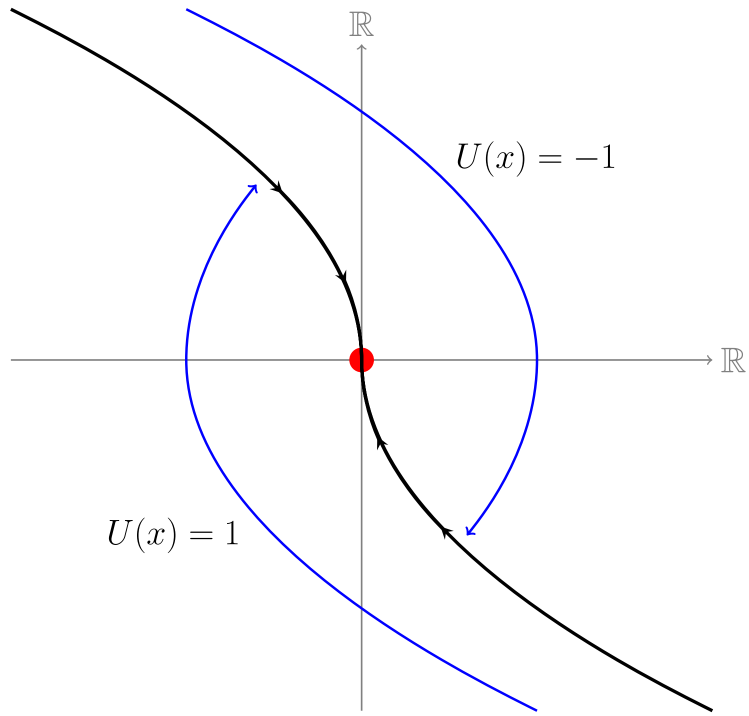


Figure 1.4: The optimal synthesis of the soft landing problem.

interesting idea called regular synthesis was introduced by Boltyanskii in [24] and subsequently generalized to broader settings by many authors; see for instance Brunovsky [36], Piccoli-Sussmann [102] and Piccoli [101], the latter dealing with state-constrained problems. This concept rises as an alternative to our approach because, instead of considering discontinuous feedback as we are going to do shortly, it deals with a collection of extremals that cover the whole state space and fit together in an appropriate way. We also mention that behind the idea of regular synthesis there is as well an underlying stratified structure.

A different way to proceed consists in calculating the Value Function and then use the HJB equation to compute the optimal feedback as one of the controls that realizes the maximum on the Hamiltonian, that is,

- **Infinite horizon problem:** $U(x) \in \operatorname{argmin}_{u\in\mathcal{U}}\langle f(x,u), \nabla\vartheta(x)\rangle + \ell(x,u)$.
- **Bolza problem:** $U(t,x) \in \operatorname{argmin}_{u\in\mathcal{U}}\langle f(x,u), \nabla_x\vartheta(t,x)\rangle + \ell(x,u)$.
- **Minimum problem:** $U(x) \in \operatorname{argmin}_{u\in\mathcal{U}}\langle f(x,u), \nabla T^{\Theta}(x)\rangle$.

Notice that, taking into consideration the above rules, an optimal feedback is in reality a *selection* map. Therefore, disregarding the fact that $U(\cdot)$ may not be well-defined, there is no reason to assume that it is uniquely determined, or that it can be chosen in such a way it defines a regular function.

On the other hand, in order to overcome the presumable existence issues, we may enlarge the set of minimizers by allowing almost optimal ones. The advantage of doing so, is that suboptimal controls exist whenever the Value Function is finite. Formally, given a precision parameter $\varepsilon > 0$, an *ε-suboptimal control* is an admissible control $u_\varepsilon \in \mathbb{U}_t^T(x)$, for which

$$\int_t^T e^{-\lambda s}\ell(y_{t,x}^{u_\varepsilon}(s), u_\varepsilon(s))ds + e^{-\lambda T}\psi(y_{t,x}^{u_\varepsilon}(T)) \leq \vartheta(t,x) + \varepsilon.$$

Hence, setting $\Omega = \operatorname{dom}\vartheta$ and, as done for the optimal policies, we can define a map $U^\varepsilon : \Omega \to \mathcal{U}$, called an *$\varepsilon$-suboptimal feedback* law, via the condition

$$U^\varepsilon(s, y_{t,x}^{u_\varepsilon}(s)) = u_\varepsilon(s), \quad \text{for a.e. } s \in [t,T],\ \forall (t,x) \in \Omega.$$

Remark 1.2.2. *If there is an end-point constraint involved in the problem, as in the minimum time control process, the suboptimality condition can also be relaxed to just reach a neighborhood of the target*

$$y_{t,x}^{u_\varepsilon}(T) \in \Theta + \mathbb{B}(0,\varepsilon), \quad \forall (t,x) \in \operatorname{dom}\vartheta.$$

1.2.1 Discontinuous ODEs and robustness

It is by now well-known that *optimal feedback laws* are in general discontinuous functions on the state; see for instance the discussion in [39]. Notice that Example 1.2.1 shows that even for linear systems it is likely that optimal feedback are discontinuous. There are indeed topological obstructions that block the existence of continuous feedback policies such as the Brockett's condition introduced in [33]; see also the notes of Clarke [39] or the book of Sontag [122]. The latter was firstly conceived for stabilization problems (reach the origin asymptotically on time), but it can apply to some classes of optimal problems as well. Topological obstructions, such as the Brockett's condition, are so significant that they may even preclude the existence of continuous suboptimal strategies.

Example 1.2.2. *We take under consideration the Artstein's circles system whose dynamics is given by*

$$f(x,u) = (u(x_1^2 - x_2^2), 2ux_1x_2), \quad \forall x \in \mathbb{R}^2, u \in \mathcal{U} = [-1,1].$$

We readily check that for $x \in \mathbb{R}^2$ and $u \in \mathbb{U}(x)$, if $\sigma(s) = \int_0^t u(\tau)d\tau$, we have

$$y_x^u(s) = \frac{1}{(1 + x_1\sigma(s))^2 + \sigma^2(s)x_2^2}(x_1 + |x|^2\sigma(s), x_2), \quad \forall s \geq 0.$$

Furthermore, any trajectory, for which $x_2 \neq 0$, remains on the circle centered at $\left(0, \frac{1}{2x_2}|x|^2\right)$ that contains $(0,0)$. For $x_2 = 0$, the arcs stay on the x_1-axis (see Figure 1.5a), this is because

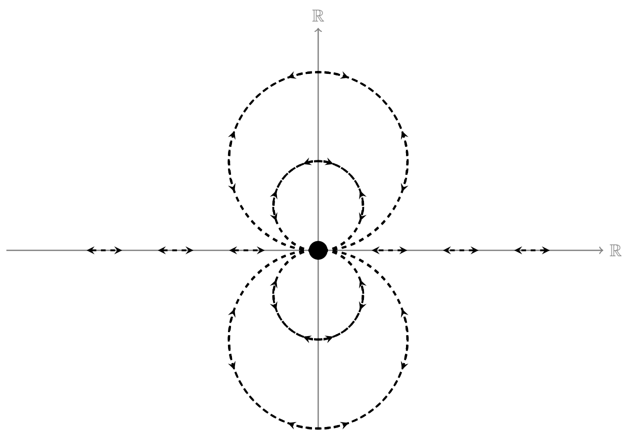
$$\left| y_x^u(s) - \left(0, \frac{1}{2x_2}|x|^2\right)\right|^2 = \frac{1}{4x_2^2}|x|^4, \quad \forall x \in \mathbb{R}^2,\ u \in \mathbb{U}(x),\ s \geq 0.$$

Consider the infinite horizon problem with $\lambda = 1$ and running cost $\ell(x,u) = |x|$ for any $x \in \mathbb{R}^2$. Notice that no trajectory can reach the origin in finite time, but they can approach to $(0,0)$ as much as wanted.

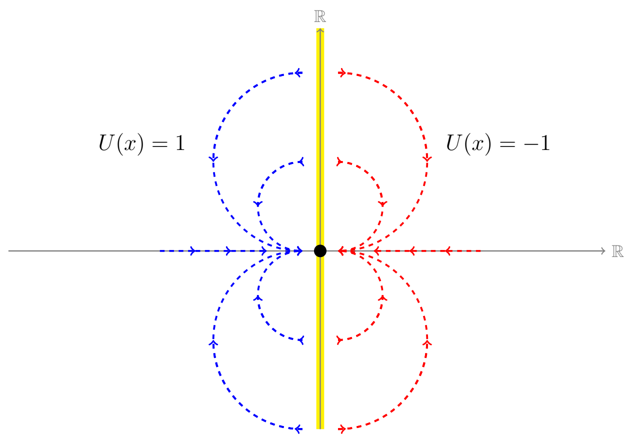
By simple inspection, we verify that the Value Function is real-valued and that an optimal synthesis for this problem (see Figure 1.5b) should satisfy

$$U(x) = \begin{cases} -1 & x_2 > 0, \\ 1 & x_2 < 0, \end{cases} \quad x \in \mathbb{R}^2.$$

Thereby, any optimal feedback will have a discontinuity on the x_2-axis. Notably, by virtue of the structure of the trajectories, any suboptimal strategy will have a discontinuity at some point near to the x_2-axis.



(a) Dynamics.



(b) Optimal synthesis.

Figure 1.5: The Artstein's Circles.

The inherent discontinuity on feedback policies yields to several theoretical and practical issues when dealing with the closed-loop system

$$\dot{y}(s) = f(y(s), U(s, y(s))), \quad \text{a.e. on } [t, T]. \tag{1.5}$$

Notice that, once computed the optimal (or suboptimal) feedback, we are compelled to consider an ODE, as the preceding one, in order to reconstruct the optimal trajectories. However, as long as the righthand side is a discontinuous function on the state, the classical theory of ODEs can not be applied. Hence, the mere existence of an absolutely continuous function that satisfies (1.5) is not guaranteed. Not to mention that continuous dependence upon the initial data and robustness with respect to perturbations are puzzling issues.

The most classical approach to deal with discontinuous ODEs consists in replacing the righthand side on (1.5) with a regularized dynamics that fits into the standard framework of differential equations or inclusions. One of the most typical examples is the *Filippov* regularization which is defined as follows

$$F_f(s, x) := \bigcap_{\varepsilon > 0} \bigcap_{\text{meas}(R)=0} \overline{\text{co}} \left\{ f(\tilde{x}, U(s, \tilde{x})) : \ \tilde{x} \in \mathbb{B}(x, \varepsilon) \cap \Omega \setminus R \right\}, \quad \forall s \in [t, T], \ x \in \mathcal{K}.$$

Another well-known case is the so-called *Krasovskii* regularization that is given by

$$K_f(s, x) := \bigcap_{\varepsilon > 0} \overline{\text{co}} \left\{ f(\tilde{x}, U(s, \tilde{x})) : \ \tilde{x} \in \mathbb{B}(x, \varepsilon) \cap \Omega \right\}, \quad \forall s \in [t, T], \ x \in \mathcal{K}.$$

Both set-valued maps satisfy standing properties that ensure the existence of solutions; this may be found in the monograph of Aubin-Cellina, more precisely [11, Chapter 1]. Thereby, a *Filippov solution* of (1.5) is an absolutely continuous function $y : [t,T] \to \mathcal{K}$ that satisfies

$$\dot{y}(s) \in F_f(s, y(s)), \quad \text{for a.e. } s \in [t,T].$$

In a similar way, we can define the *Krasovskii solution* of (1.5).

The Filippov and Krasovskii approaches have the advantage that whenever the feedback is continuous with respect to the state, the regularizations agree with the righthand side of (1.5). Nevertheless, at discontinuity points this procedure may introduce velocities which have no real meaning for the initial closed loop system producing trajectories that may not be optimal nor suboptimal.

Remark 1.2.3. *In Example 1.2.2, we have that $(0,0) \in K_f(0,x_2) = F_f(0,x_2)$ for any $x_2 \in \mathbb{R}$. Consequently, $y \equiv (0,x_2)$ is a Filippov and Krasovskii solution, notwithstanding that it is far from being suboptimal.*

Other techniques well-suited for closed loops systems have been investigated in the literature. This methodologies avoid this sort of lost information caused by the regularization scheme. Depending on the purpose at hand, we can classify these methods into two types:

- Generalized notions of solutions.
- Tame discontinuities of the feedback.

In the first class we find the *sample-and-hold* solutions which are defined in the following way: Let $\pi = \{t_n\}_{n=0}^{n_T}$ be a partition of $[t,T]$; $t_0 = t$ and, if T is finite, $n_T \in \mathbb{N}$ and $t_{n_T} = T$, otherwise, $n_T = +\infty$ and $t_n \to +\infty$ as $n \to +\infty$. A π-solution of (1.5) is an absolutely continuous arc constructed inductively by

$$\dot{y}(s) = f(y(s), U(t_n, y(t_n))), \quad \text{a.e. on } [t_n, t_{n+1}].$$

The collection of all π-trajectories will be referred as the sample-and-hold solutions to (1.5). Under reasonable assumptions, any of these curves is a trajectory of the control system (not necessarily feasible on $\mathcal{K}$). Furthermore, for finite horizon problems, Clarke-Rifford-Stern have shown in [43] that for any precision parameter $\varepsilon > 0$, there exist a feedback laws which is an ε-suboptimal and $\delta > 0$ such that, any π-trajectory with maximal step size smaller than δ is ε-suboptimal trajectory. If state-constraints are taken into account, their construction works under an IPC. Almost at the same time, Ishii-Koike exhibited a similar construction for the infinite horizon problem in [79] requiring an IPC as well.

In the approach described previously, the singularities of the feedback do not play any role. This is explained by the different notion of solution adopted. Now, if we seek anyhow to work with classical solutions, we are compelled to consider the singularities of the feedback.

Among the methods that reckon with tame singularities we have the *patchy* strategies introduced by Ancona-Bressan in [6]. Let $(\Lambda, \preccurlyeq)$ be a partially ordered set, $\{\Omega_\alpha\}_{\alpha\in\Lambda}$ a subordinated locally finite collection of open domains with smooth boundary and a family of continuous maps $\{U_\alpha : \Omega_\alpha \to \mathcal{U}\}_{\alpha\in\Lambda}$ which satisfy

$$\langle f(x, U_\alpha(x)), \mathbf{n}_{\text{ext}}(x)\rangle < 0 \quad \forall x \in \partial\Omega_\alpha,\ \forall \alpha \in \Lambda.$$

A map $U : \mathcal{K} \to \mathcal{U}$ is called a *patchy feedback* provided

$$U(x) = U_\alpha(x), \quad \forall x \in \Omega_\alpha \setminus \bigcup_{\tilde{\alpha} \preccurlyeq \alpha,\ \tilde{\alpha} \neq \alpha} \Omega_{\tilde{\alpha}}.$$

This type of feedback provides existence of solutions to (1.5) under general hypotheses. Moreover, if no state-constraints are involved, it can be constructed in such a way that any patchy feedback trajectory is suboptimal; we refer to analysis made by Ancona-Bressan in [8] or to the book of Bressan-Piccoli [32, Chapter 9]. We refer to the recent work of Priuli [104] for an extension to state-constrained problems operating under an IPC.

On the other hand, due to uncertainties on the model or on calculations, it is of concern to introduce perturbations to (1.5) and study the behavior of this new equation. Under these circumstances, we can consider internal or external disturbances, so in general we study a differential equation of the type

$$\dot{y}(s) = f(y(s), U(s, y(s) + \xi_i(s))) + \xi_e(s), \quad \text{a.e. on } [t, T], \tag{1.6}$$

where the *internal error* $\xi_i : [t, T] \to \mathbb{R}^N$ is a bounded measurable function and the *external perturbation* $\xi_e : [t, T] \to \mathbb{R}^N$ is an integrable function.

A feedback law is said to be *robust with respect to measurement errors* if for any solution to (1.6) with $\xi_e \equiv 0$ and ξ_i having small $L^\infty([t, T]; \mathbb{R}^N)$ norm, there exists a trajectory of (1.5) which is close to the perturbed trajectory in $L^\infty([t, T]; \mathbb{R}^N)$. Likewise, the feedback is *robust with respect to external disturbances* if for any solution to (1.6) with $\xi_i \equiv 0$ and ξ_e having small $L^1([t, T]; \mathbb{R}^N)$ norm, there exists a trajectory of (1.5) which is close to the perturbed trajectory in $L^\infty([t, T]; \mathbb{R}^N)$. If the feedback law satisfies both robustness axioms, we refer to it as a *fully robust* strategy.

It was shown in [7] that patchy feedback are always, notwithstanding it may not be suboptimal, fully robust if the total variation of ξ_i is small. On the other hand, in [43] the authors proved that it is possible to construct a sample-and-hold suboptimal feedback robust with respect to measurement errors if the partition is chosen in adequate fashion.

1.2.2 Singularities of optimal feedbacks

Notice that in the approach described earlier (sample-and-hold and patchy) we end up with trajectories that have a particular structure, namely, piecewise differentiable. Therefore, there may be circumstances where the optimal feedback is not patchy (Example 1.2.1 for instance) and others where the optimal trajectory is not a π-trajectory.

Example 1.2.3. *Consider the Fuller's problem*

$$\min \int_0^T y_1^2(s)ds \quad s.t.\ \dot{y}(s) = \begin{pmatrix} y_2(s) \\ u(s) \end{pmatrix},\ u(s) \in [-1, 1],\ a.e.\ on\ [0, T],\ y(0) = x,\ y(T) = (0, 0).$$

Let $\sigma > 0$ such that $24\sigma^2 + 1 = \sqrt{33}$. The optimal synthesis for this problem (see Figure 1.6) satisfies

$$U(x) = \begin{cases} 1 & x \in \{x_1 < -\sigma\ sign(x_2)x_2^2\}, \\ -1 & x \in \{x_1 > -\sigma\ sign(x_2)x_2^2\}, \\ 0 & x = (0, 0), \end{cases} \quad x \in \mathbb{R}^2.$$

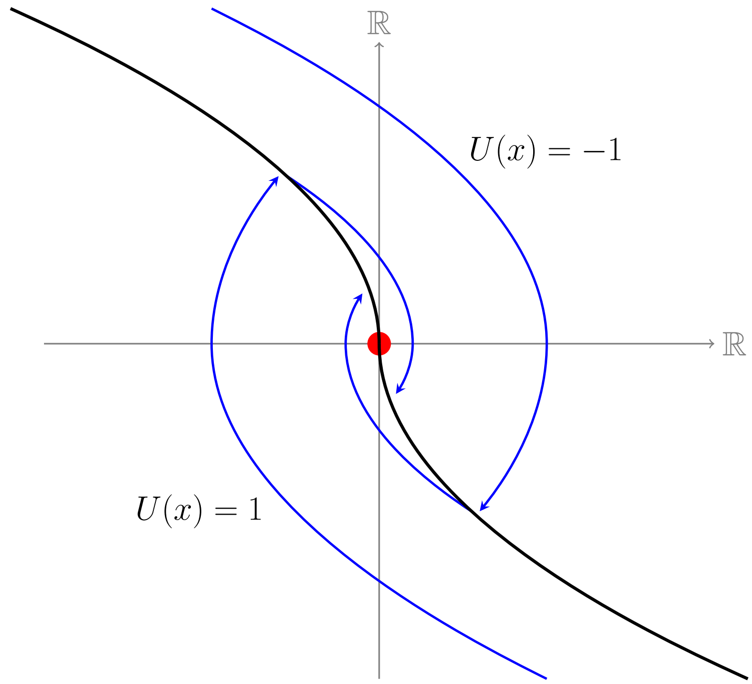


Figure 1.6: The optimal synthesis of the Fuller's problem

Now, since $\sigma < \frac{1}{2}$, no trajectory remains on $\{x_1 + \sigma\, sign(x_2)x_2^2 = 0\}$, the switching curve for the synthesis. Hence, any optimal solution is the outcome of an infinite concatenation of piecewise constant controls. In particular, no optimal trajectory can be a π- trajectory.

It still remains to figure out what is the real structure of optimal feedbacks. In this respect, several works have pointed out that the singularities of these laws have in many situations a tame assemblage. For linear systems this have been studied by Hajek in [63], Brunovsky in [36] and Meeker in [93]. In the framework of nonlinear control problems, this was investigated also by Brunovsky in [37], Sussmann in [126] and by Boscain-Piccoli in [30].

All these works have in common a stratified structure behind the optimal synthesis, because the singularities of the feedback form a partition of the state space. This means that the state-space can be decomposed into a locally finite family of submanifolds, in such a way that in each of these sets the feedback is smooth or it is not defined at all. Therefore, it is suitable to study a theory which cover such situation.

> **Question:** *Can we construct a theory that provides well-posedness for the closed-loops system* (1.5) *if the singularities of the feedback laws are stratified set?*

We study this issue on Chapter 7 where we extrapolate the idea of stratified feedback and study discontinuous differential equations associated with a piecewise continuous vector field. The main contribution of the theory we develop is the analysis of conditions that ensure the existence of solutions and also, a study for the robustness of the system with respect to external perturbations. The concept of *relative wedgedness* is momentous for the exposition.

This notion is a generalization of the so-called *epi-Lipschitz* sets studied by Rockafellar in [112], and it appeared first in [17], where Barnard-Wolenski introduced it for embedded manifolds whose closure is proximally smooth. The definition we adopt in this dissertation considers any arbitrary embedded manifold.

The aforementioned question has been consigned in the literature from other points of view. For instance, in [90] Marigo-Piccoli study the properties of a discontinuous ODE starting from an axiomatic definition of stratified solutions inspired by the idea of regular syntheses. Other contributions, written by Teixeira [128] and Jeffrey-Colombo [81], deal with a qualitative analysis in presence of a switching surface; the last two quoted works are focused on 3-dimensional piecewise smooth dynamical systems.

There is another important aspect in feedback synthesis over which we have not spoken so far. This is the optimality of trajectories associated with optimal feedbacks. It is noticeable that any optimal trajectory is an arc associated with an optimal feedback. However, the converse in not true as the example below shows. We point out that if the Value Function is locally Lipschitz then it is known that the converse does holds; we refer to the works of Frankowska [50] and Berkovitz [20], to the construction exhibited in Rowland-Vinter [117] and to the discussion in [13, Section 3.2.5].

Example 1.2.4 ([102, Example 5.3])**.** *Consider the minimum time problem defined below*

$$\min T \quad s.t.\ \dot{y} = \begin{pmatrix} 1 - y_2(s)\dfrac{u(s)+1}{2} \\ \\ (y_1(s)+1)\dfrac{u(s)+1}{2} \end{pmatrix},\ u(s) \in [-1,1], \quad y(0) = x,\ y(T) = (0,0).$$

The optimal synthesis for this problem is described in Figure 1.7. Around the origin, the optimal feedback has the form

$$U(x) = \begin{cases} 1 & x_2 = 0, \\ -1 & \textit{otherwise}, \end{cases} \qquad \forall x \sim 0.$$

Notice that there are infinitely many solutions starting at $x = (-1,0)$ which may turn around in the circle centered at $(-1,1)$ of radius 1 with the control $u \equiv 1$ as long as wanted and afterwards, using the control $u \equiv -1$ they can reach the origin from x. Of course, none of these trajectories is time-optimal regardless that they are all curves of the closed loop system.

The preceding example also exposes that we need to restraint our attention to suboptimal feedback in order to avoid the existence of undesired arcs associated with the feedback laws.

We want to emphasis that the constructions done for the sample-and-hold and patchy cases do not use directly the possible information coming from an optimal feedback synthesis. Hence, it is plausible to ask whether or not we can produce a suboptimal feedback from an optimal one, so that we can understand the relation between both strategies.

> **Question:** *Can we construct a suboptimal feedback law from an optimal one in such a way that any curve related to the nearly optimal strategy is a suboptimal trajectory of the control system?*

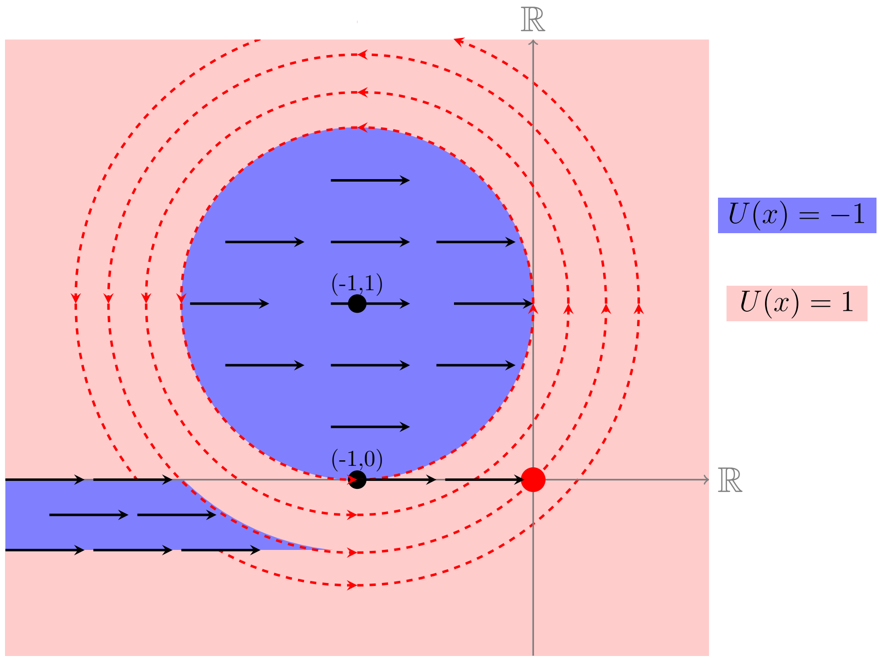


Figure 1.7: An Illustration of Example 1.2.4.

We investigate this question in Chapter 8 where we propose a methodology to construct suboptimal continuous feedbacks starting from the very structure of the optimal syntheses. This construction allows to avoid robustness issues because of the regularity of the produced suboptimal strategy. In particular, we get full robustness. In the chapter we first illustrate the nature of the construction through the soft landing example and we provide some numerical tests as well.

1.3 Organization of the Manuscript

The dissertation is organized in 9 chapters (including this introduction), divided into four parts in the following way:

- **Part I:** Chapter 2 and Chapter 3.
- **Part II:** Chapter 4, Chapter 5 and Chapter 6.
- **Part III:** Chapter 7 and Chapter 8.
- **Part IV:** Chapter 9.

The first part is devoted to provide some general results and definitions required in the next parts. Chapter 2 is an insight into Nonsmooth Analysis as well as Dynamical Systems. In Chapter 3 we revisit the notions of manifold and stratification. In the latter we also study the concept of relatively wedged set.

The second part is concerned with the characterization of the Value Function of an optimal control problem with state-constraints. In Chapter 4 we study the case with stratifiable state-constraints. Chapter 5 and Chapter 6 consider only convex state-constraints. In particular, Chapter 5 is dedicated to linear-like systems and the approach of Chapter 6 is based on a penalization approach. In the latter situation, the dynamics are nonlinear and verify an absorbing property at the boundary of the state-constraints.

The third part is focused on discontinuous feedbacks laws whose singularities form a stratified set on the state-space. In Chapter 7 we present a theory for treating the existence of solutions and the robustness with respect to external perturbations of the corresponding closed-loop system. In Chapter 8 we deal with the construction of a suboptimal continuous feedback from an optimal one.

The fourth and last part is dedicated to investigate optimal control problems on networks. So Chapter 9 addresses the problem in the standard setting of (1-dimensional) networks. Furthermore, the approach adopted allows to extend the analysis to the case of generalized d-dimensional networks.

Publications of the thesis

The contents of the thesis are based on some publications or some that are being prepared to be submitted. The list of publications is described below.

Accepted or submitted publications

[67] *Legendre transform and applications to finite and infinite optimization*, submitted. 2015

[68] *Stratified discontinuous differential equations and sufficient conditions for robustness*, Discret. Contin. Dyn. S.-A, 35(9): 4415–4437, 2015.

[70] (with P. Wolenski and H. Zidani) *The Mayer and minimum time problems with stratified state constraints.*, submitted. 2015.

[71] (with H. Zidani) *Infinite horizon problems on stratifiable state constraints sets*, J. Differential Equations, 258(4): 1430-1460, 2015.

In preparation

[9] (with F. Ancona) *On the construction of nearly time-optimal continuous feedback around switching manifolds.*

[62] (with J. Graber and H. Zidani) *Discontinuous solutions of Hamilton-Jacobi equations on networks.*

[69] (with R. Vinter and H. Zidani) *Optimal control process with convex state- constraints*

Publications and chapters

The relation between the publications and the chapters is described in the next list.

- Chapter 2: the strong invariance criterion (Proposition 2.4.6) was proved in [71].
- Chapter 3: the study about relatively wedged sets is based on [68].
- Chapter 4: the results for the infinite horizon problem and the Mayer problem were reported in [71] and [70], respectively.
- Chapter 5: the results are being summarized in [69].
- Chapter 6: the exposition is derived from the results stated in [67].
- Chapter 7: the definitions and results are based on [68].
- Chapter 8: the results are being summarized in [9].
- Chapter 9 : the results are being summarized in [62].

PART I

GENERAL RESULTS AND MATHEMATICAL TOOLS

Abstract. In this part we provide some technical tools required for a good understanding of the entire manuscript. Some results presented are new, and consequently, are part of the contributions of the dissertation. We first give a brief survey on some mathematical theories which are well-suited for control theory and for the purposes of the thesis. Later on we make a short insight into the notions of manifolds and stratification. We emphasis that this last concept is essential for the rest of the present manuscript.

Resumé. Dans cette partie nous fournissons quelques outils mathématiques nécessaires pour mieux comprendre les chapitres suivants de la thèse. Parmi les résultats que nous énoncerons certains sont noveaux et représentent une contribution de cette thèse. D'abord nous faisons des rappels sur quelques théories mathématiques utiles à bien formuler la théorie de la commande optimale. Ensuite, nous revisserons les notions de variétés lisses et d'ensembles stratifiés. Nous insistons sur le fait que ce dernier concept est essentiel pour ce qui va suivre dans le manuscript.

CHAPTER 2

Nonsmooth Analysis and Dynamical Systems

Abstract. In this chapter we present a brief survey on variational and nonsmooth analysis, as well as on differential inclusions. We also provide a criterion for strong invariance suitable for dynamical systems evolving on manifolds.

2.1 Introduction

It is by now well-understood that the Hamilton-Jacobi-Bellman approach for optimal control problems can be studied from different points of views. For example, in a purely viscosity setting, the most common technique is the so-called *doubling of variables*; we refer to the book of Bardi- Capuzzo-Dolcetta [13] or Barles [14] for further details.

The methodology we have adopted for this thesis is based on the invariance of dynamical systems, which is very close to the formalism used by Clarke-Ledyaed-Stern-Wolenski in [41]. Roughly speaking, invariance refers to the study of trajectories of a dynamical system that remain in a given set. We will explain this in more details later on in Section 2.4.2.

The theory of invariance is intrinsically related to other theories such as nonsmooth and variational analysis. The principal objects that connect them are the *subdifferentials* and the *normal cones*, which also allow to link the classical theory of viscosity solutions (via test functions) and the approach followed in the dissertation; this is thanks to a density theorem that implies in particular that any viscosity test functions can be approximated by a sequence of quadratic functions (see Proposition 2.3.9 for a precise statement).

On the other hand, the theory of invariance usually requires certain regularity of the dynamical systems. These requirements are often formulated in the language of set-valued analysis. Depending on the issue at hand, we may be interested in the *upper semicontinuity* or in the *Lipschitz continuity* of the multifunction that defines the dynamical system.

In this chapter we provide precise definitions of the aforementioned concepts (subdifferentials, normal cones, continuity of set-valued maps, differential inclusions) as well as some of their more relevant properties. We also make a brief review of convex analysis, this is done for the sake of the exposition of Chapter 5 and Chapter 6.

We stress that the utility of definitions and results exposed in this chapter is not limited to the study done for the Hamilton-Jacobi-bellman approach but also for the the rest of the manuscript. In particular, variational analysis plays an important role in Chapter 7.

The main sources for this chapter are the books of Aubin-Cellina [11], Aubin-Frankowska [12], Borwein-Vanderwerff [28], Clarke-Ledyaev-Stern-Wolenski [41], Clarke [40], Rockafellar [111], Rockafellar-Wets [114], and the paper of Wolenski-Zhuang [132].

2.2 Set-valued analysis

A *set-valued* map from X into Y is a relation that associates any $x \in X$ with a set $\Gamma(x) \subseteq Y$. We write it as $\Gamma : X \rightrightarrows Y$ and we call it *multifunction* or *multivalued function* as well.

A multifunction may have empty images at some points, and so, it is useful to identify the subset of X where its images are nonempty. This set is the *effective domain* and is defined via

$$\operatorname{dom}\Gamma := \{x \in X \mid \ \Gamma(x) \neq \emptyset\}.$$

Another object that plays an important role in set-valued analysis is the *graph* of Γ

$$\operatorname{gr}(\Gamma) := \{(x, y) \mid \ y \in \Gamma(x)\} \subseteq X \times Y.$$

2.2.1 Continuity

Consider two metric spaces (X, d_X) and (Y, d_Y). We write, for $Z = X, Y$

$$\mathbb{B}_Z(z, r) := \{\tilde{z} \in Z \mid \ d_Z(z, \tilde{z}) < r\}, \quad \forall z \in Z,\ r > 0.$$

The concepts of semicontinuity can be adapted for multivalued functions. A multifunction $\Gamma : X \rightrightarrows Y$ is called *lower semicontinuous* at $x \in \operatorname{dom}\Gamma$ if for any $y \in \Gamma(x)$ and $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\forall \tilde{x} \in \mathbb{B}_X(x, \delta), \quad \Gamma(\tilde{x}) \cap \mathbb{B}_Y(y, \varepsilon) \neq \emptyset.$$

Remark 2.2.1. *Notice that if $y \in \Gamma(x)$ with Γ being lower semicontinuous on a neighborhood of x, then*

$$\Gamma_x(\tilde{x}) := \begin{cases} \Gamma(\tilde{x}) & \tilde{x} \neq x \\ \{y\} & \tilde{x} = x \end{cases} \quad \forall \tilde{x} \in X$$

is lower semicontinuous as well around x.

The lower semicontinuity of a multifunction can also be described in terms of sequences.

Proposition 2.2.1. *A set-valued map Γ is lower semicontinuous at $x \in \operatorname{dom}\Gamma$ if and only if for any $y \in \Gamma(x)$ and any sequence $\{x_n\} \subseteq \operatorname{dom}\Gamma$ with $x_n \to x$, there exists a sequence of elements $y_n \in \Gamma(x_n)$ with $y_n \to y$.*

On the other hand, a multivalued function $\Gamma : X \rightrightarrows Y$ is said to be *upper semicontinuous* at $x \in \operatorname{dom}\Gamma$ provided for any open set $\mathcal{O} \subseteq X$ for which $\Gamma(x) \subseteq \mathcal{O}$, there is $\delta > 0$ such that

$$\forall \tilde{x} \in \mathbb{B}_X(x, \delta), \quad \Gamma(\tilde{x}) \subseteq \mathcal{O}.$$

The upper semicontinuity is a property suitable for compact-valued multifunctions, otherwise, it becomes a rather strong condition. In particular, if the images of Γ are cones, we have the next result.

Proposition 2.2.2. *Let X and Y be Banach spaces. Let $\Gamma : X \rightrightarrows Y$ be a set-valued map for which $\lambda\Gamma(x) \subseteq \Gamma(x)$ for any $x \in \operatorname{dom}\Gamma$ and $\lambda > 0$. Suppose that $\Gamma(x) \neq \{0\}$ for some $x \in \operatorname{dom}\Gamma$, then Γ is upper semicontinuous at x if and only if there exists $\delta > 0$ such that $\Gamma \equiv \Gamma(x)$ on $\operatorname{dom}\Gamma \cap \mathbb{B}_X(x, \delta)$, that is, Γ is constant on $\mathbb{B}_X(x, \delta)$.*

Proof. Let us just focus on the necessity part, the sufficiency is evident. By virtue of [11, Theorem 1.1.2], there exists a bounded set $\mathcal{S} \subseteq \Gamma(x)$ and $\delta > 0$ so that

$$\Gamma(\tilde{x}) \subseteq \Gamma(x) \cup \mathcal{S}, \quad \tilde{x} \in \operatorname{dom}\Gamma \cap \mathbb{B}_X(x,\delta).$$

Suppose there exist $\tilde{x} \in \operatorname{dom}\Gamma \cap \mathbb{B}_X(x,\delta)$ and $y \in Y$ such that

$$y \in \Gamma(\tilde{x}) \cap \mathcal{S} \setminus \Gamma(x).$$

Since Γ is lower semicontinuous and $\Gamma(x) \neq \{0\}$, by taking δ smaller if necessary we can assume that $|y| \geq \frac{1}{2}$, and due to the fact that Γ is cone-valued, $\lambda v \in \Gamma(\tilde{x}) \setminus \Gamma(x)$ for any $\lambda > 0$. However, since $\mathcal{S}$ is bounded, we get a contradiction, so the conclusion follows. □

A more suitable notion of continuity for maps with unbounded images is the one we define next. A set valued-map $\Gamma : X \rightrightarrows Y$ is called *compactly upper semicontinuous* at $x \in \operatorname{dom}\Gamma$ provided for every compact subset $\mathcal{S} \subseteq Y$, the map $x \mapsto \Gamma(x) \cap \mathcal{S}$ is upper semicontinuous.

In contrast with the lower semicontinuity, the upper semicontinuity can not be expressed in terms of sequences, unless extra hypotheses are made.

Proposition 2.2.3 ([12, Proposition 1.4.8 and Proposition 1.4.9])**.** *Let Γ be an upper semicontinuous set-valued map with closed images and with $\operatorname{dom}\Gamma$ closed as well. Then $gr(\Gamma)$ is closed in $X \times Y$. The converse holds true provided Y is compact and, in any case, Γ is compactly upper semicontinuous.*

When a multivalued function is lower and upper semicontinuous at the same time on its domain, it is said to be *continuous*. This notion is not often used in the literature but a stronger version does so. We say that $\Gamma : X \rightrightarrows Y$ is *locally Lipschitz continuous* if for any $x \in X$ there exist $L, \delta > 0$ such that

$$\forall \tilde{x}, \hat{x} \in \mathbb{B}_X(x,\delta), \quad \Gamma(\tilde{x}) \subseteq \bigcup_{y \in \Gamma(\hat{x})} \mathbb{B}_Y(y, L d_X(\tilde{x},\hat{x})).$$

The Lipschitz continuous character of the set-valued map can be written in a simpler way by means of the *Hausdorff distance* which is defined via

$$d_H(\mathcal{S}_1,\mathcal{S}_2) := \max\left\{ \sup_{y\in\mathcal{S}_2} \inf_{\tilde{y}\in\mathcal{S}_1} d_Y(y,\tilde{y}), \sup_{y\in\mathcal{S}_1} \inf_{\tilde{y}\in\mathcal{S}_2} d_Y(y,\tilde{y}) \right\}, \quad \forall \mathcal{S}_1,\mathcal{S}_2 \subseteq Y.$$

We adopt the convention that $d_H(\emptyset,\emptyset) = 0$ and $d_H(\emptyset,\mathcal{S}) = +\infty$ if $\mathcal{S} \neq \emptyset$.

Hence, Γ is locally Lipschitz continuous if and only if for any $x \in X$ there exist $L, \delta > 0$

$$d_H(\Gamma(\tilde{x}),\Gamma(\hat{x})) \leq L d_X(\tilde{x},\hat{x}), \quad \forall \tilde{x},\hat{x} \in \mathbb{B}_X(x,\delta).$$

If X is paracompact, then we also have that Γ is locally Lipschitz continuous provided for any compact subset $\mathcal{S} \subseteq X$, we can find $L > 0$ such that

$$d_H(\Gamma(x),\Gamma(\tilde{x})) \leq L d_X(x,\tilde{x}), \quad \forall x,\tilde{x} \in \mathcal{S}.$$

2.2.2 Selection theorems

Given a multivalued function $\Gamma : X \rightrightarrows Y$, a *selection* of Γ is a function $\gamma : X \to Y$ that satisfies $\gamma(x) \in \Gamma(x)$ for any $x \in \operatorname{dom}\Gamma$.

In many cases, it turns out that if the set-valued maps verifies some regularity property, it admits an equivalently regular selection. The proposition below is commonly quoted in the literature and establishes the existence of a continuous selection, it is the so-called *Michael's Selection Theorem.*

Proposition 2.2.4 ([11, Theorem 1.11.1])**.** *Suppose Y is a Banach space and $\Gamma : X \rightrightarrows Y$ is lower semicontinuous with closed convex images. Then there is a continuous selection of Γ.*

By Remark 2.2.1, for any $x \in \operatorname{dom}\Gamma$ and $y \in \Gamma(x)$, there exists a continuous selection given by the Michael's Theorem such that $\gamma(x) = y$. Moreover, if Γ is even more regular than lower semicontinuous, say Lipschitz continuous, the selection can be taken Lipschitz continuous as well provided $\dim(Y) \in \mathbb{N}$.

Proposition 2.2.5 ([12, Theorem 9.4.3])**.** *Suppose Y is a Banach space of finite dimension and $\Gamma : X \rightrightarrows Y$ is Lipschitz continuous with closed convex values. Then, for any $x \in \operatorname{dom}\Gamma$ and $y \in \Gamma(x)$, there exists a Lipschitz continuous selection γ of Γ such that $\gamma(x) = y$.*

On the other hand, a specially designed selection theorem, known as the Filippov's Selection Theorem, connects open-loop control systems with differential inclusions. This is summarized in the following proposition.

Proposition 2.2.6 ([11, Corollary 1.14.1])**.** *Suppose that X is a finite dimensional Banach space and $\mathcal{U}$ is a compact separable metric space. Let $f : X \times \mathcal{U} \to X$ be a continuous function and let $y : [a, b] \to X$ be an absolutely continuous function such that $\dot{y}(t) \in f(y(t), \mathcal{U})$ for almost all $t \in [a, b]$. Then there exists a measurable function $u : [a, b] \to \mathcal{U}$ satisfying*

$$\dot{y}(t) = f(t, u(t)) \qquad a.e.\ t \in [a, b].$$

2.3 Nonsmooth and variational analysis

In this section we review some definitions and results from the nonsmooth analysis and variational analysis which are intrinsically related to the scope of this thesis. We first recall some notions from convex analysis and later we review several cones and some of their properties. Finally, we focus our attention on the concept of subgradient.

Henceforth, $(X, \langle \cdot, \cdot \rangle, |\cdot|)$ is a finite dimensional Hilbert space and the *distance function* to $\mathcal{S} \subseteq X$ is written as

$$\operatorname{dist}_{\mathcal{S}}(x) := \inf_{\tilde{x} \in \mathcal{S}} |x - \tilde{x}|, \quad \forall x \in X.$$

Many of the result stated have been presented originally for the model space $\mathbb{R}^N$. However, by means of the canonical isomorphism between X and the former space, all the properties can be readily transported from one into the other.

2.3.1 Elements of convex analysis

A set $\mathcal{S} \subseteq X$ is called *convex* provided

$$\lambda x + (1-\lambda)\tilde{x} \in \mathcal{S}, \quad \forall x, \tilde{x} \in \mathcal{S},\ \forall \lambda \in [0,1].$$

Given a convex set $\mathcal{S} \subseteq X$, its *affine hull* is the hyperplane of X that contains all the possible linear combination of elements of $\mathcal{S}$

$$\{\lambda x + (1-\lambda)\tilde{x} \mid \ x, \tilde{x} \in \mathcal{S},\ \lambda \in \mathbb{R}\}.$$

The relative interior of $\mathcal{S}$, written $\mathrm{ri}(\mathcal{S})$, is the interior of $\mathcal{S}$ in the induced topology of its affine hull. This set is always nonempty provided $\mathcal{S}$ is convex, this is mainly due to the Accessibility Lemma stated below. This result will also be of utility in Chapter 5 because it will allow us to obtain two NFT-type statements (Proposition 5.2.1 and Proposition 5.3.1).

Proposition 2.3.1 ([111, Theorem 6.1]). *For any convex subset $\mathcal{S} \subseteq X$, $x \in ri(\mathcal{S})$ and $\tilde{x} \in \overline{\mathcal{S}}$ we have*

$$\lambda x + (1-\lambda)\tilde{x} \in ri(\mathcal{S}), \quad \forall \lambda \in (0,1].$$

In particular we have that if $\mathcal{S}$ is convex then $\mathrm{ri}(\overline{\mathcal{S}}) = \mathrm{ri}(\mathcal{S})$ and $\overline{\mathcal{S}} = \overline{\mathrm{ri}(\mathcal{S})}$.

On the other hand, a function $g : X \to \mathbb{R} \cup \{+\infty\}$ is *convex* provided $\mathrm{dom}\, g$ is a convex set and the algebraic criterion below is satisfied:

$$g(\lambda x + (1-\lambda)\tilde{x}) \le \lambda g(x) + (1-\lambda) g(\tilde{x}), \quad \forall x, \tilde{x} \in \mathrm{dom}\, g,\ \forall \lambda \in [0,1].$$

If the effective domain of g is nonempty, then it is said to be *proper*.

Given a function $g : X \to \mathbb{R} \cup \{+\infty\}$, g^* stands for its *Fenchel-Legendre conjugate*

$$g^*(y) := \sup\{\langle x, y\rangle - g(x) \mid x \in \mathrm{dom}\, g\}, \quad \forall y \in X.$$

This function is always convex and lower semicontinuous. If g is convex, proper and lower semicontinuous then $(g^*)^*(x) = g(x)$ for $x \in X$. Furthermore, the *convex subdifferential* of a proper function is the set-valued map given by

$$\partial g(x) = \{\zeta \in X \mid \forall \tilde{x} \in X,\ g(x) + \langle \zeta, \tilde{x} - x\rangle \le g(\tilde{x})\}, \quad \forall x \in X.$$

Legendre functions

The study we are going to present in Chapter 6 is based on a suitable class of penalization functions called *of Legendre type*. These functions were introduced by Rockafellar in [111, Chapter 26] and has been continuously studied by other authors; see for instance Borwein-Varderwerff [28, Chapter 7]. Formally, a convex, proper and lower semicontinuous function $g : X \to \mathbb{R} \cup \{+\infty\}$ is called *essentially smooth* if it verifies

- $\mathrm{int}(\mathrm{dom}\, g) \neq \emptyset$ and g is differentiable on $\mathrm{int}(\mathrm{dom}\, g)$.
- $|\nabla g(x_k)| \to +\infty$ for every $\{x_k\} \subseteq \mathrm{int}(\mathrm{dom}\, g)$ with $x_k \to \bar{x}$ for some $\bar{x} \in \mathrm{bdry}(\mathrm{dom}\, g)$.

This property has a dual interpretation in terms of the Legendre-Frenchel conjugate. Indeed, a necessary and sufficient condition for g being essentially smooth is that g^* is strictly convex on every convex subset of $\operatorname{dom} \partial g$. This last condition is known as *essential strictly convexity.*

Proposition 2.3.2 ([111, Theorem 26.3])**.** *Let $g : X \to \mathbb{R} \cup \{+\infty\}$ be a convex, proper and lower semicontinuous function. Then, g is essentially smooth if and only if g^* is essentially strictly convex.*

In view of the previous proposition, a convex, proper and lower semicontinuous function $g : X \to \mathbb{R} \cup \{+\infty\}$ which is essentially smooth and essentially strictly convex at the same time is named *Legendre function*

Remark 2.3.1. *Some well-known examples of Legendre functions on $\mathbb{R}$ for which the interior of their domains agrees with $(0, +\infty)$ are the log-barrier g_{log} and the Boltzmann-Shannon entropy g_{ent}; cf. [28, 29]. These functions are respectively given by:*

$$g_{log}(x) := \begin{cases} -\log(x) & x > 0 \\ +\infty & x \leq 0 \end{cases} \quad \textit{and} \quad g_{ent}(x) := \begin{cases} x\log(x) - x & x \geq 0 \\ +\infty & x < 0. \end{cases}$$

Consequently, g is a Legendre function if and only if g^* it is as well. In this case, we have the following result.

Proposition 2.3.3 ([111, Theorem 26.5])**.** *If $g : X \to \mathbb{R} \cup \{+\infty\}$ is a Legendre function then $\nabla g : int(dom\, g) \to int(dom\, g^*)$ is a bijection. Furthermore, we have*

$$(\nabla g)^{-1}(y) = \nabla g^*(y), \quad \forall y \in int(dom\, g^*). \tag{2.1}$$

In this situation ∇g is known as the Legendre transform.

The proposition above implies that if g and g^* are twice differentiable on the interior of their domains, the Hessian matrices are invertible and satisfy

$$[\nabla^2 g(x)]^{-1} = \nabla^2 g^*(y), \quad \forall x \in \operatorname{int}(\operatorname{dom} g),\ y = \nabla g(x).$$

2.3.2 Tangent and normal cones

We are interested in several notions of tangent and normal cones to a locally closed set of X, which we denote generically by $\mathcal{S}$ from beginning to end of this section. All these cones, when regarded as set-valued maps, have nonempty images on $\mathcal{S}$ because they always contain the zero vector.

We begin with some concepts of tangentiality. The *Bouligand* or *Contingent* cone to $\mathcal{S}$, which we denoted by $\mathcal{T}_{\mathcal{S}}^B(\cdot)$, is defined as

$$\mathcal{T}_{\mathcal{S}}^B(x) = \left\{ v \in X \,\middle|\, \liminf_{t \to 0^+} \frac{\operatorname{dist}_{\mathcal{S}}(x + tv)}{t} \leq 0 \right\}, \quad \forall x \in \mathcal{S}.$$

This cone, seen as a multifunction, has always closed images.

Let us write $\mathcal{T}_{\mathcal{S}}^{C}(\cdot)$ for the *Clarke tangent* cone to $\mathcal{S}$. This tangent cone is convex and closed-valued, and can be represented in several ways depending upon the needs. In this case we have chosen the following definition:

$$\mathcal{T}_{\mathcal{S}}^{C}(x) = \left\{ v \in \mathbb{R}^N \,\middle|\, \limsup_{y \to x,\ t \to 0^+} \frac{\mathrm{dist}_{\mathcal{S}}(y + tv)}{t} \leq 0 \right\}, \quad \forall x \in \mathcal{S}.$$

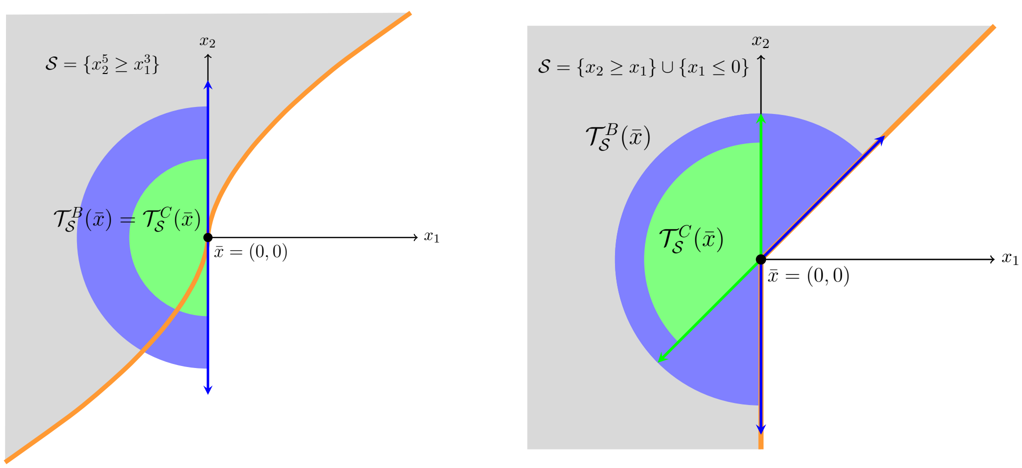
In the next proposition we exhibit some useful properties of these two tangent cones.

Proposition 2.3.4 ([41, Theorem 3.6.12 and Corollary 3.6.13])**.** *Let $\mathcal{S} \subseteq X$ be a locally closed subset. Then, for any $x \in \mathcal{S}$*

$$\mathcal{T}_{\mathcal{S}}^{C}(x) = \left\{ v \in X \middle| \ \forall \{x_n\} \subseteq \mathcal{S} \text{ with } x_n \to x, \exists v_n \in \mathcal{T}_{\mathcal{S}}^{B}(x_n) \text{ with } v_n \to v \right\}.$$

In particular, $\mathcal{T}_{\mathcal{S}}^{C}(x) \subseteq \mathcal{T}_{\mathcal{S}}^{B}(x)$ for any $x \in \mathcal{S}$ and equality holds whenever $\mathcal{T}_{\mathcal{S}}^{B}(\cdot)$ is lower semicontinuous at x.

Example 2.3.1. *To illustrate the statement of Proposition 2.3.4, let us consider two situations in $\mathbb{R}^2$. First, let $\mathcal{S} = \{x_2^5 \geq x_1^3\}$ as in Figure 2.1a, under these circumstances both cones coincide at $\bar{x} = (0,0)$ and we can also see that $\mathcal{T}_{\mathcal{S}}^{B}(\cdot)$ is lower semicontinuous at $\bar{x}$. However, if $\mathcal{S} = x_2 \geq x_1 \cup \{x_1 \leq 0\}$ as in Figure 2.1b we get the strict inclusion at $\bar{x}$.*



(a) $\mathcal{T}_{\mathcal{S}}^{C}(\bar{x}) = \mathcal{T}_{\mathcal{S}}^{B}(\bar{x})$.

(b) $\mathcal{T}_{\mathcal{S}}^{C}(\bar{x}) \subsetneq \mathcal{T}_{\mathcal{S}}^{B}(\bar{x})$.

Figure 2.1: Some examples of tangent cones.

We now turn our attention into some notions of normal cones. The *Proximal normal cone* to $\mathcal{S}$ at x, denoted by $\mathcal{N}_{\mathcal{S}}^{P}(x)$, is the set of all $\eta \in X$ such that

$$\delta |x - \tilde{x}|^2 \geq \langle \eta, \tilde{x} - x \rangle \quad \forall \tilde{x} \in \mathcal{S},$$

for some $\delta = \delta(x, \eta) \geq 0$. This cone has convex images but not necessarily closed. In Figure 2.2 we exhibit the proximal normal cone to $\mathcal{S} = \{x_1 \leq \min\{x_2, x_2^2\}\}$ at $\bar{x} = (0,0)$.

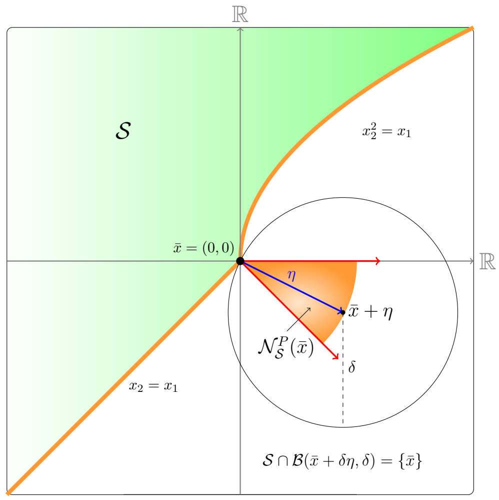


Figure 2.2: An example of proximal normal to a set in $\mathbb{R}^2$.

Before revisiting other normal cones, let us consider the particular case in which $\mathcal{S}$ is the *epigraph* of an extended real-valued function $\omega(\cdot)$, that is,

$$\mathrm{epi}(\omega) := \{(x, r) \in X \times \mathbb{R} \mid \ \omega(x) \le r\}.$$

Consequently, $\eta \in \mathcal{N}^P_{\mathrm{epi}(\omega)}(x)$ if and only if there exist $\xi \in X$ and $\lambda \ge 0$ such that $\eta = (\xi, -\lambda)$. The situations when $\lambda = 0$ is of particular concern and in these circumstances η is called a *horizontal* proximal normal. The following proposition, attributed to Rockafellar, shows that any of these normals can be approximated by a sequence of non horizontal ones.

Proposition 2.3.5 ([40, Theorem 11.30])**.** *Let $\omega : X \to \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous function. Consider $x \in dom\,\omega$ and $\xi \in X \setminus \{0\}$ such that $(\xi, 0) \in \mathcal{N}^P_{epi(\omega)}(x)$. Then, for every $\varepsilon > 0$ there exist $x_\varepsilon \in \mathbb{B}_X(x, \varepsilon)$, $\lambda_\varepsilon \in (0, \varepsilon)$ and $\xi_\varepsilon \in \mathbb{B}_X(x, \varepsilon)$ such that*

$$(\xi_\varepsilon, \lambda_\varepsilon) \in \mathcal{N}^P_{epi(\omega)}(x_\varepsilon) \quad \text{and} \quad |\omega(x) - \omega(x_\varepsilon)| < \varepsilon.$$

The *Limiting normal cone* to $\mathcal{S}$, denoted by $\mathcal{N}^L_{\mathcal{S}}(\cdot)$, is given by

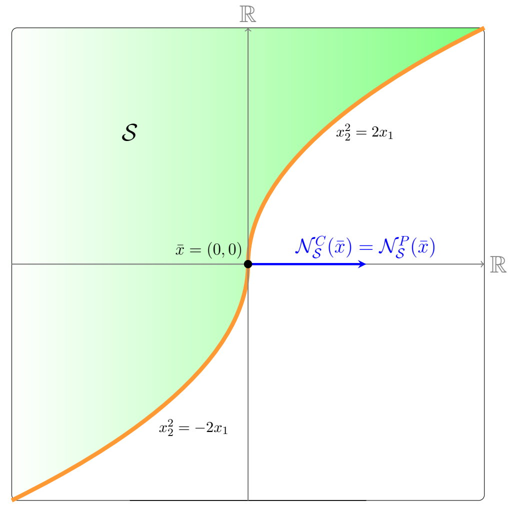
$$\mathcal{N}^L_{\mathcal{S}}(x) := \left\{ \lim_{n\to\infty} \eta_n : \ \exists \{x_n\} \subseteq \mathcal{S} \text{ with } x_n \to x, \exists \eta_n \in \mathcal{N}^P_{\mathcal{S}}(x_n) \right\}, \quad \forall x \in \mathcal{S}.$$

By definition, this cone has closed images, possibly non convex, and always contains a nonzero vector unless $x \in \mathrm{int}(\mathcal{S})$. Additionally, its graph $\mathrm{gr}\left(\mathcal{N}^L_{\mathcal{S}}(\cdot)\right)$ is locally closed in $\mathcal{S} \times X$. The *Clarke normal cone* to $\mathcal{S}$ at x, written as $\mathcal{N}^C_{\mathcal{S}}(x)$, is exactly the convex closed hull of $\mathcal{N}^L_{\mathcal{S}}(x)$.

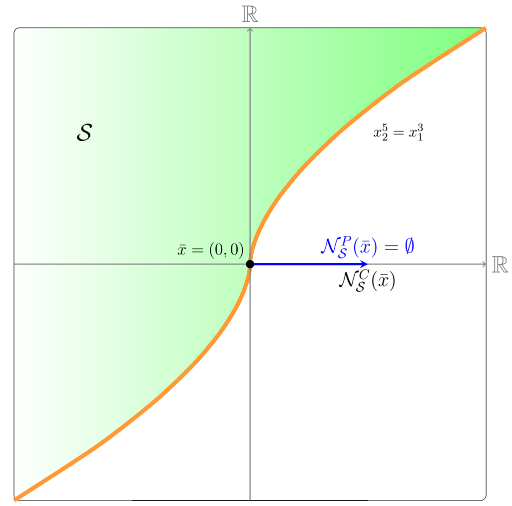
These three normal cones satisfy, no matter what, the next inclusion:

$$\mathcal{N}^P_{\mathcal{S}}(x) \subseteq \mathcal{N}^L_{\mathcal{S}}(x) \subseteq \mathcal{N}^C_{\mathcal{S}}(x) \quad \forall x \in \mathcal{S}.$$

Example 2.3.2. *To exemplify the last affirmation, let us consider again two cases in $\mathbb{R}^2$. First of all let $\mathcal{S} = \{2x_1 \le sign(x_2)x_2^2\}$ as in Figure 2.3a, then the three cones coincide at $\bar{x} = (0,0)$. However, if we take $\mathcal{S} = \{x_2^5 \ge x_1^3\}$ as in Figure 2.3b we get that $\mathcal{N}^P_{\mathcal{S}}(\bar{x}) = \{0\}$ but $\mathcal{N}^L_{\mathcal{S}}(\bar{x})$ and $\mathcal{N}^C_{\mathcal{S}}(\bar{x})$ coincide containing both a nonzero normal.*



(a) $\mathcal{N}_{\mathcal{S}}^P(\bar{x}) = \mathcal{N}_{\mathcal{S}}^C(\bar{x})$.



(b) $\mathcal{N}_{\mathcal{S}}^P(\bar{x}) = \{0\} \subsetneq \mathcal{N}_{\mathcal{S}}^L(\bar{x}) = \mathcal{N}_{\mathcal{S}}^C(\bar{x})$.

Figure 2.3: Some examples of normal cones.

The proposition below shows that there exists a polarity relationship that connects both Clarke cones.

Proposition 2.3.6 ([41, Proposition 2.5.4]). *Let $\mathcal{S} \subseteq X$ be a locally closed subset. Then*

$$\mathcal{T}_{\mathcal{S}}^C(x) = \left\{ v \in X : \ \langle v, \eta \rangle \leq 0, \ \forall \eta \in \mathcal{N}_{\mathcal{S}}^C(x) \right\}, \quad \forall x \in \mathcal{S}.$$

We conclude this section by exposing a result that will be of particular utility when we study stratification in section 3.3.3.

Proposition 2.3.7 ([114, Theorem 6.42]). *Let $\mathcal{S}_1, \ldots, \mathcal{S}_n \subseteq X$ be closed subsets and set $\mathcal{S} = \mathcal{S}_1 \cap \ldots \cap \mathcal{S}_n$,*

$$\bigcap_{i=1}^{n} \mathcal{T}_{\mathcal{S}_i}^C(x) \subseteq \mathcal{T}_{\mathcal{S}}^C(x) \quad \text{and} \quad \mathcal{N}_{\mathcal{S}}^C(x) \subseteq \sum_{i=1}^{n} \mathcal{N}_{\mathcal{S}_i}^C(x), \quad \forall x \in \mathcal{S}.$$

2.3.3 Subdifferentials

These concepts arise in nonsmooth analysis as a way to generalize the idea of gradient of a function when it is not well-defined. Subdifferentials will allow us to write in a rather simple way the Hamilton-Jacobi-Bellman equations in Part II and Part IV. Furthermore, they will allow us to link the Hamilton-Jacobi-Bellman equation with some criterions for invariance.

All along this section $\omega(\cdot)$ stands for a lower semicontinuous functions defined on X with values in $\mathbb{R} \cup \{+\infty\}$ whose effective domain is nonempty.

We begin the exposition with a very well-structured class of functions, the convex ones. We recall that in this situation the *convex subdifferential* is the set-valued map defined via

$$\partial\omega(x) = \{\zeta \in X \mid \forall \tilde{x} \in \operatorname{dom}\omega, \ \omega(x) + \langle \zeta, \tilde{x} - x \rangle \leq \omega(\tilde{x})\}, \quad \forall x \in X.$$

This subdifferential has closed and convex images, seen as multifunction. Furthermore, it can be described in terms of the directional derivative

$$\omega'(x;v) = \lim_{t\to 0^+} \frac{\omega(x+tv)-\omega(x)}{t}, \quad \forall x, v \in X.$$

The next is a well-known formula that is verified whenever $\omega : X \to \mathbb{R}\cup\{+\infty\}$ is a convex lower semicontinuous function. We refer for further details to [111, Theorem 23.2].

$$\partial\omega(x) = \{\zeta \in X \mid \ \omega'(x;v) \geq \langle \zeta, v\rangle,\ \forall v \in X\}, \quad \forall x \in \operatorname{dom}\omega.$$

We now switch the discussion to functions that are merely lower semicontinuous. There is a notion os subdifferential that is closely related to the theory of viscosity solution.

A vector $\zeta \in \mathbb{R}^N$ is called a viscosity subgradient of ω at $x \in \operatorname{dom}\omega$ provided there exists a continuous function $\varphi : \mathbb{R}^N \to \mathbb{R}$ differentiable at x such that $\nabla\varphi(x) = \zeta$ and $\omega - \varphi$ attains a local minimum at x. The set of all viscosity subgradients of ω at x is denoted by $\partial_V\omega(x)$.

The foregoing notion of subgradient is closely related to the *Fréchet subdifferential* which is given by

$$\partial_F\omega(x) := \left\{\zeta \in X \mid \ \liminf_{\tilde{x}\to x} \frac{\omega(\tilde{x})-\omega(x)-\langle\zeta,\tilde{x}-x\rangle}{|\tilde{x}-x|} \geq 0\right\}, \quad \forall x \in \operatorname{dom}\omega.$$

In the literature, $\partial_F\omega(\cdot)$ is also known as the *Dini subdifferential* ([40] for instance) or simply *subdifferential* [13, 12, 41], and its notation may also vary ($\partial_D\omega(\cdot)$, $D^-\omega(\cdot)$, $\partial_-\omega(\cdot)$, etc.)

Likewise in the convex framework, the set of all Fréchet subgradients can be expressed in terms of a directional derivative, which in this case corresponds to the *lower Dini derivative* (also referred as the *contingent epiderivative*)

$$D\omega(x;v) = \liminf_{t\to 0^+,\ \tilde{v}\to v} \frac{\omega(x+t\tilde{v})-\omega(x)}{t}, \quad \forall x, v \in X.$$

Moreover, in general, $\partial_V\omega(x) \subseteq \partial_F\omega(x)$ for any $x \in \operatorname{dom}\omega$. Nevertheless, in the finite dimensional context we are setting the analysis, both coincide. These two facts are sumed up in the ensuing proposition.

Proposition 2.3.8 ([41, Proposition 3.4.10 and Proposition 3.4.12])**.** *Let $\omega : X \to \mathbb{R}\cup\{+\infty\}$ be a lower semicontinuous function, then $\partial_V\omega(x) = \partial_F\omega(x)$ for any $x \in dom\,\omega$ and*

$$\partial_V\omega(x) = \{\zeta \in X \mid \ D\omega(x;v) \geq \langle\zeta, v\rangle,\ \forall v \in X\}, \quad \forall x \in dom\,\omega.$$

The last notion of generalized gradient we review is the *proximal subdifferential* to ω, denoted by $\partial_P\omega(\cdot)$, and that agrees with the collection of all $\zeta \in X$ for which there exist $\sigma, \delta > 0$ such that

$$\omega(\tilde{x}) \geq \omega(x) + \langle\zeta, \tilde{x}-x\rangle - \sigma|\tilde{x}-x|^2, \quad \forall \tilde{x} \in \mathbb{B}_X(x,\delta).$$

Note that $\partial_P\omega(x) \subseteq \partial_V\omega(x)$ for any $x \in \operatorname{dom}\omega$. Indeed, the test function is a quadratic one:

$$\varphi(\tilde{x}) := \langle\zeta, \tilde{x}-x\rangle - \sigma|\tilde{x}-x|^2.$$

The proximal subdifferential of a function is intrinsically connected with the proximal normal to its epigraph. The following proposition is the key result we use in the manuscript to link the Hamilton-Jacobi-Bellman approach with the theory invariance of dynamical systems.

Proposition 2.3.9 ([40, Theorem 11.31]). *Let $\omega : X \to \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous function and $x \in dom\,\omega$. Then, $\zeta \in \partial_P \omega(x)$ if and only if $(\zeta, -1) \in \mathcal{N}^P_{epi\,\omega}(x, \omega(x))$.*

On the other hand, the utility of the proximal subdifferential lies in a density result that is summarized below. It says that a viscosity subgradient can always be approximated by a sequence of proximal subgradients. This result allows us to link the classical approach of viscosity solutions (via test functions) and the invariance approach we have adopted.

Proposition 2.3.10 ([41, Proposition 3.4.5]). *Let $\omega : X \to \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous function and $x \in dom\,\omega$. Then for any $\zeta \in \partial_V \omega(x)$ and $\varepsilon > 0$, there exist $x_\varepsilon \in \mathbb{B}_X(x, \varepsilon)$ and $\zeta_\varepsilon \in \partial_P \omega(x_\varepsilon)$ such that*

$$|\zeta - \zeta_\varepsilon| < \varepsilon \quad \textit{and} \quad |\omega(x) - \omega(x_\varepsilon)| < \varepsilon.$$

2.4 Differential inclusions

A Differential inclusion, in a normed space X with dynamics $\Gamma : X \rightrightarrows X$, is a generalization of an ordinary differential equation usually written

$$\dot{y}(t) \in \Gamma(y(t)) \quad \text{a.e. } t \in [a, b]. \tag{2.2}$$

Solutions to (2.2) are understood in the sense of *Carathéodory*, that is, they are absolutely continuous functions $y : [a, b] \to X$ whose derivative belongs to $\Gamma(y(t))$ except on a negligible set of $[a, b]$. Furthermore, several of the properties of differential equations are easily transferred to the differential inclusions. The most remarkable cases are the existence of solutions and the Gronwall's Lemma.

Proposition 2.4.1 ([41, Proposition 4.1.4]). *Suppose that $\Gamma : X \rightrightarrows X$ has linear growth, that is, there exists $c_\Gamma > 0$ such that*

$$\Gamma(x) \leq c_\Gamma(1 + |x|), \quad \forall x \in X.$$

Then any solution of (2.2) satisfies

$$|y(t) - y(a)| \leq (e^{c_\Gamma(t-a)} - 1)\,(|y(a)| + 1)\,, \quad \forall t \in [a, b].$$

2.4.1 Existence of solutions

There is a simple way to deal with existence of solution for the lower semicontinuous framework in a finite dimensional context. This is mainly due to the continuous selection theorems and to the *Nagumo's Theorem.*

Proposition 2.4.2 ([11, Theorem 4.2.2]). *Suppose X is a Hilbert space of finite dimension and $\mathcal{S} \subseteq X$ is a locally compact set. Consider $f : \mathcal{S} \to X$ a continuous vector field and suppose that*

$$f(x) \in \mathcal{T}^B_{\mathcal{S}}(x), \qquad \forall x \in \mathcal{S}. \tag{2.3}$$

Then for all $x \in \mathcal{S}$ there exists $T > 0$ such that the differential equation

$$\dot{y}(t) = f(y(t)), \quad \forall t \in (0, T), \quad y(0) = x$$

has a solution lying in $\mathcal{S}$ on the interval of time $[0, T)$.

Nevertheless, in many situations the dynamics is only upper semicontinuous, which, in general, does not have continuous selections. In this case, the existence of solution remain valid provided the dynamics has nonempty, convex and compact images.

Proposition 2.4.3 ([11, Theorem 2.1.3])**.** *Suppose X is a Hilbert space and $\Gamma : X \rightrightarrows X$ is upper semicontinuous on a neighborhood of $x \in X$ with nonempty, convex and compact images. Then there exists $T > 0$ such that (2.2) has a solution $y : [0, T] \to X$ with $y(0) = x$.*

In the context of Proposition 2.4.3, the set of solutions of (2.2), besides of being nonempty, is compact. This is a direct consequence of the *Convergence Theorem* whose statement we have adapted to the present framework.

Proposition 2.4.4 ([11, Theorem 1.4.1])**.** *Suppose X is a Hilbert space and $\Gamma : X \rightrightarrows X$ is upper semicontinuous with nonempty, convex and compact images. Let $y_n, v_n : [a, b] \to X$ satisfying for a.e. $t \in [a, b]$:*

$$\forall \varepsilon > 0, \ \exists n_0 \in \mathbb{N}, \ s.t. \quad (y_n(t), v_n(t)) \in gr(\Gamma) + \mathbb{B}_X(0, \varepsilon), \quad \forall n \geq n_0.$$

If $\{y_n\}$ converges a.e. to $y : [a, b] \to X$, $\{v_n\} \subseteq L^1([a, b]; X)$ and converges weakly in $L^1([a, b]; X)$ to $v \in L^1([a, b]; X)$. Then,

$$v(t) \in \Gamma(y(t)) \quad \text{for a.e. } t \in [a, b].$$

2.4.2 Invariance of dynamical systems

From a theoretical point of view, the invariance of a control system is a powerful tool for optimal control theory because it allows to link the Value Function with a Hamilton-Jacobi-Bellman equation. The underlying idea is that the epigraph of the Value Function is invariant with respect to an augmented dynamical system. In this case, the Hamilton-Jacobi-Bellman equation can be interpreted as a criterion of invariance.

Depending on the issue at hand, the invariance of a dynamical system can be understood in a weak or in a strong sense. The difference between these two approaches relies upon the number of trajectories solution to (2.2) in which we are interested. The term *weakly* connotes *at least one*, and *strongly* refers to *all of them*.

Definition 2.4.1. *Let $\mathcal{S} \subseteq X$ nonempty, $\mathcal{O} \subseteq X$ open and $\Gamma : X \rightrightarrows X$ a given multifunction. The system $(\mathcal{S}, \Gamma)$ is called weakly invariant in $\mathcal{O}$ if for all $x \in \mathcal{S} \cap \mathcal{O}$, there exists a solution of (2.2) which remains in $\mathcal{O}$ on a maximal interval $[0, T)$ and that satisfies*

$$y(0) = x \quad \text{and} \quad y(t) \in \mathcal{S} \quad \forall t \in [0, T).$$

Furthermore, $(\mathcal{S}, \Gamma)$ is said to be strongly invariant in $\mathcal{O}$ provided every solution of (2.2) satisfies the above conditions.

A very useful characterization of weakly invariance can be stated in term of minimized Hamiltonians and proximal normals.

Proposition 2.4.5. *[132, Theorem 3.1(a)] Suppose $\mathcal{S} \subseteq X$ is nonempty and closed, $\mathcal{O} \subseteq X$ is open and $\Gamma : X \rightrightarrows X$ is a compactly upper semicontinuous multifunction with locally bounded images on X. Suppose in addition that Γ has convex and nonempty images on $\mathcal{S} \cap \mathcal{O}$. Then $(\mathcal{S}, \Gamma)$ is weakly invariant in $\mathcal{O}$ if and only if*

$$\min_{v \in \Gamma(x)} \langle \eta, v \rangle \leq 0 \qquad \forall x \in \mathcal{S} \cap \mathcal{O},\ \forall \eta \in \mathcal{N}^P_{\mathcal{S}}(x). \tag{2.4}$$

We present a criterion for strong invariance adapted to embedded manifolds. This proposition is similar in spirit to Theorem 4.1 in [17] and is an extension of the classical criterion for strong invariance found in the current literature; e.g. [41, Chapter 4.3] or [40, Chapter 12.3]. We also mention that this result is seemingly new and appears first in [71]

Proposition 2.4.6 ([71, Proposition 4.2])**.** *Suppose $M \subseteq X$ is locally closed, $\mathcal{S} \subseteq X$ is closed with $\mathcal{S} \cap \overline{M} \neq \emptyset$ and $\Gamma : \overline{M} \rightrightarrows X$ is locally Lipschitz continuous with locally bounded images. Let $r > 0$ and set $M^r = M \cap \mathbb{B}_X(0, r)$. Assume that the following condition holds: there exists $\kappa = \kappa(r) > 0$ such that*

$$\sup_{v \in \Gamma(x)} \langle x - s, v \rangle \leq \kappa\, dist_{\mathcal{S} \cap \overline{M}}(x)^2, \qquad \forall x \in M^r,\ \forall s \in proj_{\mathcal{S} \cap \overline{M}}(x). \tag{2.5}$$

Then for any $y : [0, T] \to X$ solution of (2.2) with $y(t) \in M^r$ for any $t \in (0, T)$, we have

$$dist_{\mathcal{S} \cap \overline{M}}(y(t)) \leq e^{\kappa t} dist_{\mathcal{S} \cap \overline{M}}(y(0)) \qquad \forall t \in [0, T].$$

Proof. Let c_Γ and L_Γ stands for the corresponding bound for the velocities of Γ and the Lipschitz constant of Γ on $\overline{M} \cap \mathbb{B}(0, r)$. We take $C_1 > 0$ such that

$$\max_{t \in [0,T]} \operatorname{dist}_{\mathcal{S} \cap \overline{M}}(y(t)) \leq C_1$$

Let $\varepsilon > 0$ and set $t_0 = 0$, we construct inductively a partition of $[0, T]$ in the following way: Given $t_i \in [0, T)$ take $t_{i+1} \in (t_i, T]$ satisfying

$$t_{i+1} \leq t_i + \varepsilon \quad \text{and} \quad |y((1-s)t_i + st_{i+1}) - y(t_i)| \leq \frac{1}{L_\Gamma}\varepsilon,\ \forall s \in [0, 1].$$

Note that $|y((1-s)t_i + st) - y(t_i)| \leq c_\Gamma(t - t_i)$ for any $s \in [0, 1]$ and $t > t_i$, so the choice of such t_{i+1} is possible. Moreover, we can do this in such a way it produces a finite partition of $[0, T]$ which we denote $\pi_\varepsilon = \{0 = t_0 < t_1 < \ldots < t_n < t_{n+1} = T\}$. Notice that $\|\pi_\varepsilon\| = \max_{i=0,\ldots,n}(t_{i+1} - t_i) \leq \varepsilon$. For any $i \in \{0, \ldots, n+1\}$, we set $y_i = y(t_i)$ and choose $s_i \in \operatorname{proj}_{\mathcal{S} \cap \overline{M}}(y_i)$ arbitrary. Suppose first that $y(0) \in M$. We will show the inequality only for $t = T$. For $t \in (0, T)$ the proof is similar.

Let $s \mapsto \omega(s) := y((1-s)t_i + st_{i+1})$ defined on $[0, 1]$. Hence, ω is an absolutely continuous function with $\dot{\omega}(s) = \dot{y}((1-s)t_i + st_{i+1})(t_{i+1} - t_i)$ a.e. $s \in [0, 1]$. Thus

$$\omega(1) - \omega(0) = y_{i+1} - y_i = (t_{i+1} - t_i) \int_0^1 \dot{y}((1-s)t_i + st_{i+1}) ds$$

On the other hand, since Γ is locally Lipschitz continuous

$$\Gamma(y((1-s)t_i + st_{i+1})) \subseteq \Gamma(y_i) + L_\Gamma |y((1-s)t_i + st_{i+1}) - y(t_i)|\mathbb{B}_X, \quad \forall s \in [0,1].$$

By construction $L_\Gamma |y((1-s)t_i + st_{i+1}) - y(t_i)| \leq \varepsilon$. Therefore, there exist two measurable functions $v_i : [0,1] \to \Gamma(y_i)$ and $b_i : [0,1] \to \mathbb{B}_X$ such that

$$\dot{y}((1-s)t_i + st_{i+1}) = v_i(s) + \varepsilon b_i(s), \quad \text{a.e. } s \in [0,1].$$

Consequently,

$$\begin{aligned}
&\operatorname{dist}_{\mathcal{S}\cap\overline{M}}(y_{i+1})^2 \leq |y_{i+1} - s_i|^2 \\
&= |y_i - s_i|^2 + 2(t_{i+1} - t_i)\int_0^1 \langle y_i - s_i, v_i(s) + \varepsilon b_i(s)\rangle ds + |y_{i+1} - y_i|^2 \\
&\leq (1 + 2(t_{i+1} - t_i)\kappa)\operatorname{dist}_{\mathcal{S}\cap\overline{M}}(y_i)^2 + \varepsilon(t_{i+1} - t_i)[2C_1 + c_\Gamma^2],
\end{aligned}$$

where this last comes from (2.5), the definition of b_i and the choice of t_i.

Let us denote $\sigma_i = \operatorname{dist}_{\mathcal{S}\cap\overline{M}}(y_i)$ and $\delta_i = t_{i+1} - t_i$. Then, using an inductive argument it is not difficult to state the next inequalities:

$$\begin{aligned}
\sigma_{n+1}^2 &\leq \prod_{i=0}^{n}(1 + 2\delta_i\kappa)\sigma_0^2 + \varepsilon[2C_1 + c_\Gamma^2]\sum_{j=0}^{n}\prod_{i=j+1}^{n}(1 + 2\delta_i\kappa)\delta_j. \\
&\leq \left(\prod_{i=0}^{n}(1 + 2\delta_i\kappa)\right)\left(\sigma_0^2 + \varepsilon[2C_1 + c_\Gamma^2]\sum_{j=0}^{n}\delta_j\right).
\end{aligned}$$

Note that

$$\sum_{j=0}^{n}\delta_j = T \quad \text{and} \quad \prod_{i=0}^{n}(1 + 2\delta_i\kappa) \leq e^{2\kappa T}.$$

In particular, this implies that

$$\sigma_{n+1}^2 \leq e^{2\kappa T}(\sigma_0^2 + \varepsilon[2C_1 + c_\Gamma^2]T).$$

Since $\sigma_{n+1} = \operatorname{dist}_{\mathcal{S}\cap\overline{M}}(y(T))$ and $\sigma_0 = \operatorname{dist}_{\mathcal{S}\cap\overline{M}}(y(0))$, letting $\varepsilon \to 0$ we get the desired result.

Suppose now that $y(0) \notin M$. Then it is clear that for any $\delta > 0$ small enough the trajectory $\tilde{y} = y|_{[\delta,T]}$ satisfies the previous assumptions, so the inequality is valid on the interval $[\delta, T]$ for any $\delta > 0$. Finally, since the distance function is continuous, we can extend the inequality up to $t = 0$ by taking limits and the conclusion follows. □

CHAPTER 3

Manifolds and stratifications

Abstract. In this chapter we review the notions of manifolds and stratifications, as well as the concept of *relatively wedged set* and its relation with stratifications and variational analysis.

3.1 Introduction

The principal feature of this dissertation is that it deals with some issues in control theory from a well-structured point of view; this means, for instance, that when considering a problem with state-constraints, this set is not an arbitrary closed set but one with a recognizable tame structure. The motivation to do so is that in full generality many wild situations may occur. For example, it is well-known that if $\mathcal{S} \subseteq \mathbb{R}^N$ is a compact set, then the collection of *continuous nowhere differentiable* functions is dense in the space of continuous functions $\mathcal{C}(\mathcal{S})$; see for instance the book of Hewitt-Stromberg [72, Chapter 5]. This implies that, for state-constraints of the form $\mathcal{K} = \{x \in \mathcal{S} \mid g(x) \leq 0\}$ with g being merely continuous, the normal and tangent cones are given, in very few occasions, by the following formulas

$$\mathcal{N}_{\mathcal{K}}^{C}(x) = \{\lambda \nabla g(x) \mid \lambda \geq 0\} \quad \text{and} \quad \mathcal{T}_{\mathcal{K}}^{B}(x) = \{v \in \mathbb{R}^N \mid \langle v, \nabla g(x)\rangle \leq 0\}, \quad \forall x \in \mathcal{K}.$$

On the other hand, the Rademacher's theorem show that if g is Lipschitz continuous, then the former expression might have sense for almost all $x \in \mathcal{K}$. However, as shown by Borwein-Wang in [26], given a L-Lipschitz continuous function $g : \mathcal{S} \to \mathbb{R}$ defined on a compact set $\mathcal{S} \subseteq \mathbb{R}^N$, it typically verifies

$$\partial_C g(x)^{1} = \overline{\text{co}} \left\{ \lim_{n\to+\infty} \nabla g(x_n) \,\middle|\, g \text{ differentiable at } x_n \right\} = \mathbb{B}(0, L), \quad \forall x \in \mathcal{S}.$$

This yields in particular to assert that, in a generic way, we have that $\mathcal{N}_{\mathcal{K}}^{C}(x) = \mathbb{R}^N$ as long as $\mathcal{K} = \{x \in \mathcal{S} \mid g(x) \leq 0\}$ with g being Lipschitz continuous but without any additional structure. It is clear that this formula provides absolutely no information about the set $\mathcal{K}$; see also the discussion in Borwein-Zhu [29, Chapter 5].

These two examples show that full generality often considers unnecessarily broad classes of sets and functions.[2] Nevertheless, in many applications it is usual to find well-structured objects more than these pathological ones.

[1] this set is usually referred as the generalized gradient of g at x; see for instance [41, Chapter 2].

[2] Grothendieck explained this in his sketch of a programme saying: *general topology was developed by analysts and in order to meet the needs of analysis, not for topology per se, i.e. the study of the topological properties of the various geometrical shapes...*

In the last few years an increasing number of publications have been dedicated to well-structured optimization; see for instance the survey of Ioffe [76], the book of Lasserre [82], the article by Lasserre-Henrion-Prieur-Trélat [83], and the references therein.

In this dissertation we are concerned with well-structured state-constraints and feedback controls, in particular, in the case in which a stratification of the state-space can be associated with one of the aforementioned objects. For this reason, in this chapter we provide a brief overview on manifolds and stratifications.

Furthermore, to study closed-loop systems generated by a stratified feedback it is necessary to take into account the evolution of the corresponding differential equation on the boundary of an embedded manifold. To provide a well-posed framework to do so, we revisit the notion of *relatively wedged* set. This has been reported in Section 3.4 and in [68], thereby, the contents of this section are new and can be considered as part of the contribution of the thesis.

The definitions and results of this chapter, unless a proof is provided, have been chiefly taken from monograph by Clarke-Ledyaev-Stern-Wolenski [41], Clarke [40], Lee [86], Van den Dries-Miller [130], and the notes of Mather [91]. The rest of the results are part of the contribution of the thesis, and as aforesaid, most of them have been reported in [67, 68] or are being considered for future publications.

3.1.1 Notation

In the forthcoming, the letter $\mathcal{M}$ is reserved to denote a manifold, $k \in \mathbb{N}\cup\{\infty\}$ and $(X, \langle\cdot,\cdot\rangle, |\cdot|)$ stands for a finite dimensional real Hilbert space. We denote by $\mathcal{L}(X, Y)$ the space of linear operators from X into Y. We indicate by $\mathrm{iso}(X, Y)$ the collection of *isomorphisms* from X into Y, that is, $P \in \mathrm{iso}(X, Y)$ if and only if $P \in \mathcal{L}(X, Y)$ and it is bijective. If $X = Y$, $\mathrm{aut}(X)$ stands for $\mathrm{iso}(X, X)$, the *automorphisms* on X. Additionally, we denote by $\mathbb{O}(X)$ the orthogonal group, that is, all the $P \in \mathrm{aut}(X)$ satisfying

$$\langle Px, P\tilde{x}\rangle = \langle x, \tilde{x}\rangle, \quad \forall x, \tilde{x} \in X.$$

3.2 Embedded Manifolds

We begin the exposition with the definition of one of the fundamental objects we use all along the dissertation, we refer to the notion of *embedded manifold* of a finite dimensional Hilbert space. Depending on the issue at hand, we might consider an abstract definition in terms of embedding maps. Nonetheless, for the purpose of the discussion we adopt first a level-set approach and afterwards we make the link with embedding maps.

Formally, given $(X, \langle\cdot,\cdot\rangle, |\cdot|)$ a finite dimensional Hilbert space and $k \in \mathbb{N}\cup\{\infty\}$, a subset $\mathcal{M} \subseteq X$ is a $\mathcal{C}^k$-embedded manifold of X of codimension d if and only for every point on $\mathcal{M}$ there exist an open subset $\mathcal{O} \subseteq X$ and a $\mathcal{C}^k$ *submersion* [3] $h : \mathcal{O} \to \mathbb{R}^d$ such that

$$\mathcal{O} \cap \mathcal{M} = \{x \in \mathcal{O} \mid \ h(x) = 0\}.$$

The function h is called a *local defining map* for $\mathcal{M}$ with domain $\mathcal{O}$. In this case, the integer $\dim(X) - d$ is known as the *dimension* of $\mathcal{M}$ and it is usually represented by $\dim(\mathcal{M})$.

[3] that is, $h : \mathcal{O} \to \mathbb{R}^d$ is a $\mathcal{C}^k$ map so that at each $x \in \mathcal{O}$ its differential $d_x h \in \mathcal{L}(X, \mathbb{R}^d)$ is surjective

Example 3.2.1. *A canonical example of $\mathcal{C}^\infty$-embedded manifold on a N-dimensional Hilbert space X is the sphere of radius 1; see Figure 3.1. Indeed in this case, its codimension is $d = 1$, the local defining map can be taken as $h(x) = |x|^2 - 1$ and its domain $\mathcal{O} = X \setminus \{0\}$. Therefore, the dimension of the sphere is $\dim(\mathcal{M}) = N - 1$.*

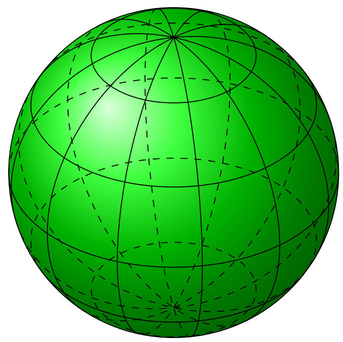


Figure 3.1: The sphere of radius 1.

3.2.1 An alternative definition

In a rather more abstract setting we might consider embedded manifolds of a given smooth manifold instead of a vectorial space. This definition is purely geometric and, as aforesaid, it does not require an ambient space such as a vectorial space to make sense.

The contents of this section are standard and aim to make the link between the definition of embedded manifolds as level-sets and the more classical one through embedding maps. We begin with the geometric definition of smooth manifold. Consider a Hausdorff, second-countable topological space[4] $\mathcal{M}$. We say that $\mathcal{M}$ is a *topological manifold* of dimension N provided for each $x \in \mathcal{M}$ we can find $\mathcal{O} \subseteq \mathcal{M}$, an open neighborhood of x, which is homeomorphic to an open subset of $\mathbb{R}^N$. In this case, if $\phi : \mathcal{O} \to \phi(\mathcal{O})$ is the corresponding homeomorphism,[5] then the pair $(\mathcal{O}, \phi)$ is called a *chart* on $\mathcal{M}$ around x.

Example 3.2.2. *Notice that if $\mathcal{M}$ stands for the sphere of radius 1 from Example 3.2.1, then $\mathcal{M}$ is a topological manifold. Indeed, since this is a closed subset of X, it is not difficult to see that it is a Hausdorff, second-countable topological space. Consider an orthonormal basis $\{\mathbf{x}_1, \ldots, \mathbf{x}_N\}$ of X and define the family of open sets (relative to $\mathcal{M}$)*

$$\mathcal{O}_{i,j} := \{x \in X \mid (-1)^j \langle \mathbf{x}_i, x\rangle > 0\} \cap \mathcal{M}, \quad i = 1, \ldots, N, \; j = 1, 2.$$

It is not difficult to see that $\{\mathcal{O}_{i,j}\}$ is an open covering of $\mathcal{M}$. Furthermore, for every $i = 1, \ldots, N$ and $j = 1, 2$ the function $\phi_{i,j} : \mathcal{O}_{i,j} \to \mathbb{R}^{N-1}$ defined via:

$$\phi_{i,j}(x) := (\langle \mathbf{x}_1, x\rangle, \ldots, \langle \mathbf{x}_{i-1}, x\rangle, \langle \mathbf{x}_{i+1}, x\rangle, \ldots, \langle \mathbf{x}_N, x\rangle), \quad \forall x \in \mathcal{O}_{i,j},$$

[4] that is, a topological space with a countable basis for its topology and in which two different points can be separated by two disjoint open sets; see for instance [96, Chapter 4].

[5] continuous bijective map with continuous inverse.

is a homeomorphism between $\mathcal{O}_{i,j}$ *and* $\{y \in \mathbb{R}^{N-1} \mid \ |y| < 1\}$ *because we can easily check that* $|\phi_{i,j}(x)| < 1$ *for any* $x \in \mathcal{O}_{i,j}$ *and* $\phi_{i,j}$ *is injective. Moreover, for any* $y \in \mathbb{R}^{N-1}$ *with* $|y| < 1$,

$$\sum_{l=1}^{i-1} y_l \mathbf{x}_l + \sum_{l=i+1}^{N} y_{l-1}\mathbf{x}_l + (-1)^j \left(\sqrt{1-|y|^2}\right)\mathbf{x}_i \in \mathcal{O}_{i,j}.$$

Consequently, $\mathcal{M}$ *is a topological manifold with the collection of charts* $\{(\mathcal{O}_{i,j}, \phi_{i,j})\}$.

On the other hand, a topological manifold is said to be a $\mathcal{C}^k$*-smooth manifold* for some $k \in \mathbb{N} \cup \{\infty\}$ provided there is a collection of charts Λ that satisfies the following conditions:

- **Covering:** For every $x \in \mathcal{M}$ there exists $(\mathcal{O}, \phi) \in \Lambda$ for which $x \in \mathcal{O}$.
- **Maximality:** A chart $(\mathcal{Q}, \varphi)$ belongs to Λ if and only if for any $(\mathcal{O}, \phi) \in \Lambda$ such that $\mathcal{Q} \cap \mathcal{O} \neq \emptyset$ we have that $\phi \circ \varphi^{-1} : \varphi(\mathcal{Q} \cap \mathcal{O}) \to \phi(\mathcal{Q} \cap \mathcal{O})$ is a $\mathcal{C}^k$ diffeomorphism.[6]

The collection Λ is referred to as *a smooth structure* on $\mathcal{M}$, and, sometimes, if the degree of smoothness k of $\mathcal{M}$ is understood from the context, we just say that $\mathcal{M}$ is a *smooth manifold.*

Example 3.2.3. *In Example 3.2.3 we have seen that the sphere of radius 1 is a topological manifold with the collection of charts* $\{(\mathcal{O}_{i,j}, \phi_{i,j})\}$. *As a matter of fact, it is a* $\mathcal{C}^\infty$*-smooth manifold. Indeed, we only need to remark that the transition maps* $\phi_{i,j} \circ \phi_{n,m}^{-1}$ *verifies*

$$\phi_{i,j} \circ \phi_{n,m}^{-1}(y) = (y_1, \dots, y_{i-1}, y_{i+1}, \dots, y_n, (-1)^m\sqrt{1-|y|}, y_{n+1}, \dots, y_{N-1}),\ \forall y \in \mathbb{R}^{N-1}, |y| < 1.$$

Example 3.2.4. *Let* X *be a vectorial space of dimension* N *and* $(e_1, \dots, e_N)$ *be the canonical basis of* $\mathbb{R}^N$. *For any basis* $\mathbf{x} = (x_1, \dots, x_N)$ *of* X, *there exists a unique* $P_{\mathbf{x}} \in iso(\mathbb{R}^N, X)$ *such that* $P_{\mathbf{x}}(e_i) = x_i$ *for every* $i = 1, \dots, N$. *Consequently,* $(X, P_{\mathbf{x}}^{-1})$ *is a chart on* X.

Let $\Lambda = \{(X, P_{\mathbf{x}}^{-1}) \mid \mathbf{x}$ *is a basis for* $X\}$, *and take* $\mathbf{x}$ *and* $\tilde{\mathbf{x}}$ *two basis for* X. *Then, the map* $P_{\mathbf{x}}^{-1} \circ P_{\tilde{\mathbf{x}}} : \mathbb{R}^N \to \mathbb{R}^N$ *is* $\mathcal{C}^\infty$ *because*

$$P_{\mathbf{x}}^{-1} \circ P_{\tilde{\mathbf{x}}}(y) = P_{\mathbf{x}}^{-1}\left(\sum_{i=1}^{N} y_i \tilde{x}_i\right) = \sum_{i=1}^{N} y_i P_{\mathbf{x}}^{-1}(\tilde{x}_i), \quad \forall y \in \mathbb{R}^N.$$

Furthermore, $P_{\mathbf{x}}^{-1} \circ P_{\tilde{\mathbf{x}}} \in aut(\mathbb{R}^N)$ *and in particular it is a* $\mathcal{C}^\infty$ *diffeomorphism. Hence,* X *is a* $\mathcal{C}^\infty$*-smooth manifold with the smooth structure* Λ.

The smooth structure of a $\mathcal{C}^k$-smooth manifold allows to extend the classical notions of differentiability to this context in the fashion we describe hereafter. Let $k, l \in \mathbb{N} \cup \{\infty\}$ with $l \leq k$. Consider two $\mathcal{C}^k$-smooth manifolds $\mathcal{M}_1$ and $\mathcal{M}_2$. A map $\Psi : \mathcal{M}_1 \to \mathcal{M}_2$ is said to be of class $\mathcal{C}^l$ provided for each $x \in \mathcal{M}_1$ there are a chart $(\mathcal{O}, \phi)$ around x and a chart $(\mathcal{Q}, \varphi)$ around $\Psi(x)$ so that $\Psi(\mathcal{O}) \subseteq \mathcal{U}$ and the map $\varphi \circ \Psi \circ \phi^{-1}$ is of class $\mathcal{C}^l$ from $\phi(\mathcal{O})$ into $\varphi(\mathcal{Q})$[7].

We recall that the differential of a function $\Psi : X \to Y$ (at a given point) between normed vectorial spaces is an element of $\mathcal{L}(X, Y)$. However, since a manifold does not necessarily have the structure of vectorial space, the differential of a map between manifolds acts on some

[6] $\mathcal{C}^k$ bijective map with $\mathcal{C}^k$ inverse in the standard framework of normed vectorial spaces.

[7] in the standard framework differential calculus on normed vectorial spaces

vectorial spaces that represent locally the corresponding manifold. Formally, let $\mathcal{M}$ be a $\mathcal{C}^k$-smooth manifold. Consider $x \in \mathcal{M}$ and denote by Γ_x the set of curves $\gamma : (-1,1) \to \mathcal{M}$ of class $\mathcal{C}^1$ around $t = 0$ and so that $\gamma(0) = x$. Given $\gamma \in \Gamma_x$, we define a tangent vector to $\mathcal{M}$ at x as the following equivalence class

$$[\gamma] = \left\{ \tilde{\gamma} \in \Gamma_x \middle| \ \exists(\mathcal{O}, \phi) \text{ a chart with } x \in \mathcal{O} \text{ for which } \left.\frac{d}{dt}\right|_{t=0} \phi \circ \gamma(t) = \left.\frac{d}{dt}\right|_{t=0} \phi \circ \tilde{\gamma}(t) \right\}.$$

Using the maximality condition, it is not difficult to see that $[\gamma]$ does not depend on the chart involved in its definition. Consequently, the *tangent space* to $\mathcal{M}$ at x, denoted by $\mathcal{T}_{\mathcal{M}}(x)$ is the collection of all the classes of equivalence, that is

$$\mathcal{T}_{\mathcal{M}}(x) := \{[\gamma] \mid \ \gamma \in \Gamma_x\}, \quad \forall x \in \mathcal{M}.$$

It is not obvious from its definition but the tangent space $\mathcal{T}_{\mathcal{M}}(x)$ is in reality a vectorial space; this is basically due to the fact that, if $(\mathcal{O}, \phi)$ is a chart around $x \in \mathcal{M}$ and, $\gamma_1, \gamma_2 \in \Gamma_x$ and $\lambda \in \mathbb{R}$ are given, then $[\gamma] = [\gamma_1] + \lambda[\gamma_2]$, where $\gamma(\cdot)$ is the curve given by

$$t \mapsto \gamma(t) := \phi^{-1}\left(\phi(x) + t\left(\left.\frac{d}{ds}\right|_{s=0} \phi \circ \gamma_1(s) + \lambda \left.\frac{d}{ds}\right|_{s=0} \phi \circ \gamma_2(s)\right)\right).$$

Moreover, the dimension of $\mathcal{T}_{\mathcal{M}}(x)$ always agrees with $\dim(\mathcal{M})$; to see this it is enough to note that for any $(\mathcal{O}, \phi)$ chart around $x \in \mathcal{M}$ and any $\{v_1, \ldots, v_n\}$ basis of $\mathbb{R}^N$, the curves $\gamma_1(\cdot), \ldots, \gamma_N(\cdot)$ defined below determined N linearly independent equivalent classes on Γ_x:

$$t \mapsto \gamma_i(t) := \phi^{-1}(\phi(x) + tv_i)), \quad \forall i = 1, \ldots, N.$$

Hence, if $\Psi : \mathcal{M}_1 \to \mathcal{M}_2$ is of class $\mathcal{C}^l$ with $l \leq k$, its *differential* at $x \in \mathcal{M}_1$ is the linear map $d_x\Psi : \mathcal{T}_{\mathcal{M}_1}(x) \to \mathcal{T}_{\mathcal{M}_2}(\Psi(x))$ given by

$$d_x\Psi(v) := [\Psi \circ \gamma] \in \mathcal{T}_{\mathcal{M}_2}(\Psi(x)), \quad \forall v \in \mathcal{T}_{\mathcal{M}_1}(x), \forall \gamma \in \Gamma_x \text{ with } v = [\gamma].$$

Before going further we need to introduce the concept that gives the name to the embedded manifolds. Consider two smooth manifolds $\mathcal{M}_1$ and $\mathcal{M}_2$. A map $\Psi : \mathcal{M}_1 \to \mathcal{M}_2$ is said to be a $\mathcal{C}^k$*-embedding* if and only if Ψ is a map of class $\mathcal{C}^k$ such that $d_x\Psi$ is injective for every $x \in \mathcal{M}$ and Ψ is a homeomorphism onto its image.

We are now in position to state the result that links our initial definition of embedded manifold and the one of the present subsection.

Proposition 3.2.1 ([86, Proposition 5.16])**.** *A subset $\mathcal{M} \subseteq X$ is a $\mathcal{C}^k$-embedded manifold of X if and only $\mathcal{M}$ is a topological manifold for the induced topology and there exists a smooth structure on $\mathcal{M}$ for which the inclusion map $\Im : \mathcal{M} \hookrightarrow X$ is a $\mathcal{C}^k$-embedding.*

Remark 3.2.1. *We recall that as shown in Example 3.2.4, X can be endowed with a canonical smooth structure that turns it into a $\mathcal{C}^\infty$-smooth manifold as well. Consequently, the notion of $\mathcal{C}^k$-embedding between $\mathcal{M}$ and X is well defined.*

3.2.2 Tangent and normal spaces

The particular structure of an embedded manifold allows us to describe the tangent space to $\mathcal{M}$ in terms of local defining maps in a rather simple way. Consider a curve $\gamma : (-1,1) \to \mathcal{M}$ of class $\mathcal{C}^1$ so that $\gamma(0) = x$, if h is a local defining map of $\mathcal{M}$, then we have that $h \circ \gamma(t) = 0$ for any $t \in (-1,1)$ close to 0. Therefore, we get that

$$\left.\frac{d}{dt}\right|_{t=0} h \circ \gamma(t) = 0, \quad \forall \gamma : (-1,1) \to \mathcal{M} \text{ of class } \mathcal{C}^1 \text{ so that } \gamma(0) = x.$$

Hence, it is not difficult to see that we can make the following identification (see for instance Gallot-Hulin-Lafontaine [55, Theorem 1.23])

$$\mathcal{T}_{\mathcal{M}}(x) \cong \{v \in X \mid \ d_x h(v) = 0\}, \quad \forall x \in \mathcal{O} \cap \mathcal{M}.$$

Accordingly, from now on we identify $\mathcal{T}_{\mathcal{M}}(x)$ with $\ker(d_x h)$ for every $x \in \mathcal{O} \cap \mathcal{M}$ and for an arbitrary local defining map h whose domain is $\mathcal{O}$. Furthermore, now $\mathcal{T}_{\mathcal{M}}(\cdot)$ can be understood as a set-valued map from X into itself whose effective domain is $\mathcal{M}$. In particular, the Bouligand and Clarke tangent cones are agree with the tangent space.

Proposition 3.2.2 ([40, Theorem 10.45]). *Let $\mathcal{M}$ be a $\mathcal{C}^k$-embedded manifold of X, then $\mathcal{T}^C_{\mathcal{M}}(x) = \mathcal{T}^B_{\mathcal{M}}(x) = \mathcal{T}_{\mathcal{M}}(x)$ for any $x \in \mathcal{M}$.*

On the other hand, since we are now considering $\mathcal{T}_{\mathcal{M}}(x)$ as a vectorial subspace of X it makes sense to define its orthogonal space. The *normal space* to $\mathcal{M}$ at x is defined via

$$\mathcal{N}_{\mathcal{M}}(x) = \{\eta \in X \mid \ \langle \eta, v\rangle = 0, \ \forall v \in \mathcal{T}_{\mathcal{M}}(x)\}, \quad \forall x \in \mathcal{M}.$$

If $h = (h_1, \ldots, h_d)$ is a local defining map with domain $\mathcal{O}$, then

$$\mathcal{N}_{\mathcal{M}}(x) = \operatorname{span}\{\nabla h_1(x), \ldots, \nabla h_d(x)\}, \quad \forall x \in \mathcal{O} \cap \mathcal{M}.$$

By Proposition 3.2.2 and 2.3.6, we get that $\mathcal{N}^C_{\mathcal{M}}(x) = \mathcal{N}_{\mathcal{M}}(x)$ for any $x \in \mathcal{M}$. Furthermore, if $\mathcal{M}$ is at least of class $\mathcal{C}^2$, we get an akin result for the proximal normal cone.

Proposition 3.2.3. *For any $\mathcal{C}^k$-embedded manifold $\mathcal{M}$ of X, $\mathcal{N}^P_{\mathcal{M}}(x) \subseteq \mathcal{N}^L_{\mathcal{M}}(x) = \mathcal{N}^C_{\mathcal{M}}(x) = \mathcal{N}_{\mathcal{M}}(x)$ for any $x \in \mathcal{M}$. Additionally, if $k \geq 2$, the equality holds and for every $x \in \mathcal{M}$, we can find $\delta = \delta(x) > 0$ for which*

$$\frac{|\eta|}{2\delta}|x - \tilde{x}|^2 \geq \langle \eta, \tilde{x} - x\rangle \qquad \forall \eta \in \mathcal{N}_{\mathcal{M}}(x), \forall \tilde{x} \in \mathcal{M}.$$

Remark 3.2.2. *Notice that if the manifold is just $\mathcal{C}^1$ then the $\mathcal{N}^P_{\mathcal{M}}(\cdot)$ may have trivial images on $\mathcal{M}$. Indeed, if $\mathcal{M} = \{x_2^5 = x_1^3\}$, the boundary of $\mathcal{S}$ in Example 2.3.2; see Figure 3.2. Then $\mathcal{N}^P_{\mathcal{M}}((0,0)) = (0,0)$ but $\mathcal{M}$ is a $\mathcal{C}^1$ manifold because $h(x_1, x_2) = x_1 - x_2^{5/3}$ is a local defining map for $\mathcal{M}$ which is $\mathcal{C}^1$ but not $\mathcal{C}^2$ on $\mathbb{R}^2$.*

To prove Proposition 3.2.3, we require the following lemma.

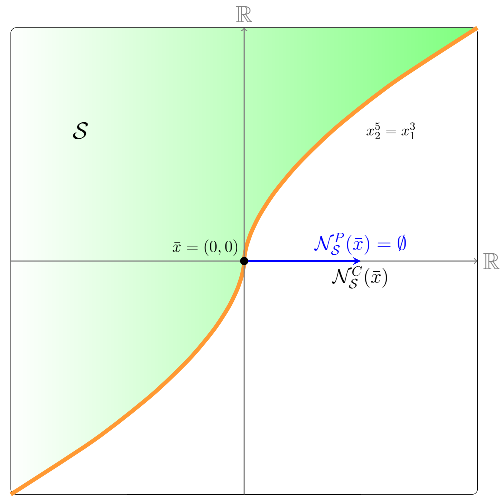


Figure 3.2: The set $\mathcal{S}$ in Example 2.3.2.

Lemma 3.2.1. *Consider $\mathcal{O} \subseteq X$ and let $\omega_1, \ldots, \omega_d : \mathcal{O} \to X$ be given continuous functions such that the collection $\{\omega_1(x), \ldots, \omega_d(x)\}$ is linearly independent for every $x \in \mathcal{O}$. Then, for every $\mathcal{S}$ compactly contained in $\mathcal{O}$, there exists $\sigma > 0$ such that*

$$|\mu| \leq \sigma \left| \sum_{i=1}^{d} \mu_i \omega_i(x) \right|, \quad \forall x \in \mathcal{S},\ \forall \mu \in \mathbb{R}^d.$$

Proof. Let $P_{ij}(x) = \langle \omega_i(x), \omega_j(x) \rangle$ for $i, j = 1, \ldots, d$ and any $x \in \mathcal{O}$. We readily realize that the matrix $P(x) = (P_{ij}(x))$ is symmetric. Notice also that, since $\omega_1(x), \ldots, \omega_d(x)$ are linearly independent, $P(x)$ is positive-definite for every $x \in \mathcal{O}$ because

$$\langle P(x)\mu, \mu \rangle = \sum_{i,j=1}^{d} \langle \mu_i \omega_i(x), \mu_j \omega_j(x) \rangle = \left| \sum_{i=1}^{d} \mu_i \omega_i(x) \right|^2, \quad \forall x \in \mathcal{O},\ \forall \mu \in \mathbb{R}^d.$$

Consequently, if $\lambda_{min}(P)$ stands for $\inf_{|\nu|=1} \langle P\nu, \nu \rangle$ for any $P \in \mathbb{M}_{d\times d}(\mathbb{R})$,

$$\lambda_{min}(P(x))|\mu|^2 \leq \langle P(x)\mu, \mu \rangle = \left| \sum_{i=1}^{d} \mu_i \omega_i(x) \right|^2, \quad \forall x \in \mathcal{O},\ \forall \mu \in \mathbb{R}^d.$$

Note that by definition the map $P \mapsto \lambda_{min}(P)$ is concave, and accordingly, locally Lipschitz continuous on the interior of its domain. Furthermore, $\lambda_{min}(P(\cdot))$ is continuous and strictly positive on $\mathcal{O}$ because the set of symmetric and positive-definite matrices is open on $\mathbb{M}_{d\times d}(\mathbb{R})$ and it is contained in $\operatorname{dom} \lambda_{min}$. Notice that if $\mathcal{S}$ is compactly contained in $\mathcal{O}$, then $\inf_{x\in\mathcal{S}} \lambda_{min}(P(x)) > 0$. This observation completes the proof. □

Proof of Proposition 3.2.3. By Proposition 3.2.2 and 2.3.6, combined with [40, Theorem 11.36], we get that the Limiting and Clarke normal cone coincides with the normal space. If $k \geq 2$, by virtue of [41, Proposition 1.1.9], we get the equality for the Proximal normal cone.

On the other hand, let $h = (h_1, \ldots, h_d)$ be a local defining map on $\mathbb{B}_X(x, r)$ with $r > 0$. Let us fix $r_0 \in (0, r)$ and set

$$L_0 = \max_{i=1,\ldots,d} \sup \left\{ |\nabla^2 h_i(\tilde{x})| \mid \tilde{x} \in \mathbb{B}_X(x, r_0) \right\}.$$

Remark that $L_0 \in \mathbb{R}$ because h is of class $\mathcal{C}^2$. Thus, by the Mean Value Theorem

$$|\nabla h_i(x)(\tilde{x} - x)| \leq L_0 |x - \tilde{x}|, \quad \forall \tilde{x} \in \mathcal{M} \cap \mathbb{B}_X(x, r_0).$$

Let $\eta \in \mathcal{N}_{\mathcal{M}}(x)$, then there is $\mu \in \mathbb{R}^d$ so that $\eta = \sum \mu_i \nabla h_i(x)$. Additionally, by Lemma 3.2.1 there exists $\sigma_0 > 0$ such that $|\mu| \leq \sigma_0 |\eta|$. Hence, gathering the estimates, we get

$$\langle \eta, \tilde{x} - x \rangle = \sum_{i=1}^{d} \mu_i \langle \nabla h_i(x), \tilde{x} - x \rangle \leq \sigma_0 L_0 |\eta| |\tilde{x} - x|^2, \quad \forall \tilde{x} \in \mathcal{M} \cap \mathbb{B}_X(x, r_0).$$

If $\tilde{x} \notin \mathcal{M} \cap \mathbb{B}_X(x, r_0)$, the inequality is immediate with $\frac{r_0}{2}$. Therefore, setting

$$\delta := \min \left\{ \frac{r_0}{2}, \frac{1}{2\sigma_0 L_0} \right\} \text{ if } L_0 > 0 \text{ and } \delta = \frac{r_0}{2}, \text{ otherwise,}$$

the proof is complete. □

Note that δ in Proposition 3.2.3 is the radius of a closed ball centered at $x + \frac{\delta}{|\eta|}\eta$ which intersects $\mathcal{M}$ only at x. In this sense, it is possible to interpret this number as the curvature of $\mathcal{M}$. We name *radius of curvature* of $\mathcal{M}$ at x to the quantity

$$\kappa(x) = \sup \left\{ \frac{2\langle \eta, \tilde{x} - x \rangle}{|\tilde{x} - x|^2} \,\middle|\, \eta \in \mathcal{N}_{\mathcal{M}}(x),\ |\eta| = 1,\ \tilde{x} \in \mathcal{M} \setminus \{x\} \right\}.$$

Consequently, $\mathcal{M}$ is said to have *bounded curvature* if there is a constant $\kappa_0 \in \mathbb{R}$ so that $\kappa(x) \leq \kappa_0$ for any $x \in \mathcal{M}$. Notice that, due to Remark 3.2.2, possibly $\kappa(x) = +\infty$ if $\mathcal{M}$ is merely of class $\mathcal{C}^1$.

Example 3.2.5. *Let $\mathcal{M}$ be a vectorial subspace of X, then it is an embedded manifold of X, its curvature is $\kappa \equiv 0$. If in addition, $\mathcal{M}$ agrees with the sphere of radius 1 in X, $\{x \in X \mid |x|^2 = 1\}$, then $\kappa(x) \equiv 1$.*

Additionally, the fact that the tangent space can be interpreted as the kernel of a matrix, suggests that the map $x \mapsto \mathcal{T}_{\mathcal{M}}(x)$ may be rather regular as set-valued map, however, it is rarely continuous. This is because $\mathcal{T}_{\mathcal{M}}(\cdot)$ is rarely upper semicontinuous.

Proposition 3.2.4. *For every embedded manifold $\mathcal{M}$ of X, its tangent space $\mathcal{T}_{\mathcal{M}} : X \rightrightarrows X$ is lower and compactly upper semicontinuous on $\mathcal{M}$. Moreover, $\mathcal{T}_{\mathcal{M}}(\cdot)$ is upper semicontinuous at $x \in \mathcal{M}$ if and only if we can find $\delta > 0$ so that $\mathcal{T}_{\mathcal{M}}(\tilde{x}) = \mathcal{T}_{\mathcal{M}}(x)$ for any $\tilde{x} \in \mathcal{M} \cap \mathbb{B}_X(x, \delta)$.*

Proof. Let $h : \mathcal{O} \to \mathbb{R}^d$ be a local defining map and let $x \in \mathcal{O} \cap \mathcal{M}$ fixed but arbitrary. Note that for any sequences $\{x_n\} \subseteq \mathcal{O} \cap \mathcal{M}$ $\{\eta_n\} \subseteq X$ for which $x_n \to x$, $\eta_n \to \eta$ and $\eta_n \in \mathcal{N}_{\mathcal{M}}(x_n)$, we have $\eta \in \mathcal{N}_{\mathcal{M}}(x)$, that is, $\mathcal{N}_{\mathcal{M}}(\cdot)$ is graph-closed at x. Indeed, for any $n \in \mathbb{N}$ there exists

$\mu_n \in \mathbb{R}^d$ such that $\eta_n = \sum \mu_{n,i} \nabla h_i(x_n)$. by virtue of Lemma 3.2.1, $\{\mu_n\}$ is bounded, thereby it has a converging subsequence. Hence, the affirmation holds and in particular, $\mathcal{N}^C_{\mathcal{M}}(\cdot)$ is graph-closed at x as well. Moreover, by Proposition 3.2.2 and [41, Proposition 3.6.8], $\mathcal{T}_{\mathcal{M}}(\cdot)$ is lower semicontinuous at x.

On the other hand, for any closed neighborhood $\mathcal{S}$ of x contained in $\mathcal{O}$, we denote by $\Gamma_{\mathcal{S}}$ the set-valued maps whose effective domain is $\mathcal{S} \cap \mathcal{M}$ and that coincides with $\mathcal{T}_{\mathcal{M}}(\cdot)$ on $\mathcal{S} \cap \mathcal{M}$. Therefore, since for any $\tilde{x} \in \mathcal{O}$, $v \in \mathcal{T}_{\mathcal{M}}(\tilde{x})$ if and only if $d_{\tilde{x}} h(v) = 0$, we obtain that $\Gamma_{\mathcal{S}}(\cdot)$ has closed graph, so by Proposition 2.2.3, $\mathcal{T}_{\mathcal{M}}(\tilde{x})$ is compactly upper semicontinuous at x, because it coincides with $\Gamma_{\mathcal{S}}$ on the interior of $\mathcal{S}$. Finally, the last statement is a direct consequence of Proposition 2.2.2 which ends the proof. $\square$

The previous result shows that the tangent space is seldom locally Lipschitz continuous, even if the manifold is highly differentiable. Anyhow, if the manifold is more than $\mathcal{C}^2$, then cut tangent space, $x \mapsto \mathcal{T}_{\mathcal{M}}(x) \cap \overline{\mathbb{B}}_X$ is locally Lipschitz continuous. The next result is a consequence of a generalized version of the Inverse Function Theorem, known as the Grave-Lyusternik Theorem.

Proposition 3.2.5. *For every $\mathcal{C}^k$-embedded manifold $\mathcal{M}$ of X with $k \geq 2$ and any $r > 0$, the map $x \mapsto \mathcal{T}_{\mathcal{M}}(x) \cap \overline{\mathbb{B}_X(0,r)}$ is locally Lipschitz continuous on $\mathcal{M}$.*

Proof. Let h be a local defining map with domain $\mathcal{O}$. Consider $x \in \mathcal{O} \cap \mathcal{M}$ fixed but arbitrary and $v \in \mathcal{T}_{\mathcal{M}}(x)$. Suppose d is the codimension of $\mathcal{M}$ and consider the function $f : \mathcal{O} \times X \to X \times \mathbb{R}^d$ defined via

$$f(\tilde{x}, \tilde{v}) = (x, d_x h(v)), \quad \forall \tilde{x} \in \mathcal{O} \times X.$$

Notice that $f^{-1}(\hat{x}, 0) = \{\hat{x}\} \times \mathcal{T}_{\mathcal{M}}(\hat{x})$ for any $\hat{x} \in \mathcal{O} \cap \mathcal{M}$. Besides, inasmuch as h can be taken at least $\mathcal{C}^2$, f is $\mathcal{C}^1$ on $\mathcal{O} \times X$ with

$$d_{(x,v)} f(p,q) = (p, d^2_x h(v,p) + d_x h(q)), \quad \forall p, q \in X.$$

Consequently, $d_{(x,v)} f$ is surjective as long as $d_x h$ does so, and by virtue of the Grave-Lyusternik Theorem [40, Theorem 5.32] there are $L, \delta > 0$, which depend upon x and v, so that, for any $\tilde{x}, \hat{x} \in \mathbb{B}_X(x, \delta)$, $\tilde{v} \in \mathbb{B}_X(v, \delta)$ and $|\mu| < \delta$

$$\operatorname{dist}_{f^{-1}(\hat{x},\mu)}(\tilde{x}, \tilde{v}) \leq L(|\tilde{x} - \hat{x}| + |d_{\tilde{x}} h \tilde{v} - \mu|).$$

Without lost of generality $\mathbb{B}_X(x, \delta) \subseteq \mathcal{O}$, and therefore, for $\mu = 0$ we get

$$\operatorname{dist}_{\mathcal{T}_{\mathcal{M}}(\hat{x})}(\tilde{v}) \leq L|\tilde{x} - \hat{x}|, \quad \forall \tilde{x}, \hat{x} \in \mathcal{M} \cap \mathbb{B}_X(x, \delta),\ \tilde{v} \in \mathcal{T}_{\mathcal{M}}(\tilde{x}) \cap \mathbb{B}_X(v, \delta).$$

On the other hand, let $r > 0$ and set $\mathcal{S} = \overline{\mathbb{B}_X(0,r)}$ and $\Gamma(\cdot) = \mathcal{T}_{\mathcal{M}}(\cdot) \cap \mathcal{S}$. By compactness, the last inequality holds true, with a possibly larger L and smaller δ, for any $\tilde{v} \in \Gamma(\tilde{x})$. So, we have proven that for any $x \in \mathcal{M}$ we can find $L, \delta > 0$

$$\sup_{\tilde{v} \in \Gamma(\tilde{x})} \inf_{\hat{v} \in \mathcal{T}_{\mathcal{M}}(\hat{x})} |\hat{v} - \tilde{v}| \leq L|\tilde{x} - \hat{x}|, \quad \forall \tilde{x}, \hat{x} \in \mathcal{M} \cap \mathbb{B}_X(x, \delta). \tag{3.1}$$

Finally, notice that if $\hat{v}$ stands for the projection of $\tilde{v} \in \Gamma(\tilde{x})$ over $\mathcal{T}_{\mathcal{M}}(\hat{x})$, by means of (3.1), $|\hat{v}| \leq r + L|\tilde{x} - \hat{x}|$. Hence, if $|\hat{v}| > r$ we get that

$$\left| \frac{r}{|\hat{v}|} \hat{v} - \tilde{v} \right| \leq |\hat{v} - \tilde{v}| + |\hat{v}| - r \leq 2L|\tilde{x} - \hat{x}|.$$

In particular, (3.1) is still valid (L could be larger) if the infimum is taken over $\Gamma(\hat{x})$ instead of $\mathcal{T}_{\mathcal{M}}(\hat{x})$. Accordingly, by switching the roles of $\tilde{x}$ and $\hat{x}$ in (3.1), and using the definition of the Hausdorff distance, we obtain the desired result. $\square$

3.2.3 Differentiable functions and extensions

Let $\Psi : X \to Y$ be a smooth map from X into another vectorial space. We now address the question of whether the restricted map $\Psi|_{\mathcal{M}} : \mathcal{M} \to Y$ is a smooth function as well. The answer depends of course on the degree of smoothness of the embedded manifold. An appropriate statement for the framework we are working in is described below.

Proposition 3.2.6 ([86, Theorem 5.27])**.** *Let $\Psi : X \to Y$ be a $\mathcal{C}^l$ map from X into another finite dimensional vectorial space Y. Let $\mathcal{M}$ be a $\mathcal{C}^k$ embedded manifold of X with $k \geq l$. Then $\Psi|_{\mathcal{M}} : \mathcal{M} \to Y$ is a $\mathcal{C}^l$ map.*

Reciprocally, to understand how a smooth function defined on $\mathcal{M}$ behaves with respect to the ambient space we have the following result.

Proposition 3.2.7. *Let $\Psi : \mathcal{M} \to \mathbb{R}$ be a $\mathcal{C}^l$ function and $\mathcal{M}$ a $\mathcal{C}^k$ embedded manifold of X with $k \geq l$. Then, there exist an open set $\mathcal{O} \subseteq X$ which contains $\mathcal{M}$ and a $\mathcal{C}^l$ function $\widetilde{\Psi} : \mathcal{O} \to \mathbb{R}$ so that $\widetilde{\Psi}(x) = \Psi(x)$ for any $x \in \mathcal{M}$.*

Proof. Let $\dim(X) = N$ and set the codimension of $\mathcal{M}$ as d. We take $(\mathcal{Q}, \phi)$ a local chart for $\mathcal{M}$ around $x \in \mathcal{M}$ and $\pi : X \to \mathbb{R}^{N-d}$ be a canonical projection. Let $r_x > 0$ so that $\pi(\mathbb{B}_X(x, r_x)) \subseteq \phi(\mathcal{Q})$ and consider the function $\Psi_x : \mathbb{B}_X(x, r_x) \to \mathbb{R}$ given by

$$\Psi_x(\tilde{x}) = \Psi \circ \phi^{-1} \circ \pi(\tilde{x}), \quad \forall \tilde{x} \in \mathbb{B}_X(x, r_x).$$

By definition this function is of class C^l on $\mathbb{B}_X(x, r_x)$. Furthermore, since $\mathcal{M}$ is an embedded manifold of X, by reducing r_x if necessary, $\mathcal{M} \cap \mathbb{B}_X(x, r_x) \subseteq \mathcal{M}$. So, Ψ_x is a local extension of Ψ around x.

On the other hand, let $\{\varphi_x\}_{x\in\mathcal{M}}$ be a $\mathcal{C}^\infty$ partition of the unity subordinated to the collection $\{\mathbb{B}_X(x, r_x)\}_{x\in\mathcal{M}}$ which is an open covering of $\mathcal{M}$; we refer to [3, Theorem 3.14] for the existence of the parition of the unity. Thereby, setting $\mathcal{O} = \cup_{x\in\mathcal{M}}\mathbb{B}_X(x, r_x)$ and $\widetilde{\Psi} : \mathcal{O} \to \mathbb{R}$ as defined below we conclude the proof.

$$\widetilde{\Psi}(\tilde{x}) = \sum_{x\in\mathcal{M}} \varphi_x(\tilde{x})\Psi_x(\tilde{x}), \quad \forall \tilde{x} \in \mathcal{O}.$$

$\square$

In view of the foregoing claim, we have that if $\Psi : \mathcal{M} \to Y$ is a $\mathcal{C}^l$ function from $\mathcal{M}$ into Y, a finite dimensional vectorial space, then Ψ can be extended to a smooth function defined on an open neighborhood of $\mathcal{M}$. Thus, differentiable function on embedded manifolds are understood all along this manuscript as differentiable function on the ambient space, defined on an open neighborhood around $\mathcal{M}$.

In addition, for set-valued maps defined on manifold we have similar extension properties adjusted for the continuity notions. This is mainly due to the *Tubular Neighborhood Theorem.*

For this purpose we introduce the *normal bundle*, written $\mathcal{N}_{\mathcal{M}}$, that is the collection of all the normal vectors to $\mathcal{M}$. Accordingly, $(x,\eta)\in\mathcal{N}_{\mathcal{M}}$ if and only if $\eta\in\mathcal{N}_{\mathcal{M}}(x)$.

The Tubular Neighborhood Theorem reads as follows.

Proposition 3.2.8 ([86, Theorem 6.24])**.** *Let $\mathcal{M}$ be a $\mathcal{C}^k$-embedded manifold of X with $k\geq 2$, and consider the map $E:\mathcal{N}_{\mathcal{M}}\to X$ given by*

$$E(x,\eta)=x+\eta,\quad \forall x\in\mathcal{M},\ \eta\in\mathcal{N}_{\mathcal{M}}(x).$$

Then, we can find a continuous function $\delta:\mathcal{M}\to(0,+\infty)$ and an open neighborhood $\mathcal{O}$ of $\mathcal{M}$ for which $E|_{\mathcal{Q}}:\mathcal{Q}\to\mathcal{O}$ is a $\mathcal{C}^{k-1}$ diffeomorphism, where

$$\mathcal{Q}=\{(x,\eta)\in\mathcal{N}_M\mid|\eta|<\delta(x)\}.$$

In this situation, $\mathcal{O}$ is called a tubular neighborhood of $\mathcal{M}$.

Remark 3.2.3. *It turns out that the normal bundle is a $\mathcal{C}^{k-1}$-embedded manifold of $X\times X$ provided that $\mathcal{M}$ is a $\mathcal{C}^k$-embedded manifold of X with $k\geq 2$; we refer to [86, Proposition 6.23] for more details. Hence, $E|_{\mathcal{Q}}:\mathcal{Q}\to\mathcal{O}$ being a $\mathcal{C}^{k-1}$ diffeomorphism makes sense.*

In particular, the *projection* over $\mathcal{M}$, $\pi_{\mathcal{M}}:\mathcal{O}\to\mathcal{M}$ is well-defined provided $\mathcal{O}$ is a tubular neighborhood of $\mathcal{M}$ and $k\geq 2$. In this case, $\pi_{\mathcal{M}}$ is a $\mathcal{C}^{k-1}$ submersion. Furthermore, if $\Gamma:\mathcal{M}\rightrightarrows Y$ is a set-valued maps, then $\widetilde{\Gamma}:\mathcal{O}\rightrightarrows Y$ defined via $\widetilde{\Gamma}(x)=\Gamma(\pi_{\mathcal{M}}(x))$ is an extension of Γ to $\mathcal{O}$ and we have the following result whose proof is immediate.

Proposition 3.2.9. *Let $\mathcal{M}$ be a $\mathcal{C}^k$-embedded manifold of X with $k\geq 2$ and consider $\Gamma:\mathcal{M}\rightrightarrows Y$ with Y being a prescribed metric space. Let $\mathcal{O}$ be the tubular neighborhood given by Proposition 3.2.8, then $\Gamma\circ\pi_{\mathcal{M}}$ is upper semicontinuous, lower semicontinuous or locally Lipschitz continuous on $\mathcal{O}$ provided Γ is upper semicontinuous, lower semicontinuous or locally Lipschitz continuous on $\mathcal{M}$, respectively.*

3.3 Stratifications

We now turn our attention into another of the concepts that plays an essential role in this manuscript, we mean, the notion of *stratification*.

Let $\mathcal{K}$ stand for an arbitrary subset of X. A $\mathcal{C}^k$-*stratification* of $\mathcal{K}$ is a collection $\{\mathcal{M}_i\}_{i\in\mathcal{I}}$ of $\mathcal{C}^k$-embedded manifolds of X that satisfies the following conditions:

- **Disjoint partition:** $\mathcal{K}=\bigcup_{i\in\mathcal{I}}\mathcal{M}_i$ and $\mathcal{M}_i\cap\mathcal{M}_j=\emptyset$ when $i\neq j$.
- **Locally finite:** $\forall r>0$ there exists $\mathcal{I}_r\subseteq\mathcal{I}$ finite such that
$$\mathcal{K}\cap\mathbb{B}_X(0,r)\subseteq\bigcup_{i\in\mathcal{I}_r}\mathcal{M}_i.$$
- **Filtration:** For any $i,j\in\mathcal{I}$, if $\mathcal{M}_i\cap\overline{\mathcal{M}_j}\neq\emptyset$, then $\mathcal{M}_i\subseteq\overline{\mathcal{M}_j}$ and $\dim(\mathcal{M}_i)<\dim(\mathcal{M}_j)$.

In such a case, we say that $\mathcal{K}$ is a $\mathcal{C}^k$-*stratifiable* set and each $\mathcal{M}_i$ is called a *stratum*. If in the filtration condition, we remove the condition over the dimensions of the strata, then $\mathcal{K}$ is said to be a *pre-stratifiable* set.

Remark 3.3.1. *The class of pre-stratifiable sets is strictly larger than the stratifiable one. Indeed, define*

$$\mathcal{M} = \left\{ (x_1, x_2) \in \mathbb{R}^2 \middle| \; x_2 = \sin\left(\frac{1}{x_1}\right), x_1 \neq 0 \right\},$$

and set $\mathcal{K} = \mathcal{M} \cup \{0\} \times [-1, 1]$*; see Figure 3.3. Hence, the following is an admissible* $\mathcal{C}^\infty$ *pre-stratification of* $\mathcal{K}$ *which is not a stratification, this is because of* $\mathcal{M}_2 \subsetneq \overline{\mathcal{M}}_1$ *but their dimensions are the same. The stratification at issue is:*

$$\mathcal{M}_1 = \mathcal{M}, \quad \mathcal{M}_2 = \{0\} \times (-1, 1), \quad \mathcal{M}_3 = \{(0, 1)\}, \quad \mathcal{M}_4 = \{(0, -1)\}.$$

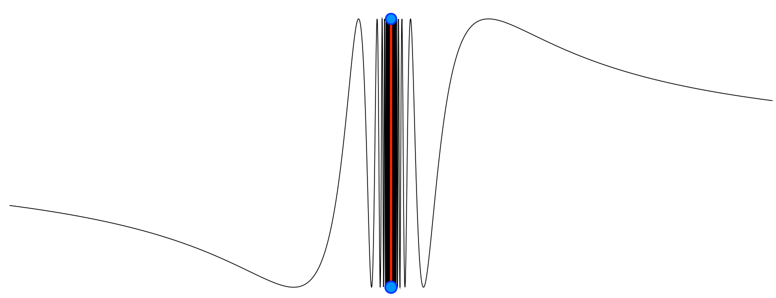


Figure 3.3: Example of pre-stratification which is not a stratification.

Notice that, a priori, the set of indexes $\mathcal{I}$ does not have any particular structure associated with. Anyhow, the Locally finite condition implies that it should be countable and, on the other hand, the Filtration condition allows us to endow it with a partial order defined via

$$\forall i, j \in \mathcal{I}, \qquad i \preceq j \text{ (or } j \succeq i) \quad \Longleftrightarrow \quad \mathcal{M}_i \subseteq \overline{\mathcal{M}}_j.$$

For a given (pre-)stratifiable set $\mathcal{K}$, we define the *index* map $\imath : \mathcal{K} \to \mathcal{I}$ as the function that links any $x \in \mathcal{K}$ with the index $i \in \mathcal{I}$ for which $x \in \mathcal{M}_i$.

3.3.1 Whitney regularity conditions

In the literature there are several regularity concepts associated with a (pre-) stratification and maybe, the most studied of them were introduced by Whitney; see for example the notes of Mather [91] or the exposition in Nicolaescu [97, Chapter 4].

To proceed further we need to introduce a notion of convergence for subspaces. The *Gap* between X_1 and X_2, two subspaces of X, is defined via

$$\mathcal{D}(X_1, X_2) = d_H(X_1 \cap \mathbb{B}_X, X_2 \cap \mathbb{B}_X).$$

Therefore, we say that $\{X_n\}$, a sequence of subspaces of X, converges to X_∞, another subspace of X, provided $\mathcal{D}(X_n, X_\infty) \to 0$ as $n \to +\infty$. To simplify the notation, we write this as $X_n \to X_\infty$. Moreover, $\mathcal{D}$ is a distance for which the set of all vectorial subspaces of X is a compact metric space.

Proposition 3.3.1. *Any sequence $\{X_n\}$ of vectorial subspaces of X has a converging subsequence to some vectorial subspace X_∞ of X. In addition, if $\dim(X_n) = p$ for $n \in \mathbb{N}$ large enough, then $\dim(X_\infty) = p$ as well.*

Proof. Set $N = \dim(X)$. Given that there are only a finite number of possible dimensions for the X_n but an infinite number of them, there must exist $p \in \{0, \ldots, N\}$ and a subsequence, which we avoid relabeling, so that $\dim(X_n) = p$. Hence, for any $n \in \mathbb{N}$ we can find $\{x_n^1, \ldots, x_n^p\}$, an orthonormal basis for X_n. Thereby, passing into another subsequence if necessary which we again eschew to relabel, each $\{x_n^l\}$ converges to some $x_l \in X$. Because of $\{x_n^1, \ldots, x_n^p\}$ is orthonormal, $\{x^1, \ldots, x^p\}$, is orthonormal too. Furthermore, we readily see that X_n converges to $X_\infty = \operatorname{span}\{x^1, \ldots, x^p\}$ because, by orthonormality,

$$\mathcal{D}(X_n, X_\infty) \leq \sum_{i=1}^{p} |x_n^i - x^i|.$$

The last affirmation on the statement is evident from the proof. □

Now, we are in position to introduce the *Whitney conditions*. Let $\mathcal{M}_i$ and $\mathcal{M}_j$ be two embedded manifolds of X and let $x \in \mathcal{M}_i \cap \overline{\mathcal{M}}_j$.

(A) The pair $(\mathcal{M}_i, \mathcal{M}_j)$ satisfies the *Whitney (a)-condition* at x if and only if for any sequence $\{x_n\} \subseteq \mathcal{M}_j$ with $x_n \to x$, if there is a vectorial subspace $\mathcal{T} \subseteq X$ for which $\mathcal{T}_{\mathcal{M}_j}(x_n) \to \mathcal{T}$, then necessarily $\mathcal{T}_{\mathcal{M}_i}(x) \subseteq \mathcal{T}$.

(B) The pair $(\mathcal{M}_i, \mathcal{M}_j)$ satisfies the *Whitney (b)-condition* at x provided that for any sequence $\{(x_n, \tilde{x}_n)\} \subseteq \mathcal{M}_j \times \mathcal{M}_i$ with $x_n, \tilde{x}_n \to x$, if there exists two vectorial subspaces of $\mathcal{T}_i, \mathcal{T}_j \subseteq X$ so that $\mathcal{T}_{\mathcal{M}_j}(x_n) \to \mathcal{T}_j$ and $\mathbb{R}(x_n - \tilde{x}_n) \to \mathcal{T}_i$, then $\mathcal{T}_i \subseteq \mathcal{T}_j$.

Consequently, we say that $\mathcal{K} \subseteq X$ is *(W_a)-stratifiable* if it admits a stratification $\{\mathcal{M}_i\}_{i \in \mathcal{I}}$ such that for any $i, j \in \mathcal{I}$ for which $i \preceq j$, the pair $(\mathcal{M}_i, \mathcal{M}_j)$ satisfies the Whitney (a)-condition all along $\mathcal{M}_i$. In a similar manner we define the *(W_b)-stratifiable* sets.

Remark 3.3.2. *We can see in the example of Remark 3.3.1, that the Whitney (a)-condition holds although the (b)-condition does not. Indeed, suppose $\mathcal{M}_i = \{0\} \times (-1, 1)$ and $\mathcal{M}_j = \mathcal{M}$. Take $x = (0, 0)$, so the sequences defined below converge to x,*

$$\tilde{x}_n = x \in \mathcal{M}_i \quad \textit{and} \quad x_n \left(\frac{1}{n\pi}, 0\right) \in \mathcal{M}_j, \quad \forall n \in \mathbb{N}.$$

Notice that $\mathbb{R}(x_n - \tilde{x}_n) = \mathbb{R} \times \{0\}$ and $\mathcal{T}_{\mathcal{M}_i}(x_n) = \{0\} \times \mathbb{R}$. Thus, the (b)-condition fails.

In view of the previous remark, it turns out that the (b)-condition is strictly stronger than the (a)-condition, as the following result states.

Proposition 3.3.2 ([91, Proposition 2.4])**.** *Let $\mathcal{M}_i$ and $\mathcal{M}_j$ be two embedded manifolds of X and let $x \in \mathcal{M}_i \cap \overline{\mathcal{M}}_j$. If $(\mathcal{M}_i, \mathcal{M}_j)$ satisfies the Whitney (b)-condition at x, then it also satisfies the Whitney (a)-condition at x.*

The importance of the Whitney (b)-condition lies in the fact that, contrary to the (a)-condition, any pre-stratification that meets it, is a stratification. This claim is an outcome of the next result.

Proposition 3.3.3 ([91, Proposition 2.4])**.** *Let $\mathcal{M}_i$ and $\mathcal{M}_j$ be two embedded manifolds of X. Suppose that the pair $(\mathcal{M}_i, \mathcal{M}_j)$ satisfies the Whitney (b)-condition at some $x \in \mathcal{M}_i \cap \overline{\mathcal{M}}_j$. Then* $\dim(\mathcal{M}_i) < \dim(\mathcal{M}_j)$.

Despite the fact that the Whitney (b)-condition always implies the Whitney (a)-condition, this last one is enough for the purpose of the forthcoming expositions as we will see later on. In particular, it provides a sort of hierarchy between the Limiting and Clarke normal cone to the closure of the strata. For this reason, we most of times work with merely (W_a)-stratifications.

3.3.2 Some favorable classes of stratifiable sets

The relevance of stratifiable sets is that there are plenty of them and they fit considerably well in applications; see for instance the discussion in [76].

Among the classes of set that admit a regular stratification, probably the most intuitive case is the collection of *semilinear* sets, that is, those which are the finite union of open polyhedron of the form

$$\left\{ x \in X \;\middle|\; \begin{array}{lll} \langle \upsilon_n, x\rangle = \alpha_n, & \upsilon_n \in X & n = 1, \dots, l, \\ \langle \eta_n, x\rangle < \alpha_n, & \eta_n \in X & n = l+1, \dots, m \end{array} \right\}.$$

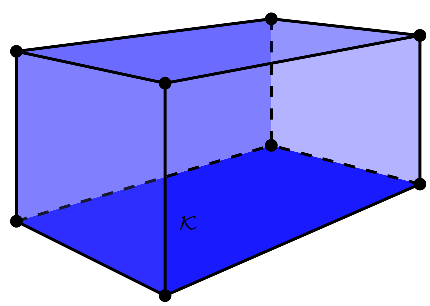


Figure 3.4: A semilinear set in $\mathbb{R}^3$.

Depending upon the issue at hand, it may be more suitable to work with curvilinear polytopes rather than polyhedron. The family of sets composed of all the finite union of open curvilinear polytopes

$$\left\{ x \in X \;\middle|\; \begin{array}{ll} p_n(x) = 0, & n = 1, \dots, l, \\ p_n(x) < 0, & n = l+1, \dots, m \end{array} \right\},$$

where each $p_n(\cdot)$ is a real polynomial, is named the *semialgebraic* sets of X.

Example 3.3.1. *Let $P : \mathbb{R}^3 \to \mathbb{M}_{3\times 3}(\mathbb{R})$ be the map defined via*

$$P(x) = \begin{pmatrix} 1 & x_1 & x_2 \\ x_1 & 1 & x_3 \\ x_2 & x_3 & 1 \end{pmatrix}, \quad \forall x \in \mathbb{R}^3.$$

The elliptope in $\mathbb{R}^3$ is the set given by $\mathcal{K} = \{x \in \mathbb{R}^3 \mid P(x)$ is positive semi-definite$\}$; see Figure 3.5. By the Sylvester's criterion, we have that $\mathcal{K}$ is semialgebraic.

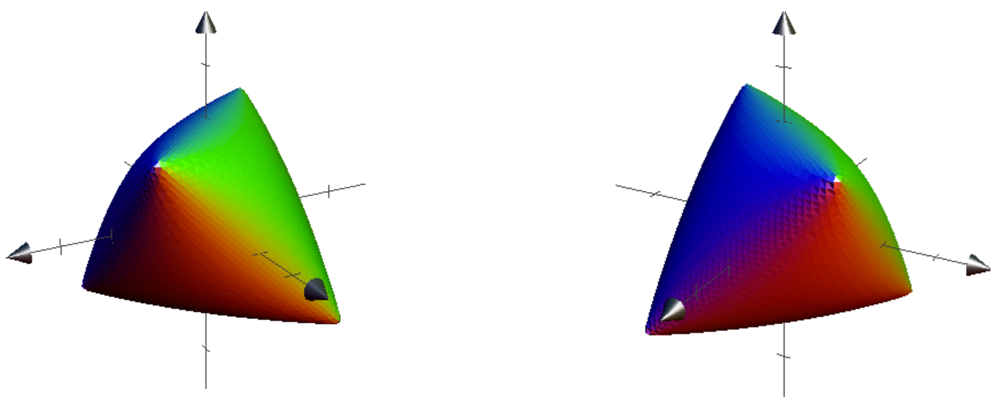


Figure 3.5: Different angles of the semialgebraic set of Example 3.3.1.

It turns out that any semialgebraic set admits a (W_b)-stratification. Besides, each stratum is itself a semialgebraic manifold.

Proposition 3.3.4. *Let $\mathcal{K} \subseteq X$ be a closet subset. If $\mathcal{K}$ is semialgebraic then it is (W_b)-stratifiable and its strata are semialgebraic as well.*

Furthermore, Proposition 3.3.4 is, as a matter of fact, a corollary of a fairly more general theorem which applies to a broader sort of sets.

A collection $\mathfrak{S} = \{\mathfrak{S}_n\}_{n\in\mathbb{N}}$ is called an *o-minimal structure* provided for every $n \in \mathbb{N}$, $\mathfrak{S}_n$ is a family of sets of $\mathbb{R}^n$ and the conditions below hold:

1. If $\mathcal{K} \subseteq \mathbb{R}^n$ is semialgebraic, then $\mathcal{K} \in \mathfrak{S}_n$.

2. The elements of $\mathfrak{S}_1$ are exactly the finite union of points and intervals.

3. $\mathcal{K}_1 \cup \mathcal{K}_2$, $\mathcal{K}_1 \cap \mathcal{K}_2$, and $\mathbb{R}^n \setminus \mathcal{K}_1$ belong to $\mathfrak{S}_n$ whenever $\mathcal{K}_1, \mathcal{K}_2 \in \mathfrak{S}_n$.

4. For every $\mathcal{K} \in \mathfrak{S}_n$, we have $\mathcal{K} \times \mathbb{R}$ and $\mathbb{R} \times \mathcal{K}$ belong to $\mathfrak{S}_{n+1}$.

5. $\pi(\mathcal{K}) \in \mathfrak{S}_n$ for any $\mathcal{K} \in \mathfrak{S}_{n+1}$, where π stands for the projection $\pi(\mathcal{S}\times\{x\}) = \mathcal{S}$ whenever $\mathcal{S} \subseteq \mathbb{R}^n$ and $x \in \mathbb{R}$.

The elements of $\mathfrak{S}_n$ are called *definable* sets on $\mathbb{R}^n$. Thereby, a definable set on X is determined via the canonical isomorphism between X and $\mathbb{R}^N$.

Proposition 3.3.5 ([130, 4.8 Whitney stratification])**.** *Let $\mathcal{K} \subseteq X$ be a closet subset. If $\mathcal{K}$ is definable then it is (W_b)-stratifiable, its strata are definable as well and can be chosen of class $\mathcal{C}^k$ for any $k \in \mathbb{N}$.*

If we denote by $\mathfrak{S}_n^{\mathrm{al}}$ the set of all the semialgebraic subsets on $\mathbb{R}^n$, then $\mathfrak{S}^{\mathrm{al}} := \{\mathfrak{S}_n^{\mathrm{al}}\}_{n\in\mathbb{N}}$ is an o-minimal structure (actually, the smaller possible). Indeed, the first four conditions hold trivially and the last one coincides with the Tarski- Seidenberg Theorem (c.f [22, Theorem 2.2.1]). Consequently, $\mathfrak{S}^{\mathrm{al}}$ is an o-minimal structure and Proposition 3.3.4 follows from Proposition 3.3.5.

On the other hand, another importance type of sets that can be seen as elements of an o-minimal structure are the so-called *finitely subanalytic* sets. To give a precise definition of this class we require the next notions.

A set $\mathcal{K} \subseteq \mathbb{R}^n$ is said to be *semianalytic* if each point of $\mathbb{R}^n$ admits a neighborhood $\mathcal{O}$ for which $\mathcal{K} \cap \mathcal{O}$ can be written as finite union of analytic varieties

$$\left\{ x \in \mathbb{R}^n \;\middle|\; \begin{array}{ll} f_p(x) = 0, & p = 1, \ldots, l, \\ f_p(x) < 0, & p = l+1, \ldots, m \end{array} \right\}$$

where $f_1(\cdot), \ldots, f_m(\cdot)$ are real analytic functions on $\mathcal{O}$. Accordingly, $\mathcal{K}$ is *subanalytic* provided for each point of $\mathbb{R}^n$ there exists a neighborhood $\mathcal{O}$, $m \in \mathbb{N}$ and a bounded semianalytic set $\widetilde{\mathcal{K}} \subseteq \mathbb{R}^n \times \mathbb{R}^m$, such that

$$\mathcal{K} \cap \mathcal{O} = \left\{ x \in \mathbb{R}^n \mid \; (x, y) \in \widetilde{\mathcal{K}} \right\}.$$

We can readily see that the collection of all the subanalytic sets of any dimension satisfies the first four conditions of an o-minimal structure, however, it does not satisfies the last one. For this is the reason we restraint the attention into a smaller class in order to ensure the o-minimality. Let $p(t) = t(1+t^2)^{-1/2}$ for any $t \in \mathbb{R}$, and consider for any $n \in \mathbb{N}$, the map $\mathbb{P}_n : \mathbb{R}^n \to \mathbb{R}^n$ defined via $\mathbb{P}_n(x_1, \ldots, x_n) = (p(x_1), \ldots, p(x_n))$. Hence, the *finitely subanalytic* sets is the following collection

$$\mathfrak{S}_n^{\mathrm{an}} := \{\mathcal{K} \subseteq \mathbb{R}^n \mid \; \mathcal{K} \text{ and } \mathbb{P}_n(\mathcal{K}) \text{ are subanalytic}\}, \quad \forall n \in \mathbb{N}.$$

Proposition 3.3.6 ([129, p.191])**.** *The family $\mathfrak{S}^{an} := \{\mathfrak{S}_n^{an}\}_{n\in\mathbb{N}}$ is an o-minimal structure, and so, any finitely subanalytic set is admits a (W_b)-stratification with subanalytic strata.*

3.3.3 Normals and tangents

Next proposition reflects the importance of the Whitney (a)-condition and its connection with nonsmooth analysis. Roughly speaking, it provides a hierarchy between the Clarke normal cones of the strata. We remark that several of the results of this section are important for the analysis we provide in Chapter 7.

Proposition 3.3.7. *If $\{\mathcal{M}_i\}_{i\in\mathcal{I}}$ is a (W_a)-stratification of $\mathcal{K} \subseteq X$, then we have $\mathcal{N}^C_{\mathcal{M}_j}(x) \subseteq \mathcal{N}_{\mathcal{M}_i}(x)$ for any $x \in \mathcal{M}_i$ whenever $i, j \in \mathcal{I}$ and $i \preceq j$.*

Proof. Note that it is enough to show that $\mathcal{N}^L_{\mathcal{M}_j}(x) \subseteq \mathcal{N}_{\mathcal{M}_i}(x)$ for any $x \in \mathcal{M}_i$; to conclude we take the convex closed hull in the inclusion.

Let $x \in \mathcal{M}_i$ arbitrary and $\eta \in \mathcal{N}^L_{\overline{\mathcal{M}}_j}(x)$. By definition, we can take two sequences $\{x_n\} \subseteq \overline{\mathcal{M}}_j$ and $\{\eta_n\} \subseteq X$ such that $\eta_n \in \mathcal{N}^P_{\overline{\mathcal{M}}_j}(x_n)$ with $x_n \to x$ and $\eta_n \to \eta$. Furthermore, since the stratification is locally finite, there is only a finite number of indices $k \in \mathcal{I}$ for which $x \in \overline{\mathcal{M}}_k$ and $i \preceq k \preceq j$. For this reason, there exists at least one of such indexes k and a subsequence of $\{x_n\}$ that is entirely contained in $\mathcal{M}_k$. We skip relabeling the subsequence.

Notice that if $\mathcal{S}_1 \subseteq \mathcal{S}_2$ then $\mathcal{N}^P_{\mathcal{S}_2}(x) \subseteq \mathcal{N}^P_{\mathcal{S}_1}(x)$ for every $x \in \mathcal{S}_1$. In view of Proposition 3.2.3 we assert that

$$\eta_n \in \mathcal{N}^P_{\overline{\mathcal{M}}_j}(x_n) \subseteq \mathcal{N}^P_{\overline{\mathcal{M}}_k}(x_n) = \mathcal{N}^P_{\mathcal{M}_k}(x_n) \subseteq \mathcal{N}^C_{\mathcal{M}_k}(x_n) = \mathcal{N}_{\mathcal{M}_k}(x_n).$$

Thanks to Proposition 3.3.1, passing into another subsequence if necessary which we again avoid to relabel, $\mathcal{T}_{\mathcal{M}_k}(x_n)$ converges to some $\mathcal{T}$. This implies that for any $v \in \mathcal{T}$ there exists $v_n \in \mathcal{T}_{\mathcal{M}_k}(x_n)$ so that $v_n \to v$. Since $\eta_n \in \mathcal{N}_{\mathcal{M}_k}(x_n)$ we have $\langle \eta_n, v_n \rangle = 0$ for any $n \in \mathbb{N}$. Thus, letting $n \to +\infty$ we establish that η is orthogonal to $\mathcal{T}$.

By virtue of the Whitney (a)-condition applied to $(\mathcal{M}_i, \mathcal{M}_k)$, one gets that $\mathcal{T}_{\mathcal{M}_i}(x) \subseteq \mathcal{T}$ and in particular, $\eta \in \mathcal{N}_{\mathcal{M}_i}(x)$. So the proof is complete. □

The above-stated claim has a very interesting consequence with respect to the regularity of the Clarke cones to closure of a stratum. This result can be seen as an extension of Proposition 3.2.4 and its relation with the lower semicontinuity of $\mathcal{T}^C_{\mathcal{M}}(\cdot)$ on $\mathcal{M}$.

Corollary 3.3.1. *If $\{\mathcal{M}_i\}_{i\in\mathcal{I}}$ is a (W_a)-stratification of $\mathcal{K} \subseteq X$, then for any $j \in \mathcal{I}$, $\mathcal{N}^C_{\overline{\mathcal{M}}_j}(\cdot)$ is compactly upper semicontinuous on $\overline{\mathcal{M}}_j$ and $\mathcal{T}^C_{\overline{\mathcal{M}}_j}(\cdot)$ is lower semicontinuous on $\overline{\mathcal{M}}_j$.*

Proof. By Proposition 2.2.3 and [41, Proposition 3.6.8], it is enough to show that $\mathcal{N}^C_{\overline{\mathcal{M}}_j}(\cdot)$ has closed graph at $x \in \overline{\mathcal{M}}_j$ arbitrary. Let $\{x_n\} \subseteq \overline{\mathcal{M}}_j$ with $x_n \to x$ and $\eta_n \to \eta \in X$ with $\eta_n \in \mathcal{N}^C_{\overline{\mathcal{M}}_j}(x_n)$. Using the same argument as in Proposition 3.3.7, we can assume that there exists $i \in \mathcal{I}$ with $i \preceq j$ such that $x_n \in \mathcal{M}_i$. Hence, by Proposition 3.3.7, $\eta_n \in \mathcal{N}_{\mathcal{M}_i}(x_n) = \mathcal{N}^L_{\overline{\mathcal{M}}_i}(x_n)$ for any $n \in \mathbb{N}$.

Since the Limiting normal cone has closed graph, we get the result.

□

In view of the previous proposition, we can provide a characterization of the normal and tangent spaces to a stratum that relies on the Clarke cones of the surrounding strata. This result is immediate if the Bouligand and Clarke tangent cones coincides on the closure of any stratum, however if this requirement fails, the conclusion is ensured by the (a)-condition.

Proposition 3.3.8. *If $\{\mathcal{M}_i\}_{i\in\mathcal{I}}$ is a (W_a)-stratification of $\mathcal{K} \subseteq X$, then for any $i \in \mathcal{I}$ such that $\overline{\mathcal{M}}_i = \bigcap \left\{ \overline{\mathcal{M}}_j \mid \ j \succeq i,\ j \neq i \right\}$ we have*

$$\sum_{j\succeq i,\ j\neq i} \mathcal{N}^C_{\overline{\mathcal{M}}_j}(x) = \mathcal{N}_{\mathcal{M}_i}(x) \text{ and } \bigcap_{j\succeq i,\ j\neq i} \mathcal{T}^C_{\overline{\mathcal{M}}_j}(x) = \mathcal{T}_{\mathcal{M}_i}(x), \quad \forall x \in \mathcal{M}_i.$$

Proof. Fix $x \in \mathcal{M}_i$, by the locally finite condition, the number of indexes $j \succeq i$ for which $x \in \overline{\mathcal{M}}_j$ is finite. Thereby, Proposition 2.3.7 implies that

$$\mathcal{N}^C_{\overline{\mathcal{M}}_i}(x) \subseteq \sum_{j\succeq i,\ j\neq i} \mathcal{N}^C_{\overline{\mathcal{M}}_j}(x) \quad \text{and} \quad \bigcap_{j\succeq i,\ j\neq i} \mathcal{T}^C_{\overline{\mathcal{M}}_j}(x) \subseteq \mathcal{T}^C_{\overline{\mathcal{M}}_i}(x), \quad \forall x \in \overline{\mathcal{M}}_i.$$

Furthermore, given that the structure of the cones relies essentially upon the local structure of the set, $\mathcal{T}^C_{\mathcal{M}_i}(x) = \mathcal{T}_{\mathcal{M}_i}(x)$ and $\mathcal{N}_{\mathcal{M}_i}(x) = \mathcal{N}^C_{\mathcal{M}_i}(x)$ for any $x \in \mathcal{M}_i$; this is thanks to Propositions 3.2.2 and 3.2.3, respectively. So, one of the inclusions, in each equality of the statement, holds independently of the Whitney condition.

On the other hand, $\bigcup \left\{ \mathcal{N}^C_{\mathcal{M}_j}(x) \mid\ j \succeq i,\ j \neq i \right\} \subseteq \mathcal{N}_{\mathcal{M}_i}(x)$ due to Proposition 3.3.7. Since the right hand side of the inclusion is a vectorial space, the sum of any finite number of elements in the left hand side are contained in $\mathcal{N}_{\mathcal{M}_i}(x)$. In particular,

$$\sum_{j \succeq i,\ j \neq i} \mathcal{N}^C_{\mathcal{M}_j}(x) \subseteq \mathcal{N}^C_{\mathcal{M}_i}(x).$$

By the argumentation done earlier, this inclusion in actually an equality. Finally, combining the polar relationship of Proposition 2.3.6 with Propositions 3.2.2 and 3.3.7, we obtain that $\mathcal{T}_{\mathcal{M}_i}(x) \subseteq \mathcal{T}^C_{\mathcal{M}_j}(x)$ for any $x \in \mathcal{M}_i$ and $j \in \mathcal{I}$ with $j \succeq i$. Inasmuch as $j \in \mathcal{I}$ is arbitrary, we conclude the proof by intersecting all these cones that leads to

$$\mathcal{T}_{\mathcal{M}_i}(x) \subseteq \bigcap \left\{ \mathcal{T}^C_{\mathcal{M}_j}(x) \mid\ j \succeq i,\ j \neq i \right\}$$

□

3.4 Relative wedgedness

In the previous section we have seen that the tangent and normal cones to embedded manifolds are, in fact vectorial subspace which satisfy quite regular conditions and, under some hypotheses, these features are inherited by the cones of the closure of the manifold. In this section we study in more detail the structure of such kind of sets (closure of the manifold), and in particular, we analyze the structure of the tangent and normal cones to them.

We begin with recalling the standard notion of wedgedness found in the literature; see for instance [41]. Given a set $\mathcal{S} \subseteq X$ we say that $\mathcal{S}$ is wedged at x if $\mathrm{int}(\mathcal{T}^C_{\mathcal{S}}(x)) \neq \emptyset$. The concept of wedged set is intrinsically related to epi-Lipschitz sets, that is, sets that are locally the epigraph of a Lipschitz continuous function; see for instance Rockafellar [112].

At present, we aim to extend this notion to the case in which $\mathrm{int}(\mathcal{T}^C_{\mathcal{S}}(x)) = \emptyset$ but $\mathrm{ri}(\mathcal{T}^C_{\mathcal{S}}(x)) \neq \emptyset$ has the same dimension everywhere and $\mathcal{S} = \overline{\mathcal{M}}$, with $\mathcal{M}$ being a manifold.

Consider a given orthogonal decomposition of the ambient space into two vectorial subspaces $X_{\mathcal{T}}$ and $X_{\mathcal{N}}$, that is, $X = X_{\mathcal{T}} \oplus X_{\mathcal{N}}$. Hence, for any embedded manifold $\mathcal{M}$ of X and any $x \in \mathcal{M}$ a *change of basis* map is an isomorphism P_x from $\mathcal{T}_{\mathcal{M}}(x) \oplus \mathcal{N}_{\mathcal{M}}(x)$ into $X_{\mathcal{T}} \oplus X_{\mathcal{N}}$ that satisfies

$$P_x(\mathcal{T}_{\mathcal{M}}(x)) = X_{\mathcal{T}} \quad \text{and} \quad P_x(\mathcal{N}_{\mathcal{M}}(x)) = X_{\mathcal{N}}.$$

Since, the map $x \mapsto \mathcal{T}_{\mathcal{M}}(x)$ is always lower semicontinuous (Proposition 3.2.4), by the Michael's Selection Theorem, we can construct a family of change of basis in such a way the map $x \mapsto P_x$ depends continuously upon x when it varies on $\mathcal{M}$. However, due to the fact that the Clarke normal cone is likely to be merely compactly upper semicontinuous, we might only have

$$P_x\left(\mathcal{T}^C_{\overline{\mathcal{M}}}(x)\right) \subseteq X_{\mathcal{T}} \quad \text{and} \quad P_x\left(\mathcal{N}^C_{\overline{\mathcal{M}}}(x)\right) \supseteq X_{\mathcal{N}}, \quad \forall x \in \overline{\mathcal{M}} \setminus \mathcal{M},$$

where P_x is any change of basis map from X into $X_{\mathcal{T}} \oplus X_{\mathcal{N}}$. Notably, the dimension of the Clarke normal at some points of $\overline{\mathcal{M}} \setminus \mathcal{M}$ may be strictly lower the codimension of $\mathcal{M}$. We sum up this idea in the upcoming statement.

Proposition 3.4.1. *Let $\mathcal{M}$ be an embedded manifold of X of codimension d, then for any $x \in \overline{\mathcal{M}}$, $\mathcal{N}^{C}_{\overline{\mathcal{M}}}(x)$ contains a (possibly non unique) vectorial space of dimension d.*

Proof. Take $x \in \overline{\mathcal{M}}$. If $x \in \mathcal{M}$ or $d = 0$, the conclusion is straightforward, the vectorial subspace coincides with the normal space and with $\{0\}$, respectively. If on the contrary, $x \in \overline{\mathcal{M}} \setminus \mathcal{M}$, then we can take $\{x_n\} \subseteq \mathcal{M}$ with $x_n \to x \in \overline{\mathcal{M}}$. In view of Proposition 3.2.3, for each $n \in \mathbb{N}$, $\mathcal{N}^{L}_{\overline{\mathcal{M}}}(x_n)$ is a subspace of dimension d because it coincides with $\mathcal{N}_{\mathcal{M}}(x_n)$. Consequently, By Proposition 3.3.1 we might assume that $\mathcal{N}^{L}_{\overline{\mathcal{M}}}(x_n) \to \mathcal{N}_\infty$ with $\dim(\mathcal{N}_\infty) = d$. Moreover, since the Limiting normal cone has closed graph, $\mathcal{N}_\infty \subseteq \mathcal{N}^{L}_{\overline{\mathcal{M}}}(x)$. By taking closed convex hull, the proof is complete. $\square$

In particular, Proposition 3.4.1 implies that for any $x \in \overline{\mathcal{M}} \setminus \mathcal{M}$ there exists $\mathcal{N}_x \subseteq X_{\mathcal{T}}$, a nontrivial convex closed cone, for which

$$\mathcal{N}^{C}_{\overline{\mathcal{M}}}(x) \cong \mathcal{N}_x \oplus X_{\mathcal{N}}.$$

Note that this decomposition is not unique and so, the normal cone may well contain a vectorial subspace of dimension strictly larger than d. Anyhow, it does not happen if $\mathcal{N}_x$ is *pointed*, that is, it does not contain a vectorial subspace of dimension greater than 1. The following definition is a generalization of the concept of wedgedness found in the literature for arbitrary closed sets (e.g. [41, Section 3.6]).

Definition 3.4.1 (Relatively wedged)**.** *Let $\mathcal{M}$ be an embedded manifold of X whose codimension is d. We say that $\overline{\mathcal{M}}$ is relatively wedged at $x \in \overline{\mathcal{M}}$ provided there is a vectorial subspace $X_{\mathcal{N}}(x)$ of dimension d and a nontrivial pointed cone $\mathcal{N}_x$ on X so that $\mathcal{N}^{C}_{\overline{\mathcal{M}}}(x) = \mathcal{N}_x \oplus X_{\mathcal{N}}(x)$*

Remark 3.4.1. *In Definition 3.4.1, $\mathcal{N}_x$ is a closed convex cone due to $\mathcal{N}^{C}_{\overline{\mathcal{M}}}(x)$ is so.*

The aforementioned definition has many different equally valid interpretations. We present some of them in the next proposition.

Proposition 3.4.2. *Let $\mathcal{M}$ be an embedded manifold of X whose codimension is d and let $x \in \overline{\mathcal{M}}$. The following are equivalent:*

a) *$\overline{\mathcal{M}}$ is relatively wedged at x.*

b) *For any d-dimensional vectorial subspace $X_{\mathcal{N}}$, there exist a pointed convex cone $\mathcal{N}_x$ and $P_x \in \mathbb{O}(X)$, so that $P_x\left(\mathcal{N}^{C}_{\overline{\mathcal{M}}}(x)\right) = \mathcal{N}_x \oplus X_{\mathcal{N}}$*

c) *$\mathcal{T}^{C}_{\overline{\mathcal{M}}}(x)$ contains $N - d$ linearly independent vectors.*

Proof. First we prove the equivalence between *a*) and *b*). Let $X_{\mathcal{N}}(x)$ be the vectorial subspace given by the relative wedgedness and $X_{\mathcal{N}}$ another arbitrary, but of dimension d as well. Consider the $\mathbf{x}$ and $\tilde{\mathbf{x}}$ two orthonormal basis for X so that $X_{\mathcal{N}}(x) = \operatorname{span}\{\mathbf{x}_1, \ldots, \mathbf{x}_d\}$ and

$X_{\mathcal{N}} = \operatorname{span}\{\tilde{\mathbf{x}}_1, \ldots, \tilde{\mathbf{x}}_d\}$. Therefore, by taking $P_x \in \mathbb{O}(X)$ that verifies $P_x(\mathbf{x}_i) = \tilde{\mathbf{x}}_i$ for any $i = 1, \ldots, N$, we get that $P_x(X_{\mathcal{N}}(x)) = X_{\mathcal{N}}$, thus $a)$ and $b)$ are equivalent.

On the other hand, if $a)$ holds the affine hull of $\mathcal{T}^C_{\overline{\mathcal{M}}}(x)$ has dimension $N - d$ and so $b)$ holds, this is due to the fact that $\mathcal{T}^C_{\overline{\mathcal{M}}}(x)$ is the polar cone to $N_x \oplus X_{\mathcal{N}}(x)$ and $\mathcal{N}_x$ is pointed. Conversely, let $X_{\mathcal{N}}(x)$ stand for the vectorial subspace of dimension d given by Proposition 3.4.1, then $\mathcal{N}^C_{\overline{\mathcal{M}}}(x) = \mathcal{N}_x \oplus X_{\mathcal{N}}(x)$ for some closed convex cone of X. If the affine hull of $\mathcal{T}^C_{\overline{\mathcal{M}}}(x)$ has dimension $N - d$, by polarity once again, $\mathcal{N}^C_{\overline{\mathcal{M}}}(x)$ can not contain a vectorial subspace of dimension larger than d. So, $\mathcal{N}_x$ is pointed and $a)$ follows. □

In Figure 3.6 we illustrate a 2-dimensional manifold embedded in $\mathbb{R}^3$. In the situations exhibited, $X_{\mathcal{N}}$ is isomorphic to $\mathbb{R}$ (red axis on the figure) and $\mathcal{N}_x$ is isomorphic to a 1-dimensional cone (blue ray on the figure) for $\bar{x}_1$ and to a 2-dimensional cone (blue cone on the figure) for $\bar{x}_2$, respectively.

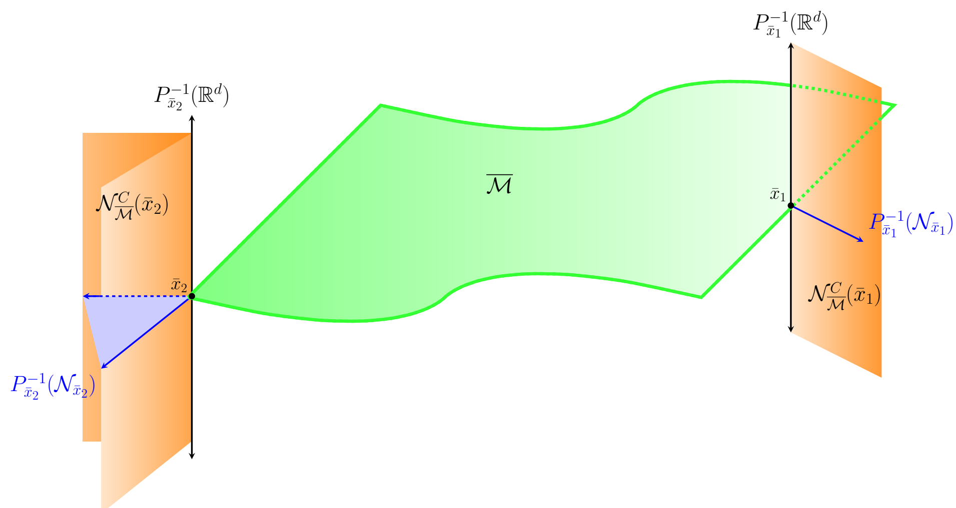


Figure 3.6: Example of relatively wedged set.

Remark 3.4.2. *A similar notion of relative wedgedness was studied in [17] for sets that are the closure of a $\mathcal{C}^2$-embedded manifold of $\mathbb{R}^N$ and that satisfy, in addition, a regularity property which generalizes the concept of epi-Lipschitz. The definition exhibited here is intrinsically related to the one introduced in [68], with the difference that here we do not assume beforehand the continuity of the change of basis.*

Now, given a vectorial subspace $X_{\mathcal{N}}$ of X of dimension d, from Definition 3.4.1 and Proposition 3.4.2, we infer that any $\eta \in \mathcal{N}^C_{\overline{\mathcal{M}}}(x)$ can be written in a unique way as the linearly independent sum of $\eta_1 \in \mathcal{N}_x$ and $\eta_2 \in X_{\mathcal{N}}$. Consequently, if $\overline{\mathcal{M}}$ is relatively wedged at x, we indicate by $\pi_x(\eta)$ the projection of η over $\mathcal{N}_x$.

The utility of the projection is reflected in the following statement which describes in particular the relative interior of the Clarke tangent cone.

Proposition 3.4.3. *Let $\mathcal{M}$ be an embedded manifold of X and suppose that $\overline{\mathcal{M}}$ is relatively wedged at $x \in \overline{\mathcal{M}}$. Then, ri$\left(\mathcal{T}^C_{\overline{\mathcal{M}}}(x)\right)$ is an embedded manifold of X with the same codimension as $\mathcal{M}$. Additionally, $v \in$ ri$\left(\mathcal{T}^C_{\overline{\mathcal{M}}}(x)\right)$ if and only if $\exists \sigma > 0$ such that*

$$\langle v, \eta\rangle \leq -\sigma|\pi_x(\eta)| \quad \forall \eta \in \mathcal{N}^C_{\overline{\mathcal{M}}}(x). \tag{3.2}$$

Proof. Let $X_{\mathcal{T}}$ be the affine hull of $\mathcal{T}^C_{\overline{\mathcal{M}}}(x)$. By Proposition 3.4.2 and due to $0 \in \mathcal{T}^C_{\overline{\mathcal{M}}}(x)$, $X_{\mathcal{T}}$ is a vectorial subspace of X of dimension $N-d$. Since, ri$\left(\mathcal{T}^C_{\overline{\mathcal{M}}}(x)\right)$ is an open subset of $X_{\mathcal{T}}$ we get that it is an embedded manifold of $X_{\mathcal{T}}$ and thereby, of X as well.

Besides, $X_{\mathcal{T}}$ is orthogonal to $X_{\mathcal{N}}(x)$, which implies that

$$\langle v, \eta\rangle = \langle v, \pi_x(\eta)\rangle, \quad \forall v \in \mathcal{T}^C_{\overline{\mathcal{M}}}(x), \eta \in \mathcal{N}^C_{\overline{\mathcal{M}}}(x).$$

Moreover, if $v \in$ ri$\left(\mathcal{T}^C_{\overline{\mathcal{M}}}(x)\right)$ if and only if $\exists \sigma > 0$ so that $v \in \mathbb{B}_X(x, \sigma) \cap X_{\mathcal{T}}$ and by Proposition 2.3.6, this is also equivalent to

$$\langle v + \sigma e, \eta\rangle = \langle v + \sigma e, \pi_x(\eta)\rangle \leq 0, \quad \forall \eta \in \mathcal{N}^C_{\overline{\mathcal{M}}}(x),\ e \in \mathbb{B}_X \cap X_{\mathcal{T}}.$$

Finally, after a few algebraic steps we get the equivalence of the statement. □

On the other hand, Definition 3.4.1 applies merely to a single point, so a priori we can not deduce if it does hold on a neighborhood of the point in question. Nevertheless, if the Clarke tangent cone of $\overline{\mathcal{M}}$ is lower semicontinuous, we can prove that the notion of relative wedgedness is, as a matter of fact, a local property.

Proposition 3.4.4. *Let $\mathcal{M}$ be an embedded manifold of X with $\mathcal{T}^C_{\overline{\mathcal{M}}}(\cdot)$ lower semicontinuous on $\overline{\mathcal{M}}$. Suppose that $\overline{\mathcal{M}}$ is relatively wedged at $x \in \overline{\mathcal{M}}$. Then, there exists $\delta > 0$ such that $\overline{\mathcal{M}}$ is relatively wedged at any $\tilde{x} \in \overline{\mathcal{M}} \cap \mathbb{B}_X(x, \delta)$. Furthermore, for any $\{x_n\} \subseteq \overline{\mathcal{M}}$ with $x_n \to x$ and, any $\{\eta_n\} \subseteq X$ with $\eta_n \in \mathcal{N}^C_{\overline{\mathcal{M}}}(x_n)$ and $\eta_n \to \eta \in X$, we have $\pi_{x_n}(\eta_n) \to \pi_x(\eta)$.*

Proof. By Proposition 3.4.2 we only need to prove that $\mathcal{T}^C_{\overline{\mathcal{M}}}(\tilde{x})$ contains $p = N - d$ linearly independent vectors for any $\tilde{x}$ sufficiently near x.

Notice that the affine hull of $\mathcal{T}^C_{\overline{\mathcal{M}}}(x)$ has dimension $p = N - d$, whereupon we can take $v_1, \ldots, v_p \in \mathcal{T}^C_{\overline{\mathcal{M}}}(x)$ linearly independent. By the Michael's Selection Theorem (Proposition 2.2.4), for any $i = 1, \ldots, p$ there is a continuous selection γ_i of $\mathcal{T}^C_{\overline{\mathcal{M}}}(\cdot)$ so that $\gamma_i(x) = v_i$.

We claim that there exists $\delta > 0$ so that $\{\gamma_1(\tilde{x}), \ldots, \gamma_p(\tilde{x})\}$ is linearly independent for any $\tilde{x} \in \overline{\mathcal{M}} \cap \mathbb{B}_X(x, \delta)$. Indeed, if this is not true there exists a sequence $\{x_n\} \subseteq \overline{\mathcal{M}}$ converging to x and another sequence $\{\mu^n\} \subseteq \mathbb{R}^p \setminus \{0\}$ satisfying

$$\sum_{i=1}^{p} \mu_i^n \gamma_i(x_n) = 0, \quad \forall n \in \mathbb{N}.$$

Dividing the preceding equation by $|\mu^n|$ if necessary, we can assume that $|\mu^n| = 1$ and so, it has a converging subsequence, which we eschew to relabel, to some $\mu \in \mathbb{R}^p \setminus \{0\}$. Letting $n \to +\infty$ and using the continuity of γ_i we get that

$$\sum_{i=1}^{p} \mu_i \gamma_i(x) = 0.$$

Since $\gamma_i(x) = v_i$ and those vectors are linearly independent, we get a contradiction. On account of this, we have proven the first part of the statement.

On the other hand, let $\{x_n\}$ and $\{\eta_n\}$ as in the statement. By [41, Proposition 3.6.8] we have that $\mathcal{N}^C_{\overline{\mathcal{M}}}(\cdot)$ is compactly upper semicontinuous at x, in particular, $\eta \in \mathcal{N}^C_{\overline{\mathcal{M}}}(x)$. Moreover, for $n \in \mathbb{N}$ large enough we have that $\operatorname{span}\{\gamma_1(x_n), \dots, \gamma_p(x_n)\}$ is orthogonal to $X_{\mathcal{N}}(x_n)$, which implies that

$$\pi_{x_n}(\eta_n) = \sum_{i=1}^{p} \mu_i^n \gamma_i(x_n), \quad \forall n \in \mathbb{N} \text{ large enough.}$$

By Lemma 3.2.1, $\{\mu^n\}$ is bounded and the collection of accumulation point of $\{\pi_{x_n}(\eta_n)\}$ is nonempty and it is contained in $\operatorname{span}\{\gamma_1(x), \dots, \gamma_p(x)\}$. Let ν be one of such cluster points. By means of the compact upper semicontinuity of Clarke normal, $\nu \in \mathcal{N}^C_{\overline{\mathcal{M}}}(x)$. Furthermore, $\langle \eta - \nu, \gamma_i(x)\rangle = 0$ for any $i = 1, \dots, p$. Hence, $\eta - \nu \in X_{\mathcal{N}}(x)$. Since the projection is unique normal on $\mathcal{N}^C_{\overline{\mathcal{M}}}(x)$ that satisfies $\eta = \pi_x(\eta) + \tilde{\eta}$ with $\tilde{\eta} \in X_{\mathcal{N}}(x)$, we get that $\pi_x(\eta) = \nu$ and therefore $\pi_x(\eta)$ is the sole accumulation point of $\{\pi_{x_n}(\eta_n)\}$. $\square$

Under the circumstances described in theforegoing statement many interesting properties can be established. These features will show their applicability later on Chapter 7.

Proposition 3.4.5. *Let $\mathcal{M}$ be an embedded manifold of X with $\mathcal{T}^C_{\overline{\mathcal{M}}}(\cdot)$ lower semicontinuous on $\overline{\mathcal{M}}$. Assume that $\overline{\mathcal{M}}$ is relatively wedged at $x \in \overline{\mathcal{M}}$. Let $g : \overline{\mathcal{M}} \to X$ be a continuous vector field with $g(\tilde{x}) \in \mathcal{T}_{\mathcal{M}}(\tilde{x})$ for every $\tilde{x} \in \mathcal{M}$. Suppose that $g(x) \in ri\big(\mathcal{T}^C_{\overline{\mathcal{M}}}(x)\big)$, then there exists $r > 0$ such that*

$$g(\tilde{x}) \in \mathcal{T}^C_{\overline{\mathcal{M}}}(\tilde{x}), \quad \forall \tilde{x} \in \mathbb{B}(x, r) \cap \overline{\mathcal{M}}.$$

Proof. Reasoning by contradiction, there are two sequences $\{x_n\} \subseteq \overline{\mathcal{M}}$ and $\{\tilde{\eta}_n\} \in X$ with $x_n \to x$ and $\tilde{\eta}_n \in \mathcal{N}^C_{\overline{\mathcal{M}}}(x_n)$ so that $\langle g(x_n), \tilde{\eta}_n\rangle > 0$.

Since $g(\tilde{x}) \in \mathcal{T}_{\mathcal{M}}(\tilde{x})$ for every $\tilde{x} \in \mathcal{M}$, we have $\langle g(\tilde{x}), \pi_{\tilde{x}}(\eta) - \eta\rangle = 0$ for any $\tilde{x} \in \overline{\mathcal{M}}$ and $\eta \in \mathcal{N}^C_{\overline{\mathcal{M}}}(\tilde{x})$. Hence, $\langle g(x_n), \pi_{x_n}(\tilde{\eta}_n)\rangle > 0$.

Let $\eta_n = \pi_{x_n}(\tilde{\eta}_n)/|\pi_{x_n}(\tilde{\eta}_n)|$, notice that $|\pi_{x_n}(\eta_n)| = 1$ and $\eta_n \in \mathcal{N}^C_{\overline{\mathcal{M}}}(x_n)$ for any $n \in \mathbb{N}$. Let η be a cluster point of $\{\eta_n\}$, then by Proposition 3.4.4, $\eta \in \mathcal{N}_x$ and $|\pi_x(\eta)| = 1$. Furthermore, given that g is continuous, passing into the limit of the subsequence, we find out that $\langle g(x), \eta\rangle \geq 0$. However, this contradicts Proposition 3.4.3, so the proof is complete. $\square$

3.5 Discussion and perspectives

We close this chapter dedicating some words to the results we have presented throughout the current exposition. We focus on the relation between the whitney (a)-condition and the relatively wedgedness.

Notice that from the analysis developed earlier, we can conclude that $x \mapsto \mathcal{T}^C_{\mathcal{M}}(x)$ is lower semicontinuous on $\mathcal{M}$; this is due to the fact that the tangent space mapping $x \mapsto \mathcal{T}_{\mathcal{M}}(x)$ is always a lower semicontinuous multifunction and that it matches with the Clarke tangent cone on $\mathcal{M}$. One of the most essential results we have got in the chapter, and apparently not noted before, is that, whenever a stratification verifies the Whitney (a)-condition, the map $x \mapsto \mathcal{T}^C_{\overline{\mathcal{M}}}(x)$ is lower semicontinuous up to $\overline{\mathcal{M}}$ (not only on $\mathcal{M}$).

Besides its own interest as extension-type theorem, this result has some consequences that help to determine the structure of $\overline{\mathcal{M}}$ at a frontier point. First of all, since $\mathcal{T}^C_{\overline{\mathcal{M}}}(\cdot)$ is always convex-valued, the Michael's Selection theorem (Proposition 2.2.4) allows us to construct a *continuous local frame* at any point of $\overline{\mathcal{M}}\setminus\mathcal{M}$ and so, in some sense, to connect the structure of the manifold with the one of its frontier. In particular, we obtain a clear decomposition of the normal and tangent cones into the vectors generated by the tangent and normal spaces, and those purely determined by the frontier of $\mathcal{M}$.

Furthermore, the construction of the continuous local frame is also useful when studying the notion of relative wedgedness, because, as shown in Proposition 3.4.4, it allows to pass from the pointwise condition:

$\mathcal{T}^C_{\overline{\mathcal{M}}}(x)$ contains exactly $n = \dim(\mathcal{M})$ linearly independent vectors

to the local one:

$\mathcal{T}^C_{\overline{\mathcal{M}}}(\tilde{x})$ contains exactly n linearly independent vectors for any $\tilde{x}\in\overline{\mathcal{M}}$ near x.

In [68] this fact was not noted, for this reason in the very definition of the property (see Definition 3.5.1 below) the existence of a continuous local frame is required.

Definition 3.5.1 ([68, Definition 4.4])**.** *Let $\mathcal{M}$ be an embedded manifold of $\mathbb{R}^N$ of codimension d. We say that $\overline{\mathcal{M}}$ is relatively wedged around $x\in\overline{\mathcal{M}}$ provided there exists a neighborhood $\mathcal{O}$ of x and a continuous mapping $P:\overline{\mathcal{M}}\cap\mathcal{O}\to\mathbb{O}(\mathbb{R}^N)$, $\tilde{x}\mapsto P_{\tilde{x}}$ such that:*

$$P_{\tilde{x}}\left(\mathcal{N}^C_{\overline{\mathcal{M}}}(\tilde{x})\right) = \mathcal{N}_{\tilde{x}}\times\mathbb{R}^d \quad \textit{with} \quad \mathcal{N}_{\tilde{x}} \textit{ pointed in } \mathbb{R}^{N-d}, \quad \forall \tilde{x}\in\overline{\mathcal{M}}\cap\mathcal{O}.$$

Besides, as aforesaid, in [17] Barnard-Wolenski have also considered a notion of relative wedgedness but only for the manifold whose closure is *proximally smooth*, that is,

$$\exists\delta>0,\ \text{so that } x\mapsto \operatorname{dist}_{\overline{\mathcal{M}}}(x) \text{ is of class } \mathcal{C}^2 \text{ on } \overline{\mathcal{M}}+\mathbb{B}(0,\delta).$$

According to the authors of [17], if $\overline{\mathcal{M}}$ is relatively wedged, then there exists a change of coordinates *around any* $x\in\overline{\mathcal{M}}\setminus\mathcal{M}$ such that $\overline{\mathcal{M}}$ is epi-Lipschitz at x. This fact yields in particular to assert that the property of relative wedgedness is also local as in the sense described above; see [17, Appendix A] for further details.

In our analysis $\mathcal{M}$ is merely a $\mathcal{C}^1$-embedded manifold. Therefore, the definition of relative wedgedness adopted here is less restrictive than the recently quoted works. Moreover, we have also found a criterion, which is immediately satisfied if $\mathcal{M}$ is a stratum of a (W_a)-stratification (Corollary 3.3.1), for ensuring that the property is verified in a local sense as well.

We also mention that an ongoing work aims to prove the conjecture described below. The interest in this claim is that it would extend the already known result that says that $\operatorname{int}(\mathcal{T}^C_{\mathcal{S}}(x))\neq\emptyset$ is equivalent to $\mathcal{S}$ is epi-Lipschitz at x; see for instance Rockafellar [112].

> **Conjecture:** If $\mathcal{M}$ is a $\mathcal{C}^1$-embedded manifold of $\mathbb{R}^N$ whose codimension is $d < N-1$ and so that $x\mapsto\mathcal{T}^C_{\overline{\mathcal{M}}}(x)$ is lower semicontinuous at $x\in\overline{\mathcal{M}}\setminus\mathcal{M}$. Then we can find a neighborhood $\mathcal{O}$ of x, a continuous map $P:\overline{\mathcal{M}}\cap\mathcal{O}\to\mathbb{O}(\mathbb{R}^N)$ and a Lipschitz continuous function $g:\mathbb{R}^{N-d-1}\to\mathbb{R}$ so that
>
> $$P\left(\overline{\mathcal{M}}\cap\mathcal{O}\right) = \operatorname{epi}(g)\times\mathbb{R}^d.$$

PART II

HAMILTON-JACOBI-BELLMAN APPROACH FOR STATE-CONSTRAINED OPTIMAL CONTROL PROBLEMS

Abstract. In this part we address the question of characterizing the Value Function of an optimal control process with state-constraints in terms of a Hamilton-Jacobi-Bellman equation. We provide different techniques to answer this question for the cases in which the set of constraints is endowed with a stratifiable structure and when it is a convex set.

Resumé. Dans cette partie nous nous intéressons au question de caractériser la Fonction Valeur d'une problème de commande optimal sous contraintes d'état comme l'unique solution généralisé d'une equation du type Hamilton-Jacobi-Bellman. Nous montrons diverses techniques pour étudier la Fonction Valeur dans les situations où l'ensemble des contraintes est muni d'une structure stratifiée et quand c'est un ensemble convexe.

CHAPTER 4

Tame State-Constraints

Abstract. In this chapter we present an approach to study the Value Function of an optimal control problem with stratifiable state-constraints. Particularly, we provide a characterization of the Value Functions in terms of an induced stratification of the set of constraints. In this work the classical pointing condition are not considered but other type of compatibility assumptions.

4.1 Introduction

It is well-known that unless some compatibility assumption between state-constraints and dynamics is assumed, there are serious obstacles to identify the Value Function as the only solution, in a generalized sense, of a Hamilton-Jacobi-Bellman (HJB) equation.

The standard approach found in the literature (cf. [120, 94, 95, 54, 123, 42, 52] among many others) deals with the notion of constrained viscosity solution of the HJB equation, that is, being a super solution on the whole state-constraints $\mathcal{K}$ and a subsolution on its interior. Furthermore, in order to state the uniqueness of this type of solutions, it is required to impose a pointing qualification hypothesis. The standard methodology is based on a technique called the neighboring feasible trajectories (NFT), which in particular force to the state-constraints to verify the condition $\mathcal{K} = \overline{\text{int}(\mathcal{K})}$.

In this chapter we present a new approach to study the Value Function of an optimal control problem based on [71]. We exhibit a characterization of the Value Functions in terms of an induced stratification of the state-constraints set. The main contribution and novelty is that it provides a set of inequalities that completes the constrained Hamilton-Jacobi-Bellman equation. We present first the case for the infinite horizon problem and later we adapt the arguments for a problem with control-free cost. Let us stress that the technique we handle in this chapter, in particular the stratified structure behind the state-constraints set, can be also used to study optimal control problems on network as we show in Section 4.2.4 and later in Chapter 9 and Chapter 9.4.

4.1.1 Stratifiable state-constraints

The main feature of the theory we want to present is that the state-constraints set is not an arbitrary closed set, but it admits a sufficiently regular partition into smooth manifolds or *strata*. More precisely,

(H_0^4) $\qquad\qquad$ $\mathcal{K}$ is a closed and *stratifiable* subset of $\mathbb{R}^N$.

We recall that a set is called *stratifiable* if there exists a locally finite collection $\{\mathcal{M}_i\}_{i\in\mathcal{I}}$ of embedded manifolds of $\mathbb{R}^N$ such that:

- $\mathcal{K} = \bigcup_{i\in\mathcal{I}} \mathcal{M}_i$ and $\mathcal{M}_i \cap \mathcal{M}_j = \emptyset$ when $i \neq j$.
- If $\mathcal{M}_i \cap \overline{\mathcal{M}_j} \neq \emptyset$, necessarily $\mathcal{M}_i \subseteq \overline{\mathcal{M}}_j$ and $\dim(\mathcal{M}_i) < \dim(\mathcal{M}_j)$.

The class of stratifiable sets is quite broad, it includes sub-analytic and semi-algebraic sets. Definable sets of an o-minimal structure are stratifiable as well. In these cases, the stratifications are even more regular and satisfy the so-called *Whitney properties*. We refer for further details to the discussion in Chapter 3, specially to Section 3.3.2.

Remark 4.1.1. *As the reader may guess, if a set is stratifiable, the stratification is not uniquely determined and, as a matter of fact, there are many others for which* (H_0^4) *may also hold. However, with the help of the Zorn's Lemma, we might prove the existence of a minimal stratification with respect to the number of strata. Whenever the number of strata is finite, this stratification has to be unique, after possible permutations among its indices.*

Examples of stratifiable sets

One of the first and simpler is when $\mathcal{K}$ is a *closed manifold* (compact manifold without boundary); the torus embedded in $\mathbb{R}^3$, illustrated in Figure 4.1, is a good model. In this case, the minimal stratification consists of just one stratum, $\mathcal{K}$ itself.

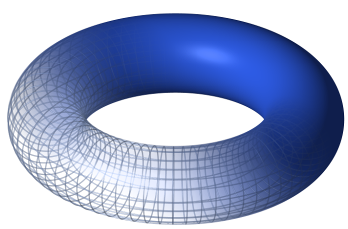


Figure 4.1: A smooth manifold without boundary (Torus embedded in $\mathbb{R}^3$).

Consider that $\text{int}(\mathcal{K}) \neq \emptyset$ and $\partial\mathcal{K}$ is smooth, as in [120], then (H_0^4) holds with simply two strata, namely, $\mathcal{M}_0 = \text{int}(\mathcal{K})$ and $\mathcal{M}_1 = \partial\mathcal{K}$; a canonical example is $\mathcal{K} = \overline{\mathbb{B}}$ as in Figure 4.2.

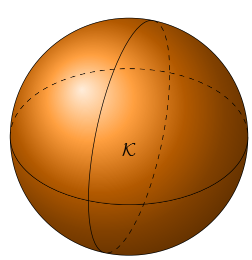
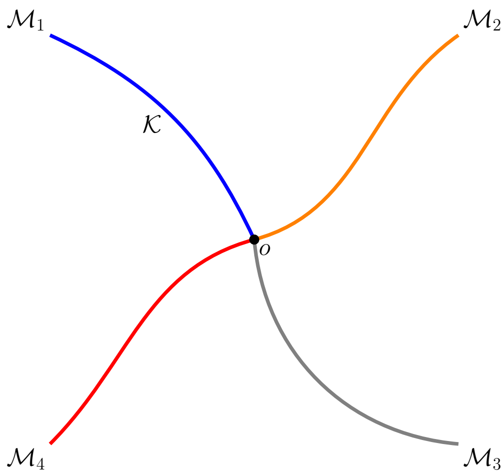


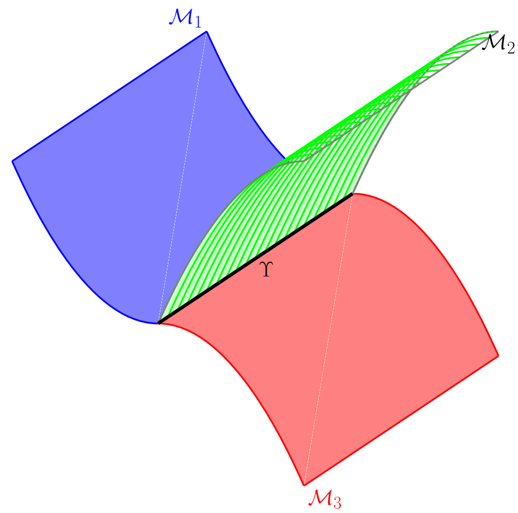
Figure 4.2: A smooth manifold with boundary (closed ball in $\mathbb{R}^3$).

Other example of interest in the nowadays literature is a network configuration. Indeed, the minimal stratification is formed by edges and junctions. Figure 4.3a illustrates an example of a network with four edges, $\mathcal{M}_1, \dots, \mathcal{M}_4$ and a single junction $\mathcal{M}_0 := \{o\}$.

More general networks can also be considered as in Figure 4.3b where the set $\mathcal{K}$ is a network embedded in the space $\mathbb{R}^3$. In the illustration we have exhibited below, the minimal stratification comprises three branches that are smooth surfaces $\mathcal{M}_1$, $\mathcal{M}_2$ and $\mathcal{M}_3$, and a junction $\mathcal{M}_0$ that corresponds to the curve Υ



(a) A network in $\mathbb{R}^2$.



(b) A generalized network in $\mathbb{R}^3$.

Figure 4.3: Some examples of networks.

4.2 Infinite horizon problems.

The theory we develop here aims to characterize the Value Function in terms of a bilateral HJB equation. We focus first on the infinite horizon problem with dynamical constraint

$$y = f(y, u), \quad \text{a.e. } t \geq 0, \quad y(0) = x.$$

The class of control problems we are considering do not necessarily satisfy any qualification hypothesis such as the pointing conditions. Nevertheless, we do assume a compatibility assumption between dynamics and state-constraints, however, of a different nature.

We recall that in this case the Value Function is given by

$$\vartheta(x) := \inf \left\{ \int_0^\infty e^{-\lambda t} \ell(y_x^u(t), u(t))dt \;\middle|\; u \in \mathbb{U}(x) \right\}, \quad \forall x \in \mathcal{K},$$

where, to simplify the notation, $\mathbb{U}$ stands for the set of admissible controls. The HJB equation in this situation takes the form

$$\lambda\vartheta(x) + H(x, \nabla\vartheta(x)) = 0 \quad x \in \mathcal{K}.$$

Throughout this section we assume that the control space $\mathcal{U}$ is compact subset of $\mathbb{R}^m$ in addition to standard hypotheses on the dynamics f and the running cost ℓ. Namely, the dynamics $f : \mathbb{R}^N \times \mathcal{U} \to \mathbb{R}^N$ are assumed to satisfy:

$$(H_f^4) \qquad \begin{cases} (i) & f(\cdot,\cdot) \text{ is continuous on } \mathbb{R}^N \times \mathcal{U}. \\ (ii) & f(\cdot,\cdot) \text{ is locally Lipschitz continuous on } \mathcal{K} \times \mathcal{U}. \\ (iii) & \exists c_f > 0 \text{ such that } \forall x \in \mathcal{K} : \\ & \quad \max\{|f(x,u)| : \ u \in \mathcal{U}\} \leq c_f(1+|x|). \end{cases}$$

And the running cost $\ell : \mathbb{R}^N \times \mathcal{U} \to \mathbb{R}$ is supposed to verify:

$$(H_\ell^4) \qquad \begin{cases} (i) & \ell(\cdot,\cdot) \text{ is continuous on } \mathbb{R}^N \times \mathcal{U}. \\ (ii) & \ell(\cdot,\cdot) \text{ is locally Lipschitz continuous on } \mathcal{K} \times \mathcal{U}. \\ (iii) & \exists c_\ell > 0, \ \lambda_\ell \geq 1 \text{ such that } \forall (x,u) \in \mathcal{K} \times \mathcal{U} : \\ & \quad 0 \leq \ell(x,u) \leq c_\ell(1+|x|^{\lambda_\ell}). \end{cases}$$

Let $x \in \mathcal{K}$ and $u \in \mathbb{U}(x)$. By (H_f^4), the control system (1.1) has a solution defined on $[0,+\infty)$ that is uniquely determined by x and u which is denoted by y_x^u. Furthermore, by the Gronwall's Lemma (Proposition 2.4.1) and (H_f^4), each solution to (1.1) satisfies:

$$1 + |y_x^u(t)| \leq (1+|x|)e^{c_f t} \qquad \forall t \geq 0; \tag{4.1}$$

$$|y_x^u(t) - x| \leq (1+|x|)(e^{c_f t} - 1) \qquad \forall t \geq 0; \tag{4.2}$$

$$|\dot{y}_x^u(t)| \leq c_f(1+|x|)e^{c_f t} \qquad \text{for a.e. } t > 0; \tag{4.3}$$

Moreover, by (H_ℓ^4) and since $\lambda_\ell \geq 1$, the cost along trajectories satisfies the following bound

$$\ell(y_x^u(t), u(t)) \leq c_\ell(1+|x|)^{\lambda_\ell} e^{\lambda_\ell c_f t}, \quad \text{for a.e. } t > 0. \tag{4.4}$$

Now, when dealing with a distributed cost, it is usual to introduce an augmented dynamics. For this end, we define

$$\beta(x,u) := c_\ell(1+|x|^{\lambda_\ell}) - \ell(x,u) \qquad \forall (x,u) \in \mathbb{R}^N \times \mathcal{U}.$$

We consider the augmented dynamics $G : \mathbb{R} \times \mathbb{R}^N \rightrightarrows \mathbb{R}^N \times \mathbb{R}$ defined by

$$G(\tau,x) = \left\{ \begin{pmatrix} f(x,u) \\ e^{-\lambda\tau}(\ell(x,u)+r) \end{pmatrix} \ \middle| \ \begin{matrix} u \in \mathcal{U}, \\ 0 \leq r \leq \beta(x,u) \end{matrix} \right\}, \quad \forall (\tau,x) \in \mathbb{R} \times \mathbb{R}^N.$$

It is not difficult to see that by (H_ℓ^4) this set-valued map has compact and nonempty images on a neighborhood of $[0,+\infty) \times \mathcal{K}$. Moreover, in order to state the lower semicontinuity of the Value Function we also suppose that

$$(H_1^4) \qquad G(\cdot) \text{ has convex images on a neighborhood of } [0,+\infty) \times \mathcal{K}.$$

Remark 4.2.1. *Suppose that $\mathcal{U}$ is a convex set of $\mathbb{R}^m$, the dynamical system is control-affine and the running cost is a convex with respect to the control ($u \mapsto \ell(x,u)$ is a convex function). Hence, under this extra structural assumptions, we check that (H_1^4) is satisfied.*

4.2.1 Compatibility assumptions

The idea of considering stratifiable sets is to take as much advantage as possible of the structure of the set including the *thin* parts. In the NFT approach this can not be done because the set of trajectories remaining on the interior of the state-constraints is required to be dense in the set of all admissible trajectories; thus, it is mandatory that $\overline{\text{int}(\mathcal{K})} = \mathcal{K}$.

We define for each index $i \in \mathcal{I}$, the multifunction $\mathcal{U}_i : \mathcal{M}_i \rightrightarrows \mathcal{U}$ which corresponds to the intersection between the original control set $\mathcal{U}$ and the tangent controls to $\mathcal{M}_i$, that is,

$$\mathcal{U}_i(x) := \{u \in \mathcal{U} \mid f(x, u) \in \mathcal{T}_{\mathcal{M}_i}(x)\}, \quad \forall x \in \mathcal{M}_i.$$

This mapping is called the *tangent controls to* $\mathcal{M}_i$ and, as the following proposition shows, it is in general merely upper semicontinuous with possibly empty images.

Proposition 4.2.1. *Assume that* (H_0^4) *and* (H_f^4) *hold. Then, for each* $i \in \mathcal{I}$*, the set-valued map of the tangent control to* $\mathcal{M}_i$ *has compact images and is upper semicontinuous on* $\mathcal{M}_i$*. Besides, it can be extended to an upper semicontinuous map defined on* $\overline{\mathcal{M}}_i$*.*

Proof. Thanks to the continuity of the dynamics, the images of $\mathcal{U}_i$ are closed.

Since $\mathcal{U}$ is compact, the images of $\mathcal{U}_i$ are compact as well. Furthermore, by Proposition 2.2.3 we only need to prove that $\mathcal{U}_i$ has closed graph. Take $x \in \mathcal{M}_i$ arbitrary. Let $\{x_n\} \subseteq \mathcal{M}_i$ with $x_n \to x$ and $\{u_n\} \subseteq \mathcal{U}$ with $u_n \to u \in \mathcal{U}$ such that $u_n \in \mathcal{U}_i(x_n)$. Since $\mathcal{M}_i$ is an embedded manifold of $\mathbb{R}^N$, $\mathcal{T}_{\mathcal{M}_i}(\cdot)$ has closed graph on $\mathcal{M}_i$. Consequently, by continuity of the dynamics, $f(x_n, u_n) \to f(x, u)$ and since $f(x_n, u_n) \in \mathcal{T}_{\mathcal{M}_i}(x_n)$, we get that the multifunction $\mathcal{U}_i$ is upper semicontinuous on $\mathcal{M}_i$.

The final conclusion ensues by considering the following limiting map which is by definition upper semicontinuous and coincides with $\mathcal{U}_i$ on $\mathcal{M}_i$:

$$\overline{\mathcal{U}_i}(x) := \left\{ u \in \mathcal{U} \;\middle|\; \begin{array}{c} \exists \{x_n\} \in \mathcal{M}_i \text{ with } x_n \to x \text{ and} \\ \forall n \in \mathbb{N}, \exists u_n \in \mathcal{U}_i(x_n) \text{ so that } u_n \to u \end{array} \right\}, \quad \forall x \in \overline{\mathcal{M}}_i.$$

□

On the other hand, the fact that $\mathcal{U}_i$ may have empty images is something we can simply not avoid without imposing a further hypothesis. For this reason, we assume that we can find a stratification of the state-constraints set in such a way the set-valued map of tangent control on each stratum has nonempty or empty images all along $\mathcal{M}_i$. In view of the convention adopted for the Hausdorff distance in Section 2.2, the hypothesis is written as follows:

$$d_H(\mathcal{U}_i(x), \mathcal{U}_i(\tilde{x})) \in \mathbb{R}, \quad \forall x, \tilde{x} \in \mathcal{M}_i. \tag{4.5}$$

Furthermore, to prove the sufficiency of the HJB equation for the characterization of the Value Function we require more regularity. However, as we will discuss later on in Section 4.4, the hypothesis (4.5) implies, in many interesting cases, the hypothesis (H_2^4) we have stated below (possibly with a finer stratification). Nevertheless, we might assume a strengthen version of (4.5) for the rest of the section, that is:

(H_2^4) Each $\mathcal{U}_i$ is locally Lipschitz continuous on $\mathcal{M}_i$ for the Hausdorff distance.

Remark 4.2.2. *Similarly as done in Proposition 4.2.1, $\mathcal{U}_i$ can be extended up to $\overline{\mathcal{M}}_i$ by density. Moreover, if $\mathcal{U}_i$ is locally Lipschitz continuous, this extension turns out to be locally Lipschitz continuous too. Without loss of generality we assume that $\mathcal{U}_i$ is defined up to $\overline{\mathcal{M}}_i$ being locally Lipchitz on its whole domain.*

Remark 4.2.3. *The assumption (H_2^4) is imposed in order to ensure the Lipschitz continuity of the set-valued map $x \mapsto f(x, \mathcal{U}_i(x))$. Therefore, if for each stratum $\mathcal{M}_i$, the tangent control map does not depend upon the state, that is, $\mathcal{U}_i(x) = \mathbf{U}_i$ all along $\mathcal{M}_i$, then the dynamics and cost can be taken locally Lipschitz continuous only with respect to the state in hypotheses (H_f^4) and (H_ℓ^4), respectively.*

Example 4.2.1. *Consider the dynamical system given by the controlled harmonic oscillator:*

$$\begin{pmatrix} \dot{y}_1(t) \\ \dot{y}_2(t) \end{pmatrix} = \begin{pmatrix} y_2(t) \\ u(t) - \sigma \sin(y_1(t)) \end{pmatrix}, \quad u(t) \in \mathcal{U} := [-1, 1] \ a.e. \ t \geq 0,$$

where $\sigma \in [0, 1]$ is a fixed parameter of the model. Let us consider for simplicity the state-constraints $\mathcal{K} = [-1, 1] \times [-1, 1]$. This set is a convex polytope, so it is stratifiable. Furthermore, many stratifications are possible. Note that (H_2^4) does not hold for the minimal stratification, which consists of the interior of the set, 4 segments and 4 single points. However, there exists a finer stratification for which the hypothesis holds.

We represent one particular stratification in Figure 4.4 for which (H_2^4) does hold. In this case, $\mathcal{M}_0 = int(\mathcal{K})$, $\mathcal{M}_3$, $\mathcal{M}_4$, $\mathcal{M}_9$, $\mathcal{M}_{10}$, $\mathcal{M}_{11}$ and $\mathcal{M}_{12}$ are segments, and $\mathcal{M}_1$, $\mathcal{M}_2$, $\mathcal{M}_5$, $\mathcal{M}_6$, $\mathcal{M}_7$ and $\mathcal{M}_8$ are single points. We can check easily (H_2^4), indeed,

$$\mathcal{U}_0(x) = \mathcal{U}, \ \forall x \in \mathcal{M}_0, \quad \mathcal{U}_i(x) = \{\sigma \sin(x_1)\}, \ \forall x \in \mathcal{M}_i, \ i = 3, 4,$$

$\mathcal{U}_1(-r, 0) = \mathcal{U}_2(r, 0) = \{0\}$ *and dom* $\mathcal{U}_i = \emptyset$ *for* $i = 5, \ldots, 12$.

It is clear in this example that neither the IPC nor the OPC condition is satisfied. In figure 4.4, the green zone corresponds to the viable set, that is, the set of points for which $\mathbb{U}(x) \neq \emptyset$. Note that in this case, the viable set (dom $\mathbb{U}$) can be decomposed into a regular stratification which satisfies (H_2^4) as well.

Finally, for technical reasons, an extra hypothesis of controllability on certain strata will be required in order to complete the proof of the main theorem. For this purpose, we denote by $\mathcal{R}(x; t)$ the reachable set at time t, that is, the set of all possible positions that can be reached by an admissible trajectory. In mathematical terms

$$\mathcal{R}(x; t) = \bigcup_{u \in \mathbb{U}(x)} \{y_x^u(t)\}, \quad \forall x \in \mathcal{K}, \forall t \geq 0.$$

On the other hand, we also consider the reachable set through the stratum $\mathcal{M}_i$ which corresponds to the set of all possible positions that can be reached, at time t, by an admissible trajectory lying in $\mathcal{M}_i$ on the whole interval $[0, t)$:

$$\mathcal{R}_i(x; t) = \bigcup_{u \in \mathbb{U}(x)} \{y_x^u(t) \mid \ y_x^u(s) \in \mathcal{M}_i \ \forall s \in [0, t)\}.$$

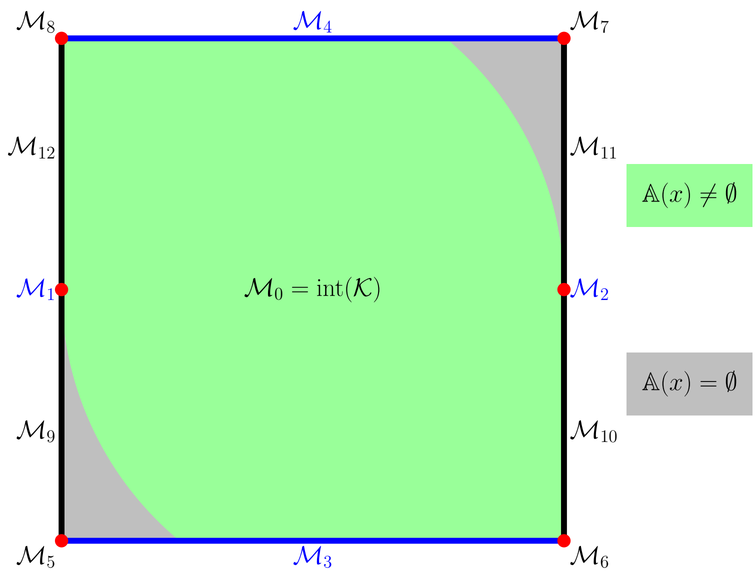


Figure 4.4: The stratification of Example 4.2.1.

Hence the controllability hypothesis we require can be stated in the following manner: for every $i \in \mathcal{I}$

$$(H_3^4) \qquad \begin{cases} \text{If } \operatorname{dom} \mathcal{U}_i \neq \emptyset, \text{ then } \forall r > 0,\ \exists \varepsilon_i, \Delta_i > 0 \text{ such that} \\ \mathcal{R}(x;t) \cap \overline{\mathcal{M}}_i \subseteq \displaystyle\bigcup_{s \in (0, \Delta_i t]} \mathcal{R}_i(x;s) \quad \forall x \in \mathcal{M}_i \cap \mathbb{B}(0,r),\ \forall t \in [0, \varepsilon_i]. \end{cases}$$

This assumption is made in order to approximate trajectories that may switch between two or more strata infinitely many times on a short interval (this could happen only if $\operatorname{dom} \mathcal{U}_i \neq \emptyset$).

Note that (H_3^4) is trivial if $\mathcal{M}_i$ is an open set or more generally if $\mathcal{M}_i$ is of maximal dimension among the strata of $\mathcal{K}$. Furthermore, the same remark holds whenever $\mathcal{U}_i \equiv \mathcal{U}$.

On the other hand, (H_3^4) is also straightforward if $\mathcal{M}_i$ is a single point. In this situation, if $\operatorname{dom} \mathcal{U}_i \neq \emptyset$ then $\mathcal{R}(x;t) \cap \overline{\mathcal{M}}_i = \overline{\mathcal{M}}_i = \mathcal{R}_i(x;t)$ for any $x \in \mathcal{M}_i$.

Let us also point out the fact that (H_3^4) can be satisfied under a rather standard criterion of full *controllability condition on manifolds*. The most classical assumption of controllability is the following: $\forall i \in \mathcal{I}$ with $\operatorname{dom} \mathcal{U}_i \neq \emptyset$

$$\exists r_i > 0 \text{ such that } \mathcal{T}_{\mathcal{M}_i}(x) \cap \mathbb{B}(0, r_i) \subseteq f(x, \mathcal{U}_i(x)), \quad \forall x \in \mathcal{M}_i. \tag{4.6}$$

Indeed, this corresponds to the Petrov condition on manifolds. Hence, by adapting the classical arguments to this setting, we can see that (4.6) implies the Lipschitz regularity of the minimum time function of the controlled dynamics restricted to the manifold $\mathcal{M}_i$, and so (H_3^4) follows; c.f. [13, Chapter 4.1]. However, let us emphasis on that (4.6) is only a *sufficient* condition for assumption (H_3^4). Indeed, (H_3^4) is still satisfied in some cases where Petrov condition does not hold. For instance, the controlled harmonic oscillator system in Example 4.2.1 fulfills the requirement (H_3^4) and clearly does not satisfy the Petrov condition (4.6); this is basically due to the fact that the controllability hypothesis only maters around the strata $\mathcal{M}_3$ and $\mathcal{M}_4$ (otherwise it is straightforward) which are themselves trajectories of the system.

4.2.2 Characterization of the Value Function

As stated in the introduction, the main aim of this subsection is to characterize the Value Function of the infinite horizon problem in terms of a bilateral HJB equation. The definition of solution that will be introduced here is based on the classical notion of supersolution and on a new subsolution concept in a stratified sense.

This last demands us to define a new Hamiltonian associated with the tangential controls. Consequently, for each index $i \in \mathcal{I}$ we write $H_i : \mathcal{M}_i \times \mathbb{R}^N \rightrightarrows \mathbb{R}$ for the *tangential Hamiltonian on* $\mathcal{M}_i$ which is define via

$$H_i(x, \zeta) = \sup_{u \in \mathcal{U}_i(x)} \{-\langle \zeta, f(x, u)\rangle - \ell(x, u)\}, \quad \forall x \in \mathcal{M}_i,\ \forall \zeta \in \mathbb{R}^N.$$

This Hamiltonian is continuous under the assumptions we have made.

Proposition 4.2.2. *Suppose that* (H_0^4) *and* (H_2^4) *hold in addition to* (H_f^4) *and* (H_ℓ^4). *Then for each* $i \in \mathcal{I}$ *such that* $\operatorname{dom} \mathcal{U}_i \neq \emptyset$, $H_i(\cdot, \cdot)$ *is locally Lipschitz continuous on* $\mathcal{M}_i \times \mathbb{R}^N$.

Proof. Let $r > 0$ fixed and write L_f and L_ℓ for the Lipschitz constants of f and ℓ on $\mathcal{M}_i \cap \mathbb{B}(0, r) \times \mathcal{U}$, respectively. We also set L_i as the Lipschitz constant of $\mathcal{U}_i$ on $\mathcal{M}_i \cap \mathbb{B}(0, r)$.

Fix $\zeta \in \mathbb{R}^N$ and take $x, y \in \mathcal{M}_i \cap \mathbb{B}(0, r)$. Since $\mathcal{U}$ is compact, there exists $u_x \in \mathcal{U}_i(x)$ so that $H_i(x, \zeta) = -\langle \zeta, f(x, u_x)\rangle - \ell(x, u_x)$. On the other hand, thanks to (H_2^4), there exists $u_y \in \mathcal{U}_i(y)$ for which $|u_x - u_y| \le L_i|x - y|$. Gathering all the information we get that

$$\begin{aligned} H_i(x, \zeta) - H_i(y, \zeta) &\le |\zeta||f(y, u_y) - f(x, u_x)| + |\ell(y, u_y) - \ell(x, u_x)| \\ &\le |\zeta|(L_f + L_\ell)(|x - y| + |u_x - u_y|) \\ &\le |\zeta|(L_f + L_\ell)(1 + L_i)|x - y| \end{aligned}$$

Since, x and y are arbitrary, we can interchange their roles and get that $x \mapsto H_i(x, \zeta)$ is Lipschitz continuous on $\mathcal{M}_i \cap \mathbb{B}(0, r)$.

Besides, using (H_f^4) and a similar argument as above, we get

$$|H_i(x, \zeta) - H_i(x, \xi)| \le c_f(1 + |x|)|\zeta - \xi|, \quad \forall x \in \mathcal{M}_i,\ \forall \zeta, \xi \in \mathbb{B}(0, r).$$

Finally, combining both partial Lipschitz estimations we get the result. □

On the other hand, since we are considering possibly not bounded running cost, the Value Function may not be bounded either. Nonetheless, thanks to (4.4), it still has a controlled growth rate

$$0 \le \vartheta(x) \le c_\ell(1 + |x|)^{\lambda_\ell} \int_0^\infty e^{(\lambda_\ell c_f - \lambda)t} dt \qquad \forall x \in \operatorname{dom} \vartheta.$$

Hence, if $\lambda > \lambda_\ell c_f$, then ϑ has superlinear growth in the following sense.

Definition 4.2.1. *Let* $\psi : \mathbb{R}^N \to \mathbb{R} \cup \{+\infty\}$ *be a given function. We say that* ψ *has* σ*-superlinear growth on its domain if there exists* $c_\psi > 0$ *so that*

$$|\psi(x)| \le c_\psi(1 + |x|)^\sigma \qquad \forall x \in \mathit{dom}\, \psi.$$

Now we are in position to state the main result of this section.

Theorem 4.2.1. *Suppose that (H_0^4), (H_1^4), (H_2^4) and (H_3^4) hold in addition to (H_f^4) and (H_ℓ^4). Assume also that $\lambda > \lambda_\ell c_f$ (where $\lambda_\ell, c_f > 0$ are the constants given by (H_ℓ^4) and (H_f^4), respectively). Then the Value Function $\vartheta(\cdot)$ of the infinite horizon problem is the only lower semicontinuous function with λ_ℓ-superlinear growth which is $+\infty$ on $\mathbb{R}^N \setminus \mathcal{K}$ and that satisfies:*

$$\lambda\vartheta(x) + H(x,\zeta) \geq 0 \qquad \forall x \in \mathcal{K},\ \forall \zeta \in \partial_V \vartheta(x), \tag{4.7}$$

$$\lambda\vartheta(x) + H_i(x,\zeta) \leq 0 \qquad \forall x \in \mathcal{M}_i,\ \forall \zeta \in \partial_V \vartheta_i(x),\ \forall i \in \mathcal{I}, \tag{4.8}$$

where $\vartheta_i(x) = \vartheta(x)$ if $x \in \overline{\mathcal{M}}_i$ and $\vartheta_i(x) = +\infty$ otherwise.

Recall that when $\mathrm{int}(\mathcal{K})$ is a nonempty set, it is a smooth manifold of $\mathbb{R}^N$ and therefore, there is no loss of generality in assuming that it is one of the stratum, say $\mathcal{M}_0$, of the stratification of $\mathcal{K}$. Under these circumstances, $H_0 = H$, and if the Value Function is continuous on $\mathcal{K}$, we have that (4.8) implies that $\vartheta(\cdot)$ is a (classical) viscosity subsolution to the HJB equation on $\mathrm{int}(\mathcal{K})$ (cf. [13, Theorem 2.5.6]), and so, the constrained HJB equation studied by Soner in [120] (lower semicontinuous supersolution on $\mathcal{K}$ and upper semicontinuous subsolution on $\mathrm{int}(\mathcal{K})$) is included in the set of equations we propose. Hence, in this sense, the above-stated theorem completes the already known theory for state constrained problems.

Remark 4.2.4. *If for some $i \in \mathcal{I}$, $\mathcal{M}_i = \{\bar{x}\}$ and $\mathcal{U}_i(\bar{x}) \neq \emptyset$ (this is the case when for instance $\mathcal{K}$ is a network with $\bar{x}$ being one of the junctions), then $f(\bar{x},u) = 0$ for any $u \in \mathcal{U}_i(\bar{x})$ and so $H_i(x,\zeta) = -\min\{\ell(\bar{x},u) \mid\ u \in \mathcal{U}_i(\bar{x})\}$ for any $\zeta \in \mathbb{R}^N$. Consequently, (4.8) for this index agrees with the following inequality:*

$$\lambda\vartheta(\bar{x}) \leq \min_{u \in \mathcal{U}_i(\bar{x})} \ell(\bar{x},u),$$

which basically says that the cost of leaving the point $\bar{x}$ should be lower than the cost of remaining at the point.

4.2.3 Proof of the main result

From this point on we start to prove Theorem 4.2.1. In order to make the proof easier to understand we decompose it into several parts. In particular, we present an intermediate characterization of the Value Function in terms of the Dynamic Programming Principle.

Theorem 4.2.1 will be a direct consequence of Propositions 4.2.4, 4.2.5, 4.2.6 and 4.2.7, and Remarks 4.2.7 and 4.2.9.

Lower Semicontinuity and existence of optimal controls.

The next proposition is a classical type of result in optimal control and states the existence of minimizer for the infinite horizon problem under a convexity assumption over the dynamics. The same argument can used to prove that the Value Function is lower semicontinuous.

Proposition 4.2.3. *Suppose that (H_f^4), (H_ℓ^4) and (H_1^4) hold. Assume that $\lambda > \lambda_\ell c_f$. If $\vartheta(x) \in \mathbb{R}$ for some $x \in \mathcal{K}$ then there exists $u \in \mathbb{U}(x)$ a minimizer of infinite horizon problem. Furthermore, the Value Function is lower semicontinuous.*

Proof. Let $x \in \mathcal{K}$ such that $\vartheta(x) \in \mathbb{R}$. This implies that for every $n \geq 0$, we can find a control law $u_n \in \mathbb{U}(x)$ so that $\{u_n\}$ verifies:

$$\lim_{n\to+\infty} \int_0^\infty e^{-\lambda t}\ell(y_n(t), u_n(t))\,dt = \vartheta(x), \tag{4.9}$$

where y_n is the solution to (1.1) with the initial condition $y_n(0) = x$. Consider $z_n(t) = \ell(y_n(t), u_n(t))$ for a.e. $t \in [0, +\infty)$.

Define the measure $d\mu = e^{-\lambda t}dt$ and let $L^1 := L^1([0,+\infty), \mathbb{R}; d\mu)$ be the Banach space of (the class of equivalence of) integrable real-valued functions on $[0, +\infty)$ for the measure $d\mu$. Consequently, we denote by $W^{1,1}$ the Sobolev space functions $y : [0, +\infty) \to \mathbb{R}^N$ for which $|y| \in L^1$ and whose weak derivative $\dot{y}$ also verifies $|\dot{y}| \in L^1$.

Let $\omega : [0, +\infty) \to \mathbb{R}$ be given by $\omega(t) := c_f(1 + |x|)e^{c_f t}$ for any $t \geq 0$. By (H_ℓ^4), $\lambda > c_f$ because $\lambda_\ell \geq 1$. Then, (4.3) implies that $\omega(\cdot)$ is a positive function in L^1 which dominates $|\dot{y}_n|$. Moreover, by (4.1) or (4.2) the sequence $\{y_n(t)\}$ is relatively compact for any $t \geq 0$, thereby the hypothesis of theorem [11, Theorem 0.3.4] are satisfied and so, there exist a function $y \in W^{1,1}$ and a subsequence, still denoted by $\{y_n\}$, such that

$$\begin{aligned} &y_n \text{ converges uniformly to } y \text{ on compact subsets of } [0, +\infty),\\ &\dot{y}_n \text{ converges weakly to } \dot{y} \text{ in } L^1([0, +\infty), \mathbb{R}^N; d\mu). \end{aligned}$$

On the other hand, given that $\lambda > \lambda_\ell c_f$ and (4.4) holds, it is not difficult to see that $\{z_n\}$ is equi-integrable with respect to $d\mu$. The Dunford-Pettis Theorem implies the existence of $z \in L^1$ and a subsequence, still denoted by z_n, so that z_n converges weakly to z in L^1.

Let $\Gamma(x) = G(0, x) \subseteq \mathbb{R}^N \times \mathbb{R}$ for every $x \in \mathcal{K}$. Hence, by (H_f^4) and (H_ℓ^4), Γ is locally Lipschitz continuous with closed images and by (H_1^4) it has convex images. In addition, the Convergence Theorem (Proposition 2.4.4) implies that $(\dot{y}, z) \in \Gamma(y)$ for almost every $t \geq 0$. Thus, by the Measurable Selection Theorem (Proposition 2.2.6), there are two measurable functions $u : [0, +\infty) \to \mathcal{U}$ and $r : [0, +\infty) \to [0, +\infty)$ that satisfy

$$\begin{aligned} &\dot{y}(t) = f(y(t), u(t)), \quad \text{a.e. } t > 0, \qquad y(0) = x.\\ &z(t) = \ell(y(t), u(t)) + r(t), \quad \text{a.e. } t > 0. \end{aligned}$$

Since $\mathcal{K}$ is closed, $y(t) \in \mathcal{K}$ for every $t \geq 0$, which means that $u \in \mathbb{U}(x)$. Finally, given that $\phi \equiv 1 \in L^\infty([0, +\infty), \mathbb{R}; d\mu)$, we have

$$\int_0^\infty e^{-\lambda t}\ell(y(t), u(t))dt \leq \int_0^\infty e^{-\lambda t} z(t)dt = \lim_{n\to+\infty} \int_0^\infty e^{-\lambda t} z_n(t)dt = \vartheta(x).$$

Therefore, u is a minimizer of the problem.

Now let us focus on the lower semicontinuity of ϑ. Let $\{x_n\} \subseteq \mathcal{K}$ be a sequence such that $x_n \to x$. Without lost of generality we assume that $|x_n| \leq |x| + 1$. We need to prove that

$$\liminf_{n\to+\infty} \vartheta(x_n) \geq \vartheta(x).$$

Suppose that there is a subsequence, we eschew relabeling, so that $\{x_n\} \subseteq \operatorname{dom}\vartheta$. Otherwise the inequality holds immediately. Then, by the previous part, for any $n \in \mathbb{N}$ we can pick a measurable control $u_n \in \mathbb{U}(x_n)$ which is optimal. Let y_n the optimal trajectory associated with u_n and x_n. Notice that (4.1), (4.3) and (4.4) hold with x_n instead of x. Hence, given that $|x_n|$ is uniformly bounded ($|x_n| \leq |x| + 1$) we can use the same technique as in the previous part to find that there exists $u \in \mathbb{U}(x)$ such that

$$\int_0^\infty e^{-\lambda t}\ell(y_x^u(t), u(t))dt \leq \liminf_{n\to+\infty} \int_0^\infty e^{-\lambda t}\ell(y_n(t), u_n(t))dt = \liminf_{n\to+\infty} \vartheta(x_n).$$

Finally, using the definition of the Value Function we conclude the proof. □

Increasing principles along trajectories.

The Dynamic Programming Principle yields to two different monotonicity conditions along admissible arcs. Indeed, the two elementary inequalities that define it can be interpreted as a *weakly decreasing* and a *strongly increasing* principle, respectively. These two properties are also known in the literature ([13, Definition 3.2.31] for example) as the *super* and *sub-optimality principles*, respectively.

Definition 4.2.2. *Let $\varphi : \mathcal{K} \to \mathbb{R} \cup \{+\infty\}$ be a given function, we say that φ is:*

i) weakly decreasing for the control system if for all $x \in \mathrm{dom}\,\varphi$, there exists a control $u \in \mathbb{U}(x)$ such that

$$e^{-\lambda t}\varphi(y_x^u(t)) + \int_0^t e^{-\lambda s}\ell(y_x^u(s), u(s))ds \leq \varphi(x), \quad \forall t \geq 0. \tag{4.10}$$

ii) strongly increasing for the control system if $\mathrm{dom}\,\mathbb{U} \subseteq \mathrm{dom}\,\varphi$ and for any $x \in \mathcal{K}$ and $u \in \mathbb{U}(x)$ we have

$$e^{-\lambda t}\varphi(y_x^u(t)) + \int_0^t e^{-\lambda s}\ell(y_x^u(s), u(s))ds \geq \varphi(x), \quad \forall t \geq 0. \tag{4.11}$$

The importance of these definitions lies in the following comparison principle which is the fundamental type of result required to single out the Value Function among other lower semicontinuous functions.

Lemma 4.2.1. *Suppose that (H_f^4), (H_ℓ^4) and (H_1^4) hold, and that $\lambda > \lambda_\ell c_f$. Let $\varphi : \mathcal{K} \to \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous function with λ_ℓ-superlinear growth.*

1. *If φ is weakly decreasing for the control system, then $\vartheta \leq \varphi$.*
2. *If φ is strongly increasing for the control system then $\vartheta \geq \varphi$.*

Proof. First of all, note that if $\lambda > \lambda_\ell c_f$, any function φ with λ_ℓ-superlinear growth and any trajectory $y(\cdot)$ of (1.1) such that $y(t) \in \mathrm{dom}\,\varphi$ satisfy

$$\lim_{t\to+\infty} e^{-\lambda t}\varphi(y(t)) = 0. \tag{4.12}$$

Case 1. Suppose φ is weakly decreasing for the control system. Let $x \in \mathcal{K}$, if $x \notin \operatorname{dom} \varphi$ then the inequality is trivial. Let x be in $\operatorname{dom} \varphi$, there exists a control $u \in \mathbb{U}(x)$ such that for all $n \in \mathbb{N}$

$$e^{-\lambda n}\varphi(y(n)) + \int_0^\infty e^{-\lambda s}\ell(y_x^u(s), u(s))\mathbb{1}_{[0,n]}ds \leq \varphi(x).$$

Therefore, by the Monotone Convergence Theorem, (4.12) and the definition of the Value Function we obtain the desired inequality $\vartheta(x) \leq \varphi(x)$.

Case 2. Suppose φ is strongly increasing for the control system and let $x \in \mathcal{K}$. Assume that $\vartheta(x) \in \mathbb{R}$, otherwise the result is direct. Let $\bar{u} \in \mathbb{U}(x)$ be an optimal control for the infinite horizon problem and let $\bar{y}$ be the optimal trajectory associated with $\bar{u}$ and x. It follows that

$$e^{-\lambda t}\varphi(y(t)) + \int_0^t e^{-\lambda s}\ell(y(s), \bar{u}(s))ds \geq \varphi(x) \quad \forall t \geq 0.$$

Taking into account (4.12) and letting $t \to +\infty$ we conclude the proof. $\square$

In view of the previous comparison lemma we can state an intermediate characterization of the Value Function, which implies particularly that the Value Function is the sole solution to the functional equation known as the Dynamic Programming Principle:

$$\vartheta(x) = \inf_{u \in \mathbb{U}(x)} \left\{ e^{-\lambda t}\vartheta(y_x^u(t)) + \int_0^t e^{-\lambda s}\ell(y_x^u(s), u(s))ds \right\}, \quad \forall t > 0,\ \forall x \in \mathcal{K}.$$

Proposition 4.2.4. *Suppose that* (H_f^4), (H_ℓ^4) *and* (H_1^4) *hold, and that* $\lambda > \lambda_\ell c_f$. *The Value Function* $\vartheta(\cdot)$ *is the unique lower semicontinuous function with* λ_ℓ*-superlinear growth that is weakly decreasing and strongly increasing for the control system at the same time.*

Proof. Recall that the Value Function $\vartheta(\cdot)$ verifies the Dynamic Programming Principle. So, it is weakly decreasing and strongly increasing for the control system. The uniqueness and the growth condition are consequences of Lemma 4.2.1. $\square$

Characterization of the weakly decreasing principle.

We now prove that the weakly decreasing principle is equivalent to a HJB inequality. This means that a function satisfies (4.10) if and only if it is a *supersolution* of the HJB equation. The idea of the proof uses very classical arguments and requires merely standing assumptions of control theory. A proof for the unconstrained case with bounded Value Function can be found in [13, Chapter 3.2]. It is worth noting that the cited proof uses purely viscosity arguments and ours mainly nonsmooth analysis tools.

We restrict our attention to a small class of viscosity subgradients, namely the proximal subgradients, and next we extend the result to all the viscosity subgradients of the Value Function by means of a density result.

Proposition 4.2.5. *Suppose that* (H_f^4), (H_ℓ^4) *and* (H_1^4) *hold. Consider a given lower semicontinuous function with real-extended values* $\varphi : \mathcal{K} \to \mathbb{R} \cup \{+\infty\}$. *Then* φ *is weakly decreasing for the control system if and only if*

$$\lambda\varphi(x) + H(x, \zeta) \geq 0 \qquad \forall x \in \mathcal{K},\ \forall \zeta \in \partial_P \varphi(x) \tag{4.13}$$

Remark 4.2.5. *The proof of the foregoing result can be achieved under weaker assumption, namely, the Lipschitz continuity of the dynamics and cost are not at all required. Actually, their continuity and growth rates suffice.*

Proof of Proposition 4.2.5. Let us first prove the implication ($\Rightarrow$).

Suppose φ is weakly decreasing for the control system. Let $x \in \mathcal{K}$, if $\partial_P \varphi(x) = \emptyset$ then (4.13) holds by vacuity. If, on the contrary, there exists $\zeta \in \partial_P\varphi(x)$, then $x \in \operatorname{dom}\varphi$ and we can find $u \in \mathbb{U}(x)$ such that (4.10) holds. Let $y(\cdot)$ stands for the trajectory of (1.1) associated with the control u and the initial condition x. We evoke from Section 2.3.3 the proximal subgradient inequality that yields to the existence of $\sigma, \delta > 0$ such that

$$\varphi(y(t)) \geq \varphi(x) + \langle \zeta, y(t) - x\rangle - \sigma|y(t) - x|^2 \quad \forall t \in [0, \delta).$$

Using that $y(\cdot)$ is a trajectory and (4.10), we get for any t small enough

$$(1 - e^{\lambda t})\varphi(x) + \int_0^t \left[\langle \zeta, f(y(s), u(s))\rangle + \ell(y(s), u(s))\right] ds \leq \sigma|y(t) - x|^2$$

Since f and ℓ are uniformly continuous around the graph of y we get

$$\frac{(1 - e^{\lambda t})}{t}\varphi(x) + \frac{1}{t}\int_0^t \left[\langle \zeta, f(x, u(s))\rangle + \ell(x, u(s))\right] ds \leq h(t)$$

where $h(t)$ verifies $\lim_{t\to 0^+} h(t) = 0$. Therefore taking infimum over $u \in \mathcal{U}$ inside the integral and letting $t \to 0^+$ we get (4.13) after some algebraic steps.

Now, we turn to the second part of the proof ($\Leftarrow$). Let $\mathcal{O} \subseteq \mathbb{R}^{N+1}$ be the neighborhood of $[0, +\infty) \times \mathcal{K}$ given by (H_1^4) which we assume is open. Let $\psi : \mathbb{R} \times \mathbb{R}^N \times \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ be defined via

$$\psi(\tau, x, z) = \begin{cases} e^{-\lambda\tau}\varphi(x) + z & \text{if } x \in \mathcal{K},\ \tau \geq 0, \\ +\infty & \text{otherwise,} \end{cases} \qquad \forall(\tau, x, z) \in \mathbb{R} \times \mathbb{R}^N \times \mathbb{R},$$

and $\Gamma : \mathbb{R} \times \mathbb{R}^N \times \mathbb{R} \times \mathbb{R} \rightrightarrows \mathbb{R} \times \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}$ given by

$$\Gamma(\tau, x, z, w) = \{1\} \times G(\tau, x) \times \{0\}, \quad \forall(\tau, x, z, w) \in \mathbb{R} \times \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}.$$

To prove that φ is weakly decreasing for the control system let us first show that for any $\gamma_0 \in \operatorname{epi}\psi$, there exists an absolutely continuous arc $\gamma : [0, T) \to \mathcal{O} \times \mathbb{R}^2$ that satisfies

$$\dot{\gamma} \in \Gamma(\gamma) \quad \text{a.e. on } [0, T) \quad \text{and} \quad \gamma(0) = \gamma_0, \tag{4.14}$$

and verifies $\gamma(t) \in \operatorname{epi}\psi$ for every $t \in [0, T)$ as well, or in terms of Definition 2.4.1, $(\Gamma, \operatorname{epi}\psi)$ is weakly invariant on $U = \mathcal{O} \times \mathbb{R}^2$. We are going to make use of Proposition 2.4.5.

Note that $\operatorname{epi}\psi$ is closed because φ is lower semicontinuous and Γ has nonempty convex compact images on $\mathcal{O} \times \mathbb{R}^2$ because of (H_1^4). Moreover, by (H_f^4) and (H_ℓ^4), Γ has closed graph and satisfies the following growth condition:

$$\exists c_\Gamma > 0 \text{ so that } \sup\{|v| \mid v \in \Gamma(\tau, x, z, w)\} \leq c_\Gamma(1 + |x| + e^{-\lambda\tau}|x|^{\lambda_\ell}).$$

So, the hypotheses required by Proposition 2.4.5 are verified. Thus, to prove the weak invariance of $(\Gamma, \operatorname{epi}\psi)$ we only need to show that, for $\mathcal{S} = \operatorname{epi}\psi$, (4.13) implies

$$\min_{v\in\Gamma(\chi)} \langle \eta, v\rangle \leq 0 \qquad \forall \chi \in \mathcal{S}\cap U,\ \forall \eta \in \mathcal{N}^P_{\mathcal{S}}(\chi). \tag{4.15}$$

Let $(\tau, x, z, w) \in \mathcal{S} \cap U$, in particular, $x \in \operatorname{dom}\varphi$. Consider $\eta \in \mathcal{N}^P_{\mathcal{S}}(\tau, x, z, w)$, thanks to the fact that $\mathcal{S}$ is an epigraph, we can write $\eta = (\xi, -p)$ with p nonnegative. Suppose $p > 0$ then $w = \psi(\tau, x, z)$ and

$$\frac{1}{p}\xi \in \partial_P\psi(\tau, x, z) \subseteq \{-\lambda e^{-\lambda\tau}\varphi(x)\} \times e^{-\lambda\tau}\partial_P\varphi(x) \times \{1\}.$$

Therefore, for some $\zeta \in \partial_P\varphi(x)$ we have

$$\begin{aligned}\min_{v\in\Gamma(\tau,x,z,w)} \langle \eta, v\rangle &\leq \min_{\substack{u\,\in\,\mathcal{U},\\ 0\,\leq\, r\,\leq\,\beta(x,u)}} pe^{-\lambda\tau}(-\lambda\varphi(x) + \langle\zeta, f(x,u)\rangle + \ell(x,u) + r)\\ &\leq pe^{-\lambda\tau}\min_{u\in\mathcal{U}}\left(-\lambda\varphi(x) + \langle\zeta, f(x,u)\rangle + \ell(x,u)\right).\end{aligned}$$

Consequently, by (4.13) we get $\min\{\langle\eta, v\rangle \mid\ v \in \Gamma(\tau, x, z, w)\} \leq 0$.

Suppose now that $p = 0$, thereby $(\xi, 0) \in \mathcal{N}^P_{\mathcal{S}}(\tau, x, z, \psi(\tau, x, z))$ and by the Rockafellar's horizontal Theorem (Proposition 2.3.5), there are some sequences $\{(\tau_n, x_n, z_n)\} \subseteq \operatorname{dom}\psi$, $\{(\xi_n)\} \subseteq \mathbb{R}^{N+2}$ and $\{p_n\} \subseteq (0,\infty)$ such that

$$\begin{aligned}(\tau_n, x_n, z_n) &\to (\tau, x, z), \qquad & \psi(\tau_n, x_n, z_n) &\to \psi(\tau, x, z),\\ (\xi_n, p_n) &\to (\xi, 0), & \frac{1}{p_n}\xi_n &\in \partial_P\psi(\tau_n, x_n, z_n).\end{aligned}$$

Thus, using the same argument as above we can demonstrate

$$\min\{\langle(\xi_n, -p_n), v\rangle \mid v \in \Gamma(\tau_n, x_n, z_n, \psi(\tau_n, x_n, z_n))\} \leq 0.$$

Hence, due to Γ has compact and locally bounded images, and it is upper semicontinuous, we can take the liminf in the foregoing inequality and, since $\Gamma(\tau, x, z, \psi(\tau, y, z)) = \Gamma(\tau, x, z, w)$, we obtain (4.15).

By Proposition 2.4.5, for every $\gamma_0 = (\tau_0, x_0, z_0, w_0) \in \mathcal{S}\cap\mathcal{O}\times\mathbb{R}^2$ there exists an absolutely continuous arc $\gamma(t) = (\tau(t), y(t), z(t), w(t))$ which lies in $\mathcal{O}\times\mathbb{R}^2$ for a maximal period of time $[0, T)$ so that (4.14) holds and

$$e^{-\lambda\tau(t)}\varphi(y(t)) + z(t) \leq w(t) \quad \forall t \in [0, T).$$

By the Measurable Selection Theorem (Proposition 2.2.6), $y(\cdot)$ is a solution of the control system associated with some $u : [0, T) \to \mathcal{U}$. Also, $y(t) \in \operatorname{dom}\varphi \subseteq \mathcal{K}$, $\forall t \in [0, T)$.

Moreover, due to $w(t) = w_0$ and $\tau(t) = \tau_0 + t$, we have

$$z(t) = \int_0^t [e^{-\lambda(\tau_0+s)}\ell(y(s), u(s)) + r(s)]ds, \qquad \text{with } r(s) \geq 0 \text{ a.e.}$$

Notice that $\gamma_0 = (0, x, 0, \varphi(x)) \in \operatorname{epi}\psi$ for any $x \in \operatorname{dom}\varphi$, so to conclude the proof we just need to show that $T = +\infty$. By contradiction, suppose $T < +\infty$, then $(\tau(t), y(t)) \to \partial\mathcal{O}$ as $t \to T^-$. Nevertheless, given that $\mathcal{O}$ is a neighborhood of $[0, +\infty) \times \mathcal{K}$, $\tau(t) = t$ and $y(t) \in \mathcal{K}$ for any $t \in [0, T)$, this is not possible. Therefore, the conclusion follows. □

Remark 4.2.6. *As can be seen from the proof, in the foregoing result the Lipschitz continuity of the dynamics and cost is not required. This is because the proof is based on weak invariance arguments that mainly require the upper semicontinuity of the augmented dynamics.*

Remark 4.2.7. *Let φ be as in Proposition 4.2.5, then φ satisfies (4.13) if and only if it satisfies as well*

$$\lambda\varphi(x) + H(x,\zeta) \geq 0 \quad \forall x \in \mathcal{K},\ \forall \zeta \in \partial_V \varphi(x). \tag{4.16}$$

Indeed, since the proximal subgradient is always contained in the viscosity subgradient, the sufficient condition follows easily.

On the other hand, if (4.13) holds, by Proposition 2.3.10 for any $x \in dom\,\varphi$ and $\zeta \in \partial_V\varphi(x)$ there exist two sequences $\{x_n\} \subseteq dom\,\varphi$ and $\{\zeta_n\} \subseteq \mathbb{R}^N$ such that $x_n \to x$, $\varphi(x_n) \to \varphi(x)$, $\zeta_n \in \partial_P\varphi(x_n)$ and $\zeta_n \to \zeta$. Furthermore

$$\lambda\varphi(x_n) + H(x_n,\zeta_n) \geq 0 \quad \forall n \in \mathbb{N}.$$

Hence, by the compactness of $\mathcal{U}$, passing into the limit in the last inequality we get (4.16).

Strongly increasing principle and HJB inequalities, the necessary condition.

We turn our attention into the subsolution principle. We show in this part that satisfying inequality (4.11) implies to be a *subsolution* of the HJB equation on each stratum.

Before going further, we state a fundamental result for the analysis. The next lemma yields to the existence of smooth trajectories remaining on one of the stratum for any initial data.

Lemma 4.2.2. *Suppose that (H_f^4), (H_ℓ^4), (H_0^4), (H_1^4) and (H_2^4) hold. Then, for any $i \in \mathcal{I}$ such that $\mathcal{U}_i$ has nonempty images, for every $x \in \mathcal{M}_i$ and any $u_x \in \mathcal{U}_i(x)$ there exist $\varepsilon > 0$, a measurable control map $u : (-\varepsilon,\varepsilon) \to \mathcal{U}$, a measurable function $r : (-\varepsilon,\varepsilon) \to [0,+\infty)$ and a continuously differentiable arc $y : (-\varepsilon,\varepsilon) \to \mathcal{M}_i$ with $y(0) = x$ and $\dot{y}(0) = f(x,u_x)$, such that*

$$\dot{y}(t) = f(y(t),u(t)) \quad \text{and} \quad \lim_{t\to 0^-} \frac{1}{t}\int_t^0 \left(e^{-\lambda s}\ell(y(s),u(s)) + r(s)\right) ds = -\ell(x,u_x).$$

Proof. Let $r > 0$ and set $\mathcal{M}_i^r = \mathcal{M}_i \cap \mathbb{B}(x,r)$. We write $\Gamma_i : \mathcal{M}_i^r \times (-1,1) \rightrightarrows \mathbb{R}^N \times \mathbb{R}$ for the set-valued map given by

$$\Gamma_i(y,t) = \left\{ \begin{pmatrix} f(y,u) \\ e^{-\lambda t}\ell(y,u) + r \end{pmatrix} \;\middle|\; \begin{matrix} u \in \mathcal{U}_i(y), \\ 0 \leq r \leq \beta(y,u) \end{matrix} \right\}, \quad \forall (y,t) \in \mathcal{M}_i^r \times (-1,1).$$

Note that by the definition of $\mathcal{U}_i$ and thanks to (H_f^4) and (H_ℓ^4), Γ_i has closed images and since $\mathcal{U}_i$ has nonempty images, Γ_i has nonempty images as well. The definition of $\mathcal{U}_i$ and (H_1^4) imply that it also has convex images.

Besides, by virtue of (H_2^4), Γ_i is Lipschitz continuous on $\mathcal{M}_i^r \times (-1,1)$, so it admits a Lipschitz continuous selection, $g_i : \mathcal{M}_i^r \times (-1,1) \to \mathbb{R}^N \times \mathbb{R}$ such that $g_i(x,0) = (f(x,u_x),\ell(x,u_x))$ (Proposition 2.2.5). Notice too that

$$g(y,t) \in f(y,\mathcal{U}_i(y)) \times \mathbb{R} \subseteq \mathcal{T}_{\mathcal{M}_i}(y) \times \mathbb{R}, \quad \forall (y,t) \in \mathcal{M}_i^r \times (-1,1).$$

Hence, by the Nagumo's theorem (Proposition 2.4.2) and the Lipschitz continuity of g_i, there exists $\varepsilon > 0$ such that the differential equation

$$(\dot{y}, \dot{z}) = g_i(t, y), \quad y(0) = x, \ z(0) = 0$$

admits a unique solution which is continuously differentiable on $(-\varepsilon, \varepsilon)$ such that $y(t) \in \mathcal{M}_i$ for every $t \in (-\varepsilon, \varepsilon)$, $\dot{y}(0) = f(x, u_x)$ and $\dot{z}(0) = \ell(x, u_x)$.

On the other hand, due to $\Gamma_i(y, t) \subseteq G(t, y)$ for any $(t, y) \in (-1, 1) \times \mathcal{M}_i^r$, by the Measurable Selection Theorem (Proposition 2.2.6), there exist a measurable control $u : (-\varepsilon, \varepsilon) \to \mathcal{U}$ and a measurable function $r : (-\varepsilon, \varepsilon) \to [0, +\infty)$ such that

$$(\dot{y}, \dot{z}) = (f(y, u), e^{-\lambda t}\ell(y, u) + r), \quad \text{a.e. on } (-\varepsilon, \varepsilon).$$

Finally, given that $y(t) \in \mathcal{M}_i$, we have that $u \in \mathcal{U}_i(y)$ a.e. on $(-\varepsilon, \varepsilon)$, and so the conclusion follows, because

$$z(t) = \int_0^t \left(e^{-\lambda s}\ell(y(s), u(s)) + r(s)\right) ds, \qquad \forall t \in (-\varepsilon, \varepsilon).$$

$\square$

Remark 4.2.8. *If $\mathcal{U}_i$ has convex images, by the Lipschitz continuous Selection Theorem, we can find a locally Lipschitz continuous feedback $U_i : \mathcal{M}_i \to \mathcal{U}$ such that $U(x) = u_x$. In this situation, the proof of Lemma 4.2.2 is a direct consequence of this fact.*

In view of the previous lemma, the necessity part of the strongly increasing principle can be state as follows.

Proposition 4.2.6. *Suppose that (H_f^4), (H_ℓ^4), (H_0^4), (H_1^4) and (H_2^4) hold. Let $\varphi : \mathcal{K} \to \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous function. Suppose that φ is strongly increasing for the control system, then*

$$\lambda\varphi(x) + H_i(x, \zeta) \le 0 \qquad \forall x \in \mathcal{M}_i, \ \forall \zeta \in \partial_P \varphi_i(x), \tag{4.17}$$

where $\varphi_i(x) = \varphi(x)$ if $x \in \overline{\mathcal{M}}_i$ and $\varphi_i(x) = +\infty$ otherwise.

Proof. First of all note that $\zeta \in \partial_P \varphi_i(x)$ if and only if $\exists \sigma, \delta > 0$ such that

$$\varphi(y) \ge \varphi(x) + \langle \zeta, y - x \rangle - \sigma|y - x|^2 \quad \forall y \in \mathbb{B}(x, \delta) \cap \overline{\mathcal{M}}_i.$$

We just show (4.17) for any $(i, x) \in \mathcal{I} \times \mathcal{K}$ such that $x \in \operatorname{dom} \partial_P \varphi_i \cap \mathcal{M}_i \cap \operatorname{dom} \mathcal{U}_i$. Otherwise, the conclusion is direct.

Let $(i, x) \in I \times \mathcal{K}$ as before and take $u_x \in \mathcal{U}_i(x)$, it suffices to prove

$$-\lambda\varphi(x) + \langle \zeta, f(x, u_x) \rangle + \ell(x, u_x) \ge 0, \qquad \forall \zeta \in \partial_P \varphi_i(x). \tag{4.18}$$

Let $u : (-\varepsilon, \varepsilon) \to \mathcal{U}$, $r : (-\varepsilon, \varepsilon) \to [0, +\infty)$ and $y : (-\varepsilon, \varepsilon) \to \mathcal{M}_i$ be the measurable control and smooth arc given by Lemma 4.2.2, respectively, where $\varepsilon > 0$ is also given by that lemma. Let $\bar{u} \in \mathbb{U}(x)$, for any $\tau \in (0, \varepsilon)$ we define the control map $u_\tau : [0, +\infty) \to \mathcal{U}$ as follows:

$$u_\tau(t) := u(t - \tau)\mathbb{1}_{[0,\tau]}(t) + \bar{u}(t - \tau)\mathbb{1}_{(\tau,+\infty)}(t) \quad \text{ for a.e. } t \in [0, +\infty).$$

Let $y_\tau(\cdot)$ be the trajectory associated with u_τ starting from $y_\tau(0) = y(-\tau)$, this means that $y_\tau(t) = y(t-\tau)$ for any $t \in [0,\tau]$. Moreover, $u_\tau \in \mathbb{U}(y(-\tau))$, and since φ is strongly increasing

$$e^{-\lambda\tau}\varphi(x) + \int_0^\tau \left(e^{-\lambda s}\ell(y(s-\tau), u(s-\tau)) + r(s-\tau)\right) ds \geq \varphi(y(-\tau)).$$

Take $\zeta \in \partial_P \varphi_i(x)$ and τ small enough, so that the proximal subgradient inequality is valid. Then

$$\varphi(y(-\tau)) \geq \varphi(x) + \langle \zeta, y(-\tau) - x\rangle - \sigma|y(-\tau) - x|^2.$$

In particular,

$$\frac{e^{-\lambda\tau} - 1}{\tau}\varphi(x) + \frac{e^{-\lambda\tau}}{\tau}\int_{-\tau}^0 \left(e^{-\lambda s}\ell(y(s), u(s)) + r(s)\right)ds + \left\langle \zeta, \frac{x - y(-\tau)}{\tau}\right\rangle \geq h(\tau),$$

with $\lim_{\tau\to 0^+} h(\tau) = 0$. Hence, by Proposition 4.2.2, passing to the limit in the last inequality we obtain (4.18) and so (4.17) follows. □

Remark 4.2.9. *Let φ be as in Proposition 4.2.6, then similarly as done in Remark 4.2.7, we can prove that φ satisfies (4.17) if and only if it satisfies*

$$\lambda\varphi(x) + H_i(x,\zeta) \leq 0 \quad \forall x \in \mathcal{K},\ \forall \zeta \in \partial\varphi(x). \tag{4.19}$$

We focus exclusively on showing that (4.17) implies (4.19). Let $x \in dom\,\varphi$ and $\zeta \in \partial\varphi(x)$, by Proposition 2.3.10 we can find two sequences $\{x_n\} \subseteq dom\,\varphi$ and $\{\zeta_n\} \subseteq \mathbb{R}^N$ such that $x_n \to x$, $\varphi(x_n) \to \varphi(x)$, $\zeta_n \in \partial_P\varphi(x_n)$ and $\zeta_n \to \zeta$ for which

$$\lambda\varphi(x_n) \leq \langle f(x_n, u), \zeta_n\rangle + \ell(x_n, u) \quad \forall n \in \mathbb{N},\ \forall u \in \mathcal{U}_i(x_n).$$

Since $\mathcal{U}_i$ is in particular lower semicontinuous, if $\bar{u} \in \mathcal{U}_i(x)$ realizes the maximum in the definition of the tangential Hamiltonian H_i at (x,ζ), we can find a sequence $u_n \in \mathcal{U}_i(x_n)$ such that $u_n \to u$. Therefore, evaluating at $u = u_n$ in the previous inequality and letting $n \to +\infty$, we get (4.19).

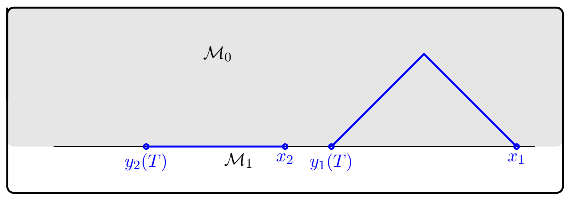
Strongly increasing principle and HJB inequalities, the sufficient condition.

In this section we prove the converse of Proposition 4.2.6 under the controllability assumption (H_3^4). The proof consists in analyzing three different types of trajectories defined on a finite interval of time $[0,T]$. The first situation corresponds to trajectories that remain on a single manifold but whose extremal points may not do so, as for instance in Figure 4.5a. This case is treated independently in Lemma 4.2.3. The second type is studied in Step 1 of the proof of Proposition 4.2.7, these trajectories have the characteristic that can be decomposed into a finite number of first type trajectories; see an example in Figure 4.5b.

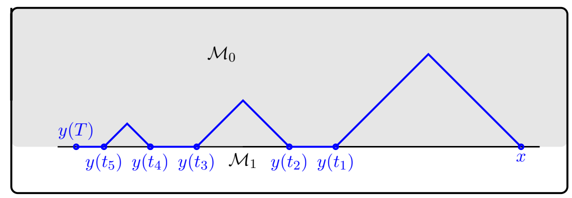
The third and more delicate type of trajectories to treat are those that switch from one stratum to another infinitely many times in a finite interval as in Figure 4.5c. The hypothesis (H_3^4) is made to handle these trajectories. It allows us to construct an approximate trajectory of type 2, as in Figure 4.5c, whose corresponding cost is almost the same.

The proof we present is based on the criterion for strong invariance adapted to smooth manifolds given in Section 2.4.2 (Proposition 2.4.6).

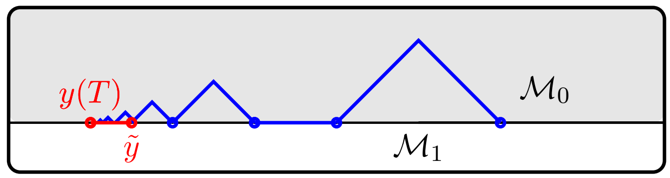
On the other hand, as aforementioned, the proof of the sufficiency part is divided itself into many steps. The step zero is the following Lemma.



(a) Extremal switching times.



(b) Finite switching times.



(c) Chattering trajectory and its approximation.

Figure 4.5: Situations to be considered.

Lemma 4.2.3. *Suppose that* (H_0^4), (H_1^4) *and* (H_2^4) *hold in addition of* (H_f^4) *and* (H_ℓ^4). *Let* $\varphi : \mathcal{K} \to \mathbb{R} \cup \{+\infty\}$ *be a lower semicontinuous function. Assume that* (4.17) *holds. Then for any* $x \in \mathcal{K}$, $u \in \mathbb{U}(x)$ *and any* $0 \leq a < b < +\infty$, *if* $y(t) := y_x^u(t) \in \mathcal{M}_i$ *for every* $t \in (a, b)$ *with* $i \in \mathcal{I}$, *we have*

$$\varphi(y(a)) \leq e^{-\lambda(b-a)}\varphi(y(b)) + e^{\lambda a}\int_a^b e^{-\lambda s}\ell(y,u)ds. \tag{4.20}$$

Proof. First of all we consider a backward augmented dynamics terms defined as follows:

$$G_i(\tau, x) = \left\{ -\begin{pmatrix} f(x,u) \\ e^{-\lambda\tau}(\ell(x,u)+r) \end{pmatrix} \;\middle|\; \begin{matrix} u \in \mathcal{U}_i(x), \\ 0 \leq r \leq \beta(x,u) \end{matrix} \right\}, \quad \forall (\tau, x) \in \mathbb{R} \times \mathcal{M}_i.$$

Thanks to (H_1^4) and the definition of $\mathcal{U}_i(\cdot)$, the mapping G_i has convex compact images. Additionally, G_i is locally Lipschitz continuous by virtue of (H_2^4).

Since $y = y_x^u \in \mathcal{M}_i$ on (a, b), then $\mathcal{U}_i$ has nonempty images and so G_i. We set $M_i = \mathbb{R} \times \mathcal{M}_i \times \mathbb{R}^2$ and define $\Gamma_i : M_i \rightrightarrows \mathbb{R}^{N+3}$ as

$$\Gamma_i(\tau, x, z, w) = \{-1\} \times G_i(\tau, x) \times \{0\}, \quad \forall (\tau, x, z, w) \in M_i.$$

Note that M_i is an embedded manifold of $\mathbb{R}^{N+3}$ and Γ_i satisfies the same conditions than G_i. Let $\mathcal{S}_i = \text{epi}(\psi_i)$ where

$$\psi_i(\tau, x, z) = \begin{cases} e^{-\lambda\tau}\varphi_i(x) + z & \text{if } \tau \geq 0,\ x \in \overline{\mathcal{M}_i}, \\ +\infty & \text{otherwise}, \end{cases} \quad \forall (\tau, x, z) \in \mathbb{R} \times \overline{\mathcal{M}_i} \times \mathbb{R}.$$

Then if (4.17) holds, the following also holds

$$\sup_{v \in \Gamma_i(\tau,x,z,w)} \langle \eta, v \rangle \leq 0 \qquad \forall (\tau, x, z, w) \in \mathcal{S}_i,\ \forall \eta \in \mathcal{N}_{\mathcal{S}_i}^P(\tau, x, z, w). \tag{4.21}$$

Indeed, if $\mathcal{S}_i = \emptyset$ it holds by vacuity. Otherwise, take $(\tau, x, z, w) \in \mathcal{S}_i$ and $(\xi, -p) \in \mathcal{N}_{\mathcal{S}_i}^P(\tau, x, z, w)$. Therefore, we have $p \geq 0$ because $\mathcal{S}_i$ is the epigraph of a function. Recall that $\Gamma_i(\tau,x,z,w) \neq \emptyset$ because $\mathcal{U}_i(x) \neq \emptyset$. Consider $p > 0$, then, by the same arguments used in Proposition 4.2.5, for any $v \in \Gamma_i(\tau,x,z,w)$ we have, for some $u \in \mathcal{U}_i(x)$, $r \geq 0$ and $\zeta \in \partial_P \varphi_i(x)$

$$\begin{aligned}\langle(\xi,-p),v\rangle &= pe^{-\lambda\tau}(\lambda\varphi_i(x) - \langle\zeta, f(x,u)\rangle - \ell(x,u) - r)\\ &\leq pe^{-\lambda\tau}(\lambda\varphi_i(x) - \langle\zeta, f(x,u)\rangle - \ell(x,u))\\ &\leq pe^{-\lambda\tau}(\lambda\varphi_i(x) + H_i(x,\zeta)).\end{aligned}$$

Since $\varphi_i(x) = \varphi(x)$, (4.17) holds and $v \in \Gamma_i(\tau,x,z,w)$ is arbitrary, we can take supremum over v to get (4.21). Similarly as done for Proposition 4.2.5, if $p = 0$ we use the Rockafellar Horizontal Theorem (Proposition 2.3.5) and the continuity of H_i (Proposition 4.2.2) to obtain (4.21) for any proximal normal η to $\mathcal{S}_i$.

Let $r > \tilde{r} > 0$ large enough so that $y_x^u([a,b]) \subseteq \mathbb{B}(0,\tilde{r})$ and

$$\sup_{X\in M\cap\mathbb{B}(0,\tilde{r})} |\mathrm{proj}_{\overline{M}_i\cap\mathcal{S}_i}(X)| < r.$$

Let L_i be the Lipschitz constant for Γ_i on $M_i \cap \mathbb{B}(0,r)$, so (4.21) implies condition (2.5) in Proposition 2.4.6 with $\kappa = L_i$. In particular, by Proposition 2.4.6 we have that, for any absolutely continuous arc $\gamma : [a,b] \to \overline{M}_i$ which satisfies (4.14) (with Γ_i instead of Γ) along with $\gamma(t) \in M_i$ for any $t \in (a,b)$, the following bound holds

$$\mathrm{dist}_{\mathcal{S}_i\cap\overline{M}_i}(\gamma(t)) \leq e^{L_i t}\mathrm{dist}_{\mathcal{S}_i\cap\overline{M}_i}(\gamma(a)) \qquad \forall t \in [a,b]. \tag{4.22}$$

Finally, take the absolutely continuous arc defined on $[a,b]$ by

$$\gamma(t) = \left(a-t, y(a+b-t), -\int_a^t e^{\lambda(s-a)}\ell(y(a+b-s), u_l(a+b-s))ds, \varphi(y(b))\right).$$

Since $\dot{\gamma} \in \Gamma_i(\gamma)$ a.e. on $[a,b]$, $\gamma(t) \in M_i$ for any $t \in (a,b)$ and $\gamma(a) \in \mathcal{S}_i$ we get that $\gamma(b) \in \mathcal{S}_i$ which implies (4.20) after some algebraic steps. ☐

Now we are in position to prove a converse result to Proposition 4.2.6.

Proposition 4.2.7. *Suppose that* (H_0^4), (H_1^4), (H_2^4) *and* (H_3^4) *hold in addition of* (H_f^4) *and* (H_ℓ^4). *Let* $\varphi : \mathcal{K} \to \mathbb{R}\cup\{+\infty\}$ *be a lower semicontinuous function with* $\mathrm{dom}\,\mathbb{U} \subseteq \mathrm{dom}\,\varphi$. *If* (4.17) *holds, then* φ *is strongly increasing for the controlled system.*

Proof. Let $x \in \mathrm{dom}\,\varphi$ and $u \in \mathbb{U}(x)$. We want to show that inequality (4.11) holds for $y = y_x^u$. For this purpose we fix $T > 0$ and we set $\mathcal{I}_T(y) = \{i \in \mathcal{I} :\ \exists t \in [0,T],\ y(t) \in \mathcal{M}_i\}$. Note that $\mathcal{I}_T(y)$ is finite because the stratification is locally finite and

$$[0,T] = \bigcup_{i\in\mathcal{I}_T(y)} J_i(y), \quad \text{with } J_i(y) := \{t \in [0,T] \mid\ y(t) \in \mathcal{M}_i\}.$$

We split the proof into two parts:

Step 1. Suppose first that each $J_i(y)$ can be written as the union of a finite number of intervals, this means that there exists a partition of $[0,T]$

$$\pi = \{0 = t_0 \leq t_1 \leq \ldots \leq t_n \leq t_{n+1} = T\},$$

so that if $t_l < t_{l+1}$ for some $l \in \{0, \dots, n\}$, then there exists $i_l \in \mathcal{I}_T(y)$ satisfying $(t_l, t_{l+1}) \subseteq J_{i_l}(y)$. Therefore, for any $l \in \{0, \dots, n\}$ such that $t_l < t_{l+1}$, Lemma 4.2.3 leads to

$$\varphi(y(t_l)) \leq e^{-\lambda(t_{l+1}-t_l)}\varphi(y(t_{l+1})) + e^{\lambda t_l}\int_{t_l}^{t_{l+1}} e^{-\lambda s}\ell(y,u)ds.$$

Hence, using inductively the previous estimation and noticing that $t_0 = 0$ and $t_{n+1} = T$ we get exactly (4.11) and the result follows.

Step 2. In general, the admissible trajectories may cross a stratum infinitely many times in arbitrary small periods of times. In order to deal with this general situation, we will use an inductive argument in the number of strata where the trajectory can pass, let us denote this number by κ. The induction hypothesis $(\mathcal{P}_\kappa)$ is:

> *Suppose $\mathcal{M}$ is the union of κ strata and $y(t) \in \mathcal{M}$ for every $t \in (a,b)$, where $0 \leq a < b \leq T$ then* (4.20) *holds.*

By Lemma 4.2.3, the induction property holds true for $\kappa = 1$ because the arc remains in only one stratum. So, let us assume that the induction hypothesis holds for some $\kappa \geq 1$ and prove that it also holds for $\kappa + 1$.

Suppose that for some $0 \leq a < b \leq t$, the arc y is contained in the union of $\kappa + 1$ strata on the interval (a,b). By the stratified structure of $\mathcal{K}$, we can always assume that there exists a unique stratum of minimal dimension (which may be disconnected) where the trajectory passes. We denote it by $\mathcal{M}_i$ and by $\mathcal{M}$ the union of the remaining κ strata. Note that, $\mathcal{M}_i \subseteq \overline{\mathcal{M}}$ and $\mathcal{M}$ is relatively open with respect to $\overline{\mathcal{M}}$. Two situations have to be considered:

Case 1: Suppose that $y([a,b]) \subseteq \mathcal{M} \cup \mathcal{M}_i$. Without loss of generality we can assume that $y(a), y(b) \in \mathcal{M}_i$. Therefore, $J := [a,b] \setminus J_i(y)$ is open, whereupon for any $\varepsilon > 0$ there exists a partition of $[a,b]$

$$b_0 := a \leq a_1 < b_1 \leq a_2 < b_2 \leq \ldots \leq a_n < b_n \leq b =: a_{n+1} \quad \text{with} \quad \text{meas}\left(J \setminus \bigcup_{l=1}^{n}(a_l, b_l)\right) \leq \varepsilon,$$

$y(a_l), y(b_l) \in J_i$ and $(a_l, b_l) \subseteq J$ for any $l = 1, \dots, n$. In particular, by the induction hypothesis we have

$$\varphi(y(a_l)) \leq e^{-\lambda(b_l-a_l)}\varphi(y(b_l)) + e^{\lambda a_l}\int_{a_l}^{b_l} e^{-\lambda s}\ell(y,u)ds. \tag{4.23}$$

Notice too that

$$\bigcup_{l=0}^{n}[b_l, a_{l+1}] \setminus J_i(y) = J \setminus \bigcup_{l=1}^{n}(a_l, b_l).$$

Hence, if we set $J^l := [b_l, a_{l+1}] \setminus J_i(y)$ and $\varepsilon_l = \text{meas}(J^l)$, we have $\sum_{l=0}^{n} \varepsilon_l \leq \varepsilon$.

We claim that there exists $L > 0$ so that

$$\varphi(y(b_l)) \leq e^{\lambda\varepsilon_l}\left(e^{-\lambda(a_{l+1}-b_l)}\varphi(y(a_{l+1})) + e^{\lambda b_l}\int_{b_l}^{a_{l+1}} e^{-\lambda s}\ell(y,u)ds\right) + L\varepsilon_l. \tag{4.24}$$

To see this, notice there exists a family of intervals $(\alpha_p, \beta_p) \subseteq [b_l, a_{l+1}]$ that is either finite or countable (in any case, pairwise disjoint with $\alpha_p < \beta_p$) such that $\varepsilon_l = \sum_{p\in\mathbb{N}}(\beta_p - \alpha_p)$, $y(t) \in \mathcal{M}$ for any $t \in (\alpha_p, \beta_p)$ and $y(\alpha_p), y(\beta_p) \in \mathcal{M}_i$. If the number of intervals turns out to be finite, then (4.24) follows by the same arguments as in Step 1. We assume then that $\{(\alpha_p, \beta_p)\}_{p\in\mathbb{N}}$ is an infinite family of pairwise disjoint intervals.

Let $r > 0$ so that $y(s) \in \mathbb{B}(0, r)$ for any $s \in [0, T]$, and $\varepsilon_i, \Delta_i > 0$ be the constants given by (H_3^4) associated with r. Reduce ε, if necessary, in order to guarantee that $\varepsilon_l < \varepsilon_i$. Given that for any $p \in \mathbb{N}$, $y(\beta_p) \in \mathcal{R}(y(\alpha_p), \beta_p - \alpha_p) \cap \overline{\mathcal{M}}_i$, by ($H_3^4$) we can pick $u_p : [0, +\infty) \to \mathcal{U}$ measurable and $s_p \in (0, \Delta_i(\beta_p - \alpha_p)]$ such that

$$y_p(t) \in \mathcal{M}_i,\ \forall t \in [\alpha_p, \alpha_p + s_p], \quad y_p(\alpha_p) = y(\alpha_p), \quad \text{and} \quad y_p(\alpha_p + s_p) = y(\beta_p)$$

where y_p is the solution to (1.1) associated with u_p.

Let $J_i^l := [b_l, a_{l+1}] \cap J_i(y)$ and the measurable function $\omega : [b_l, a_{l+1}] \to \mathbb{R}$

$$\omega(t) = \mathbb{1}_{J_i^l}(t) + \sum_{p\in\mathbb{N}} \frac{s_p}{\beta_p - \alpha_p}\mathbb{1}_{(\alpha_p,\beta_p)}(t) > 0, \quad \forall t \in [b_l, a_{l+1}].$$

Define $\nu(t) = b_l + \int_{b_l}^t \omega(s)ds$ for every $t \in [b_l, a_{l+1}]$. It is an absolutely continuous function, strictly increasing and bounded from above by $c_{l+1} := \nu(a_{l+1})$ on $[b_l, a_{l+1}]$, so it is a homeomorphism from $[b_l, a_{l+1}]$ into $[b_l, c_{l+1}]$.

Let $\tilde{u} : [b_l, c_{l+1}] \to \mathcal{U}$ measurable defined as

$$\tilde{u} = u(\nu^{-1})\mathbb{1}_{J_i^l}(\nu^{-1}) + \sum_{p\in\mathbb{N}} u_p \mathbb{1}_{(\alpha_p,\beta_p)}(\nu^{-1}), \quad \text{a.e. on } [b_l, c_{l+1}],$$

and let $\tilde{y}$ be the trajectory of (1.1) associated with u_p such that $\tilde{y}(b_l) = y(b_l)$. By construction $\tilde{y}(\nu(t)) = y(t)$ for any $t \in J_i^l$ and $\tilde{y}(t) \in \mathcal{M}_i$ for any $t \in [b_l, c_{l+1}]$. Hence by Lemma 4.2.3

$$\varphi(y(b_l)) \le e^{-\lambda(c_{l+1}-b_l)}\varphi(y(a_{l+1})) + e^{\lambda b_l}\int_{b_l}^{c_{l+1}} e^{-\lambda s}\ell(\tilde{y}(s), \tilde{u}(s))ds. \tag{4.25}$$

By the Change of Variable Theorem for absolutely continuous function (see for instance [87, Theorem 3.54]) we get

$$\int_{b_l}^{c_{l+1}} e^{-\lambda s}\ell(\tilde{y}(s), \tilde{u}(s))ds = \int_{b_l}^{a_{l+1}} e^{-\lambda\nu(s)}\ell(\tilde{y}(\nu(s)), \tilde{u}(\nu(s)))\nu'(s)ds.$$

Besides, $\ell(\tilde{y}(\nu), \tilde{u}(\nu))\nu' = \ell(y, u)$ a.e. on J_i^l and by (4.4)

$$\ell(\tilde{y}(\tau), \tilde{u}(\tau))\nu' \le L := \max\{1, \Delta_i\}c_\ell(1 + |x|)^{\lambda_\ell}e^{\lambda_\ell c_f(T+\Delta\varepsilon_l)} \quad \text{a.e. on } [b_l, a_{l+1}].$$

Due to $\ell \ge 0$ we find out that

$$\int_{b_l}^{c_{l+1}} e^{-\lambda s}\ell(\tilde{y}(s), \tilde{u}(s))ds \le \int_{b_l}^{a_{l+1}} e^{-\lambda\nu(s)}\ell(y, u)ds + L\varepsilon_l, \tag{4.26}$$

and we finally get (4.24) from (4.25) and (4.26) since

$$\nu(t) \ge b_l + \text{meas}(J_i^l \cap [b_l, t]) = t - \text{meas}([b_l, t] \cap J_l) \ge t - \varepsilon_l, \quad \forall t \in [b_l, a_{l+1}].$$

On the other hand, (4.23) and (4.24) yield to

$$\varphi(y(b_l)) \leq e^{\lambda\varepsilon_l}\left(e^{-\lambda(b_{l+1}-b_l)}\varphi(y(b_{l+1})) + e^{\lambda b_l}\int_{b_l}^{b_{l+1}} e^{-\lambda s}\ell(y,u)ds\right) + L\varepsilon_l.$$

Therefore, by using an inductive argument we can prove that

$$\varphi(y(b_0)) \leq e^{\lambda\sum_{l=0}^{n-1}\varepsilon_l}\left(e^{-\lambda(b_n-b_0)}\varphi(y(b_n)) + e^{\lambda b_0}\int_{b_0}^{b_n} e^{-\lambda s}\ell(y,u)ds\right) + L\left(\sum_{l=0}^{n-1}\varepsilon_l e^{\lambda\sum_{k=0}^{l-1}\varepsilon_k}\right),$$

and using (4.25) on the interval $[b_n, a_{n+1}]$ we get

$$\varphi(y(b_0)) \leq e^{\lambda\sum_{l=0}^{n}\varepsilon_l}\left(e^{-\lambda(a_{n+1}-b_0)}\varphi(y(a_{n+1})) + e^{\lambda b_0}\int_{b_0}^{a_{n+1}} e^{-\lambda s}\ell(y,u)ds\right) + L\left(\sum_{l=0}^{n}\varepsilon_l e^{\lambda\sum_{k=0}^{l-1}\varepsilon_k}\right).$$

Finally, by the definition of b_0 and a_{n+1} we obtain:

$$\varphi(y(a)) \leq e^{\lambda\varepsilon}\left(e^{-\lambda(b-a)}\varphi(y(b)) + e^{\lambda a}\int_{a}^{b} e^{-\lambda s}\ell(y,u)ds\right) + Le^{\lambda\varepsilon}\varepsilon.$$

Thus, letting $\varepsilon \to 0$, the induction hypothesis for $\kappa + 1$ holds true.

Case 2: We consider that $y(a) \notin \mathcal{M}\cup\mathcal{M}_i$ or $y(b) \notin \mathcal{M}\cup\mathcal{M}_i$.

Suppose first that $y(a) \notin \overline{\mathcal{M}}_i \setminus \mathcal{M}_i$ and $y(b) \notin \overline{\mathcal{M}}_i \setminus \mathcal{M}_i$, then there exists $\delta > 0$ such that $y(t) \in \mathcal{M}\cup\mathcal{M}_i$ for every $t \in [a+\delta, b-\delta]$ and $\mathrm{dist}_{\overline{\mathcal{M}}_i\setminus\mathcal{M}_i}(y(t)) > 0$ on $[a, a+\delta]\cup[b-\delta, b]$. So, we can decompose $[0, T]$ into three parts $[a, a+\delta]$, $[a+\delta, b-\delta]$ and $[b-\delta, b]$, so that $y(s) \in \mathcal{M}$ for any $s \in (a, a+\delta)\cup(b-\delta, b)$ with $y(a) \in \overline{\mathcal{M}}\setminus\mathcal{M}$ or $y(b) \in \overline{\mathcal{M}}\setminus\mathcal{M}$.

In view of Case 1 and the inductive hypothesis, (4.20) holds in each of the previous intervals. Gathering the three inequalities we get the induction hypothesis for $\kappa + 1$.

Secondly, suppose that only $y(a) \notin \mathcal{M}\cup\mathcal{M}_i$, then there exists a sequence $\{a_n\} \subseteq (a, b)$ such that $a_n \to a$ and $y([a_n, b]) \subseteq \mathcal{M}\setminus\mathcal{M}_i$. By Case 1,

$$\varphi(y(a_n)) \leq e^{-\lambda(b-a_n)}\varphi(y(b)) + e^{\lambda a_n}\int_{a_n}^{b} e^{-\lambda s}\ell(y_x^u, u)ds.$$

Furthermore, since φ is lower semicontinuous and $y(\cdot)$ is continuous we can pass to the limit to get (4.20), so the result also holds in this situation.

Finally, it only remains the case $y(b) \in \overline{\mathcal{M}}_i\setminus\mathcal{M}_i$. Similarly as above, we can take a sequence $\{b_n\} \subseteq (a, b)$ such that $b_n \to b$ and $y([a, b_n]) \subseteq \mathcal{M}\setminus\mathcal{M}_i$ such that

$$\varphi(y(a)) \leq e^{-\lambda(b_n-a)}\varphi(y(b_n)) + e^{\lambda a}\int_{a}^{b_n} e^{-\lambda s}\ell(y_x^u, u)ds.$$

By (H_3^4), for $n \in \mathbb{N}$ large enough, there exists a control $u_n : (b_n, b+\delta_n) \to \mathcal{U}$ and a trajectory $y_n : [b_n, b+\delta_n] \to \overline{\mathcal{M}}_i$ with $y_n(b_n) = y(b_n)$, $y_n(b+\delta_n) = y(b)$ and $y_n(t) \in \mathcal{M}_i$ for any $t \in [b_n, b+\delta_n)$. By Lemma 4.2.3

$$\varphi(y(b_n)) \leq e^{-\lambda(b-b_n)}\varphi(y(b)) + \varepsilon_n,$$

with $\varepsilon_n \to 0$ as $n \to +\infty$, then gathering both inequalities and letting $n \to +\infty$ we get the induction hypothesis and the proof is complete.

□

4.2.4 Application to networks.

A particular framework of interest is when $\mathcal{K}$ is a network as in Figure 4.3a. This setting has been studied for many authors on different contexts; see for instance [1, 75, 2]. To give a precise definition of a network, we first set up the notion of junction.

Definition 4.2.3. *We say that $o \in \mathbb{R}^N$ is a junction provided there exist $r > 0$ and a family $\{\mathcal{M}_1, \dots, \mathcal{M}_p\}$ of connected and pairwise disjoint embedded manifolds of $\mathbb{R}^N$ such that*

$$\{o\} = \left(\overline{\mathcal{M}}_i \setminus \mathcal{M}_i\right) \cap \mathbb{B}(o,r) \quad \text{and} \quad \dim(\mathcal{M}_i) = 1, \ \forall i \in \{1, \dots, p\}.$$

We denote by $\mathcal{B}(o) = \{\mathcal{M}_1, \dots, \mathcal{M}_p\}$ the set of branches associated with o.

Now we define a network as a collection of junctions and branches.

Definition 4.2.4. *A connected set $\mathcal{K} \subseteq \mathbb{R}^N$ is called a network provided there exists $\{o_i\}_{i \in \mathcal{I}_J}$, a locally finite and pairwise different family of junctions such that*

$$\mathcal{K} = \bigcup_{i \in \mathcal{I}_J} \left(\{o_i\} \cup \bigcup_{\mathcal{M} \in \mathcal{B}(o_i)} \mathcal{M} \right).$$

Since on any network as in Definition 4.2.4, there is (at most) a countable number of branches, we can find $\mathcal{I}_B \subseteq \mathbb{N}$ with $\mathcal{I}_J \cap \mathcal{I}_B \neq \emptyset$ with the property that for every $j \in \mathcal{I}_J$ and $\mathcal{M} \in \mathcal{B}(o_i)$ there is a unique $i \in \mathcal{I}_B$. Hence, with a slight abuse of notation, the collection of branches on a network are written as $\{\mathcal{M}_i\}_{i \in \mathcal{I}_B}$.

In the rest of the section we are going to assume that

$(H^4_{0,N})$ $\qquad$ $\mathcal{K}$ is a closed network on $\mathbb{R}^N$.

Note that in particular, $(H^4_{0,N})$ implies (H^4_0) by setting $\mathcal{I} = \mathcal{I}_J \cup \mathcal{I}_B$. Moreover, on the junctions the assumption (H^4_2) holds immediately, because they are single points and so

$$\mathcal{U}_i(o_i) = \{u \in \mathcal{U} \mid \ f(o_i, u) = 0\}, \quad \forall i \in \mathcal{I}_J.$$

Consequently, (H^4_2) can be weakened to demand the Lipschitz continuity over the tangent controls solely on the branches:

$(H^4_{2,N})$ $\qquad$ $\forall i \in \mathcal{I}_B$, $\mathcal{U}_i$ is locally Lipschitz continuous on $\mathcal{M}_i$.

Remark 4.2.10. *In many situation, it is natural to assume that for any $j \in \mathcal{I}_B$, $\exists \mathbf{U}_i \subseteq \mathcal{U}$ such that $\mathcal{U}_i(x) = \mathbf{U}_i$ for any $x \in \mathcal{M}_i$, that is, the set of tangent controls is constant. If this occurs, as we have indicated in Remark 4.2.3, $(H^4_{2,N})$ holds immediately and then, f and ℓ can be taken locally Lipschitz continuous exclusively with respect to the state in (H^4_f) and (H^4_ℓ), respectively.*

The structure of the problem allows us to take a smaller Hamiltonian to characterize the supersolution principle.

Proposition 4.2.8. *Suppose that* (H_f^4), (H_ℓ^4), $(H_{0,N}^4)$, (H_1^4) *and* $(H_{2,N}^4)$ *hold. Consider a given lower semicontinuous function* $\varphi : \mathcal{K} \to \mathbb{R} \cup \{+\infty\}$ *with* $dom\,\varphi \subseteq dom\,\mathbb{U}$. *Then* φ *is weakly decreasing for the control system if and only if*

$$\lambda\varphi(x) + H_i(x,\zeta) \geq 0 \qquad \forall x \in \mathcal{M}_j,\ \forall \zeta \in \partial_P\varphi(x),\ \forall i \in \mathcal{I}_B \tag{4.27}$$

$$\lambda\varphi(o_i) + H(o_i,\zeta) \geq 0 \qquad \forall \zeta \in \partial_P\varphi(o_i),\ \forall i \in \mathcal{I}_J. \tag{4.28}$$

Proof. First of all, if (4.27) and (4.28) hold, (4.13) is verified as well, because in the last case, the supremum is taken over a bigger set. So, by Proposition 4.2.5 φ is weakly decreasing.

Conversely, by the same arguments used for Proposition 4.2.5, for any $x \in \mathcal{K}$ for which $\partial_P\varphi(x) \neq \emptyset$ we can find $u \in \mathbb{U}(x)$. Furthermore, given that φ is weakly decreasing, for any $\zeta \in \partial_P\varphi(x)$ there exist $\delta, \sigma > 0$ such that for any $t \in (0,\delta)$

$$(1 - e^{\lambda t})\varphi(x) + \int_0^t [\langle \zeta, f(y_x^u(s), u(s))\rangle + \ell(y_x^u(s), u(s))]\, ds \leq \sigma |y_x^u(t) - x|^2.$$

Notice that $u(\cdot + s) \in \mathbb{U}(y_x^u(s))$ for every $s \in [0,+\infty)$. Hence, if $x \in \mathcal{M}_i$ for some $i \in \mathcal{I}_B$, by reducing δ if necessary, $u(s) \in \mathcal{U}_i(y_x^u(s))$ for a.e. $s \in (0,\delta)$, which implies that

$$(1 - e^{\lambda t})\varphi(x) - \int_0^t H_i(y_x^u(s), \zeta) ds \leq \sigma |y_x^u(t) - x|^2, \quad \forall t \in (0,\delta).$$

Consequently by the continuity of H_i (see Proposition 4.2.2), dividing by $t > 0$ and letting $t \to 0$ we get (4.27) so the proof is complete. $\square$

Remark 4.2.11. *In the foregoing result the dimension of* $\mathcal{M}_i$ *does not play any role, the only issue that matters is that* $\mathcal{K}$ *matches locally with* $\mathcal{M}_i$ *around any* $x \in \mathcal{M}_i$, *that is,*

$$\forall x \in \mathcal{M}_i, \exists \delta > 0 \text{ so that } \mathcal{K} \cap \mathbb{B}(x,\delta) \subseteq \mathcal{M}_i.$$

With this adapted version of the weakly decreasing principle we get the following characterization of the Value Function on networks.

Theorem 4.2.2. *Suppose that* (H_f^4), (H_ℓ^4), (H_1^4), $(H_{0,N}^4)$ *and* $(H_{2,N}^4)$ *hold and* $\lambda > \lambda_\ell c_f$. *Then the Value Function* $\vartheta(\cdot)$ *of the infinite horizon problem is the only lower semicontinuous function with* λ_ℓ*-superlinear growth which is* $+\infty$ *on* $\mathbb{R}^N \setminus \mathcal{K}$ *and that satisfies*

$$\lambda\vartheta(x) + H_i(x,\zeta) = 0 \quad \forall x \in \mathcal{M}_i,\ \forall \zeta \in \partial_V\vartheta(x),\ \forall i \in \mathcal{I}_B \tag{4.29}$$

$$\lambda\vartheta(o_i) + H(o_i,\zeta) \geq 0 \qquad \forall \zeta \in \partial_V\vartheta(o_i),\ \forall i \in \mathcal{I}_J. \tag{4.30}$$

$$\lambda\vartheta(o_i) - \inf_{u \in \mathcal{U}_i(o_i)} \ell(o_i,u) \leq 0 \qquad \forall i \in \mathcal{I}_J. \tag{4.31}$$

Proof. As discussed earlier, $(H_{0,N}^4)$ and $(H_{2,N}^4)$ imply (H_0^4) and (H_2^4), respectively. Furthermore, since each o_i is a single point, (H_3^4) holds as well. Therefore, the conclusion follows from Theorem 4.2.1, Remarks 4.2.4, 4.2.7 and 4.2.9, and by using Proposition 4.2.8 instead of Proposition 4.2.5 to characterize the supersolution principle. $\square$

Remark 4.2.12. *It is worthy to note that Theorem 4.2.2 can be compared with other notions of continuous solutions already introduced in the literature. Closest notion is the one introduced by Achdou-Camilli-Cutrì-Tchou in [1]. The equation proposed by those authors at the branches is equivalent (in the continuous framework) to* (4.29) *and the junction condition coming from the supersolution principle* (4.30) *seems to be equivalent under the geometric additional assumptions done in [1]. However, in our analysis we have shown that* (4.31)*, the junction condition associated with the subsolution principle, does not require an evaluation at test functions, so our condition appears to be weaker than that of [1].*

4.3 Mayer problems.

The present section aims to extend the technique exhibited in the last sections to optimal processes in which the control does not appear explicitly on the cost to be minimize. In this setting, it is more suitable to write the dynamical constraints as a differential inclusion rather than as a controlled ordinary differential equation.

For sake of simplicity, we only consider problems with fixed final time. If the final time is free, the arguments are similar and the hypotheses are the same.

We recall that the Value function of the Mayer problem with given final horizon $T > 0$ is

$$\vartheta(t,x) := \inf\left\{\psi(y(T)) \mid \quad y \in \mathbb{S}_t^T(x)\right\}, \quad \forall (t,x) \in [0,T] \times \mathcal{K},$$

where $\mathbb{S}_a^b(x)$ stands for the set of admissible trajectories defined on the interval $[a,b]$ with initial condition $y(a) = x$. Under these circumstances, the HJB equation is

$$-\partial_t \vartheta(t,x) + H(t,x,\nabla_x \vartheta(t,x)) = 0, \quad \text{on } (0,T) \times \mathcal{K}.$$

All along the section we assume that the final cost $\psi : \mathbb{R}^N \to \mathbb{R} \cup \{+\infty\}$ satisfies

$$(H_\psi^4) \qquad \psi(\cdot) \text{ is lower semicontinuous and bounded from below on } \mathcal{K}.$$

Remark 4.3.1. *In the formulation of the Mayer problem it is possible to consider implicitly a final constraint $\Theta \subseteq \mathbb{R}^N$ of the form*

$$y(T) \in \Theta, \quad \forall y \in \mathbb{S}_t^T(x).$$

To do this, it is enough to replace ψ with $\psi_\Theta : \mathbb{R}^N \to \mathbb{R} \cup \{+\infty\}$ defined by

$$\psi_\Theta(x) := \begin{cases} \psi(x) & \text{if } x \in \Theta, \\ +\infty & \text{otherwise}, \end{cases} \quad \forall x \in \mathbb{R}^N.$$

If Θ is a closed set and (H_ψ^4) *holds, then ψ_Θ verifies* (H_ψ^4) *as well.*

The set of dynamics $F : \mathbb{R}^N \rightrightarrows \mathbb{R}^N$ is initially taken as to verify

$$(H_F^4) \qquad \begin{cases} i) & F \text{ is upper semicontinuous on } \mathbb{R}^N. \\ ii) & F \text{ has nonempty compact convex images on a neighborhood of } \mathcal{K}. \\ iii) & \exists c_F > 0 \text{ so that } \max\{|v| \mid \ v \in F(x)\} \le c_F(1+|x|),\ \forall x \in \mathcal{K}. \end{cases}$$

In the light of Proposition 2.4.3, the assumptions over the dynamics guarantee, for every $(t, x) \in [0, T) \times \mathcal{K}$ the existence of $\delta > 0$ and an absolutely continuous curve $y : [t, t+\delta] \to \mathbb{R}^N$ which solves

$$\dot{y}(s) \in F(y(s)), \qquad \text{for a.e. } s \in [t, t+\delta], \qquad y(t) = x.$$

The foregoing trajectory may not be feasible, not even for small times. Anyhow, if the trajectory lives in $\mathcal{K}$ on $[t, t+\delta]$, then the Gronwall's Lemma (Proposition 2.4.1) leads to

$$|y(s)| \leq (1 + |x|)e^{c_F(s-t)}, \quad \forall s \in [t, t+\delta]. \tag{4.32}$$

Remark 4.3.2. *In contrast with the former section, there is not need to consider an augmented dynamics mapping; this is essentially due to the absence of the control on the objective function and to the fact that the dynamics is presupposed convex-valued.*

4.3.1 The Value Function and compatibility assumptions

We have already discussed that the Value Function is likely to be lower semicontinuous as long as the dynamics maps has convex images. The next proposition provides a precise statement for the Mayer problem.

Proposition 4.3.1. *Suppose that (H_ψ^4) and (H_F^4) hold. Then, if $\vartheta(t, x) \in \mathbb{R}$ there exists an optimal trajectory $\bar{y} \in \mathbb{S}_t^T(x)$ for the Mayer problem. Furthermore, $\vartheta : [0, T] \times \mathcal{K} \to \mathbb{R} \cup \{+\infty\}$ is lower semicontinuous.*

This proposition is well-known and its proof usually follows the same scheme as Proposition 4.2.3. For this reason we only provide a sketch of the proof.

Sketch of the proof of Proposition 4.3.1. Let $(t, x) \in [0, T] \times \mathcal{K}$ so that $\vartheta(t, x) \in \mathbb{R}$. The bound (4.32) together with [11, Theorem 0.3.4] and the Convergence Theorem (Proposition 2.4.4) yield to the existence of a minimizing subsequence that converges uniformly to some $\bar{y} \in \mathbb{S}_t^T(x)$ and whose weak derivative converges weakly to $\dot{\bar{y}}$ in $L^1([t, T], \mathbb{R}^N)$. Thus, the lower semicontinuity of ψ implies the optimality of $\bar{y}$.

For the lower semicontinuity of the Value Function, if $\{(t_n, x_n)\} \subseteq \operatorname{dom} \vartheta$ converges to some (t, x), it is enough to take $y_n \in \mathbb{S}_{t_n}^T(x_n)$ optimal and use the same compactness arguments as above to prove that y_n has a subsequence that converges to an element of $\mathbb{S}_t^T(x)$ and then we use the definition of the Value Function to conclude. □

Before going further, we require to introduce some notation and to disclose the compatibility assumptions under which the theorem of the section is stated.

We recall that $\mathcal{T}_{\mathcal{K}}^B(\cdot)$ stands for the Bouligand tangent cone to $\mathcal{K}$. We are going to relate the supersolution principle with a smaller Hamiltonian than the classical one. For this purpose, we write $F^\sharp : \mathcal{K} \rightrightarrows \mathbb{R}^N$ for the multivalued map defined via

$$F^\sharp(x) := F(x) \cap \mathcal{T}_{\mathcal{K}}^B(x), \quad \forall x \in \mathcal{K}.$$

As done for the infinite horizon problem, the subsolution principle will be associated with a different Hamiltonian on each stratum. Let us define, for each index $i \in \mathcal{I}$, the multifunction $F_i : \mathcal{M}_i \rightrightarrows \mathbb{R}^N$ as follows

$$F_i(x) := F(x) \cap \mathcal{T}_{\mathcal{M}_i}(x), \quad \forall x \in \mathcal{M}_i.$$

In accordance with the definitions introduced for the infinite horizon case, we call this set-valued map the *tangent dynamics to* $\mathcal{M}_i$. Given that $\mathcal{T}_{\mathcal{M}_i}(\cdot)$ is compactly upper semicontinuous on $\mathcal{M}_i$ (Proposition 3.2.4) and $F(\cdot)$ is upper semicontinuous all along $\mathcal{K}$, we get that $F_i(\cdot)$ is as well upper semicontinuous on $\mathcal{M}_i$. Furthermore, its images are compact convex, optionally nonempty, sets of $\mathbb{R}^N$.

At the present section, the tangent dynamics to a stratum play a similar role as the tangents controls in Section 4.2. Consequently, all the theory we develop from this point on is done under the following assumption:

(H_4^4) Each F_i is locally Lipschitz continuous on $\mathcal{M}_i$ for the Hausdorff distance.

We recall that we have adopted the convention $d_H(\emptyset, \mathcal{S}) = +\infty$ for $\mathcal{S} \neq \emptyset$. Thus, (H_4^4) implies that the images of $F_i(\cdot)$ are either empty or nonempty throughout $\mathcal{M}_i$.

Moreover, under this framework we also require a controllability condition in order to prove the final result of the section. We evoke from Section 4.2 that $\mathcal{R}(t,x;s)$ stands for the reachable set at time s of curves with initial condition $y(t) = x$. We consider likewise the reachable set through the stratum $\mathcal{M}_i$ which corresponds to the set of all possible positions that can be attained by an admissible trajectory lying entirely in $\mathcal{M}_i$:

$$\mathcal{R}_i(t,x;s) := \bigcup_{y \in \mathbb{S}_t^s(x)} \{y(s) \mid \ y(\tau) \in \mathcal{M}_i, \ \forall \tau \in [t,s)\}, \quad \forall x \in \mathcal{M}_i, \ \forall t,s \in \mathbb{R}, \ t < s.$$

Therefore, the controllability hypothesis we demand is stated as follows:

$$(H_5^4) \qquad \begin{cases} \forall r > 0, \forall\ i \in \mathcal{I}, \text{ if } \operatorname{dom} F_i \neq \emptyset, \ \text{ then } \exists \varepsilon_i, \Delta_i > 0 \text{ so that } \forall x \in \mathcal{M}_i \cap \mathbb{B}(0,r) \\ \mathcal{R}(t,x;s) \cap \overline{\mathcal{M}}_i \subseteq \displaystyle\bigcup_{r \in [t, t+\Delta_i s]} \mathcal{R}_i(t,x;r), \ \forall t \in [0,T], \forall s \in [t, t+\varepsilon_i]. \end{cases}$$

This assumption is the adapted version of hypothesis (H_3^4) for the present framework, and so, as aforementioned, the full controllability condition on manifolds

$$\forall i \in \mathcal{I} \text{ with } \operatorname{dom} F_i \neq \emptyset. \ \exists r_i > 0 \text{ such that } \mathcal{T}_{\mathcal{M}_i}(x) \cap \mathbb{B}(0, r_i) \subseteq F_i(x), \quad \forall x \in \mathcal{M}_i,$$

is a sufficient condition for (H_5^4) to be fulfilled.

Under these assumptions, we have obtained the following statement.

Theorem 4.3.1. *Suppose (H_0^4), (H_4^4) and (H_5^4) hold along with (H_ψ^4) and (H_F^4). Then the Value Function of the Mayer problem is the unique lower semicontinuous function on $[0,T] \times \mathcal{K}$ which is $+\infty$ outside $[0,T] \times \mathcal{K}$ and that verifies*

$$\begin{aligned} -\theta + \max_{v \in F^\sharp(x)} \{-\langle v, \zeta \rangle\} \geq 0, &\quad \forall (t,x) \in [0,T) \times \mathcal{K}, \ \forall (\theta, \zeta) \in \partial_V \vartheta(t,x), \\ -\theta + \max_{v \in F_i(x)} \{-\langle v, \zeta \rangle\} \leq 0, &\quad \forall i \in \mathcal{I}, \ \forall (t,x) \in (0,T] \times \mathcal{M}_i, \ \forall (\theta, \zeta) \in \partial_V \vartheta_i(t,x), \\ \vartheta(T,x) = \psi(x), &\qquad x \in \mathcal{K}, \end{aligned}$$

where ϑ_i stands for the function that agrees with ϑ on $[0,T] \times \overline{\mathcal{M}}_i$ and is $+\infty$ elsewhere.

Remark 4.3.3. *In the known literature (for instance [51, 54, 131]) it is usual to write the supersolution inequality only on $(0,T)\times\mathcal{K}$ and complement the information at time $t=0$ with the limit condition*

$$\liminf_{t\to 0^+,\ \tilde{x}\to x} \vartheta(t,\tilde{x}) = \vartheta(0,x), \quad \forall x\in\mathcal{K}.$$

In our setting, we have chosen to use another condition at time $t=0$, namely

$$-\theta + \max_{v\in F^\sharp(x)} \{-\langle v,\zeta\rangle\} \geq 0, \quad \forall x\in\mathcal{K},\ \forall(\theta,\zeta)\in\partial_V\vartheta(0,x).$$

It is worthy to note that both condition are equivalent under the right conditions; see for instance the discussion in [132, Section 4]. A similar limit condition has also been used in the literature to complete the information provided by the subsolution inequality at time $t=T$. Notice that in Theorem 4.3.1 the subsolution inequality has been written up to time $t=T$, which explain why we do not require such limit condition.

The proof of the above-stated result is composed of a part that is rather standard and another which uses stratified techniques. In any case, we make use of the monotone properties of the Value Function along trajectories which for this case are as follows.

Definition 4.3.1. *A function $\varphi:[0,T]\times\mathcal{K}\to\mathbb{R}\cup\{+\infty\}$ is said to be:*

i) weakly decreasing along trajectories of $\mathbb{S}^T$ provided for all $(t,x)\in dom\,\varphi$ we can find a curve $y\in\mathbb{S}_t^T(x)$ so that $\varphi(s,y(s))\leq\varphi(t,x)$ for all $t\leq s\leq T$.

ii) strongly increasing along trajectories of $\mathbb{S}^T$ if for each $(t,x)\in[0,T]\times\mathcal{K}$ and each $y\in\mathbb{S}_t^T(x)$, we have $\varphi(s,y(s))\geq\varphi(t,x)$ for all $t\leq s\leq T$.

Similarly as done for the infinite horizon problem, we can also state a comparison lemma for the Mayer problem.

Lemma 4.3.1. *Let $\varphi:[0,T]\times\mathcal{K}\to\mathbb{R}\cup\{+\infty\}$ satisfying $\varphi(T,x)=\psi(x)$ for all $x\in\mathcal{K}$.*

1. *If φ is weakly decreasing along trajectories of $\mathbb{S}^T$, then $\vartheta(t,x)\leq\varphi(t,x)$ on $[0,T]\times\mathcal{K}$.*

2. *If φ is strongly increasing along trajectories of $\mathbb{S}^T$, then $\vartheta(t,x)\geq\varphi(t,x)$ on $[0,T]\times\mathcal{K}$.*

Proof. It is enough to evaluate each inequality at $s=T$, use the end-point condition and the definition of the Value Function. □

We evoke that the Value Function of the Mayer problem solves the functional equation

$$\vartheta(t,x)=\inf\left\{\vartheta(s,y(s))\mid\ y\in\mathbb{S}_t^T(x)\right\},\quad\forall x\in\mathcal{K},\quad\forall 0\leq t\leq s\leq T.$$

The preceding lemma leads to assert, as in Proposition 4.2.4, that the Value Function is the unique function being weakly decreasing and strongly increasing along trajectories of $\mathbb{S}^T$ at the same time. So, to prove Theorem 4.3.1 it suffices to find equivalent formulation for the monotone properties in terms of HJB inequalities.

4.3.2 Decreasing principle

The characterization of the weakly decreasing property is stated below. As for the infinite horizon problem, this is rather classical. However, the novelty on the statement is that the equation is written with a smaller Hamiltonian that only consider the viable velocities, i.e., those that belong to the Bouligand tangent cone to $\mathcal{K}$. We want to emphasis that in Section 4.2 the weakly decreasing principle was characterized in Proposition 4.2.5 with the usual Hamiltonian, and then it was suggested in Proposition 4.2.8 that the Hamiltonian can be taken smaller. In this section we concretize this idea by exhibiting a characterization with the Hamiltonian associated with the dynamics $F^\sharp$ we have introduced earlier.

We begin with presenting (without proving) a result which is somehow classical and well-known, that is, the characterization of the weakly decreasing principle by means of the usual Hamiltonian. The next lemma can be proved using the same scheme as for Proposition 4.2.5. For the unconstrained case its prove can be found in [51, 53, 41, 132].

Lemma 4.3.2. *Suppose $\mathcal{K}$ is closed and (H_ψ^4) holds along with (H_F^4). Consider a lower semicontinuous function $\varphi : [0,T]\times\mathcal{K}\to\mathbb{R}\cup\{+\infty\}$ verifying $\varphi(T,x)=\psi(x)$ for all $x\in\mathcal{K}$. Then φ is weakly decreasing along trajectories of $\mathbb{S}^T$ if and only if*

$$-\theta + H(x,\zeta)\geq 0 \quad \textit{for all } (\theta,\zeta)\in\partial_V\varphi(t,x),\quad \forall (t,x)\in[0,T)\times\mathcal{K}. \tag{4.33}$$

We now show that a function verifies the weakly decreasing property if and only it is supersolution principle of the HJB equation associated with the dynamics $F^\sharp$.

Proposition 4.3.2. *Suppose $\mathcal{K}$ is closed and (H_ψ^4) holds along with (H_F^4). Consider a lower semicontinuous function $\varphi : [0,T]\times\mathcal{K}\to\mathbb{R}\cup\{+\infty\}$ verifying $\varphi(T,x)=\psi(x)$ for all $x\in\mathcal{K}$. Then φ is weakly decreasing along trajectories of $\mathbb{S}^T$ if and only if*

$$-\theta + \max_{v\in F^\sharp(x)}\{-\langle v,\zeta\rangle\}\geq 0 \quad \textit{for all } (\theta,\zeta)\in\partial_V\varphi(t,x),\quad \forall (t,x)\in[0,T)\times\mathcal{K}. \tag{4.34}$$

Proof. Notice first that if (4.34) holds then (4.33) is satisfied too, because of $F^\sharp(x)\subseteq F(x)$ on $\mathcal{K}$. Consequently, the sufficient implication holds immediately by means of Lemma 4.3.2.

Hence, it only remains to show that φ being weakly decreasing along trajectories of $\mathbb{S}^T$ implies that (4.34) holds. If $\varphi(t,x)=+\infty$, then $\partial_V\varphi(t,x)=\emptyset$ meaning that (4.34) is trivial. So we might exclusively assume $\varphi(t,x)<\infty$.

Let $(\theta,\zeta)\in\partial_V\varphi(t,x)$, then in particular by Proposition 2.3.8, $\partial_V\varphi(t,x)=\partial_F\varphi(t,x)$ and so we have that for any sequence $\{(s_n,x_n)\}$ converging to (t,x) the following holds true:

$$\liminf_{n\to+\infty}\frac{\varphi(s_n,x_n)-\varphi(t,x)-\theta(s_n-t)-\langle\zeta,x_n-x\rangle}{|x_n-x|+|s_n-t|}\geq 0. \tag{4.35}$$

By the weak decreasing property, there is $y\in\mathbb{S}_t^T(x)$ so that

$$\varphi\big(s,y(s)\big)\leq\varphi(t,x)\quad \text{for all } t\leq s\leq T. \tag{4.36}$$

Now choose any sequence $\{s_n\}\subseteq(t,T]$ so that $s_n\to t$ and $v_n:=\frac{y(s_n)-x}{s_n-t}\to v$. It is clear that $y(s_n)\to x$. This is always possible because F is locally bounded. We claim that $v\in F^\sharp(x)$. To see this notice that

$$\int_0^1\gamma_n(\lambda)d\lambda = v_n\to v,\quad \text{with } \gamma_n(\lambda):=\dot{y}(\lambda s_n+(1-\lambda)t)$$

Take $\varepsilon > 0$ arbitrary. Since F is upper semicontinuous at x there is $n_\varepsilon \in \mathbb{N}$ so that

$$F(y(\lambda s_n + (1-\lambda)t)) \subseteq F(x) + \overline{\mathbb{B}}(0,\varepsilon), \quad \forall n \geq n_\varepsilon,\ \forall \lambda \in [0,1].$$

Since $F(x) + \overline{\mathbb{B}}(0,\varepsilon)$ is a compact convex set and $\gamma_n(\lambda) \in F(y(\lambda s_n + (1-\lambda)t))$ a.e. on $[0,1]$, by [119, Lemma 4.2] we have that

$$v_n = \int_0^1 \gamma_n(\lambda) d\lambda \in F(x) + \overline{\mathbb{B}}(0,\varepsilon), \quad \forall n \geq n_\varepsilon.$$

Letting $n \to +\infty$ we find out that $v \in F(x) + \overline{\mathbb{B}}(0,\varepsilon)$. Moreover, since $\varepsilon > 0$ is arbitrary, we get that $v \in \overline{F(x)} = F(x)$. Furthermore, since $y(s_n) \in \mathcal{K}$ for all $n \in \mathbb{N}$, we have that $v \in F(x) \cap \mathcal{T}^B_{\mathcal{K}}(x) = F^\sharp(x)$, so the claim holds true.

Now, setting $x_n = y(s_n)$ and using (4.36) we get for any $n \in \mathbb{N}$

$$\frac{\varphi(s_n,x_n) - \varphi(t,x) - \theta(s_n - t) - \langle \zeta, x_n - x\rangle}{|x_n - x| + |s_n - t|} \leq \frac{-\theta(s_n - t) - \langle \zeta, x_n - x\rangle}{|x_n - x| + |s_n - t|} \tag{4.37}$$

Besides, it is not difficult to see that

$$\frac{-\theta(s_n - t) - \langle \zeta, x_n - x\rangle}{|x_n - x| + |s_n - t|} \to \frac{-\theta - \langle \zeta, v\rangle}{|v| + 1}, \quad \text{as } n \to +\infty.$$

Thus, by virtue of (4.35), letting $n \to \infty$ in (4.37), we find out that

$$-\theta + \max_{v \in F^\sharp(x)} \{-\langle v, \zeta\rangle\} \geq -\theta - \langle v, \zeta\rangle \geq 0.$$

Finally, given that (t,x) and (θ,ζ) are arbitrary the conclusion follows.

□

4.3.3 Increasing principle

The last step required in the proof of Theorem 4.3.1 is the characterization of strongly increasing functions along the trajectories of the controlled system.

The following result is the corresponding version for the Mayer problem of Proposition 4.2.6 and Proposition 4.2.7.

Remark 4.3.4. *We would like to emphasis that the necessary condition in the next proposition holds under weaker assumptions. Actually, the controllability assumption is not al all required (as in Proposition 4.2.6) and the Lipschitz continuity hypothesis (H_4^4) can be relaxed to lower semicontinuity with nonempty images.*

Proposition 4.3.3. *Suppose (H_0^4), (H_4^4) and (H_5^4) hold along with (H_ψ^4) and (H_F^4). Consider a lower semicontinuous function $\varphi : [0,T] \times \mathcal{K} \to \mathbb{R} \cup \{+\infty\}$ verifying $\varphi(T,x) = \psi(x)$ for all $x \in \mathcal{K}$. Then φ is strongly increasing along trajectories of $\mathbb{S}^T$ if and only if*

$$-\theta + \max_{v \in F_i(x)} \{-\langle v, \zeta\rangle\} \leq 0 \quad \forall i \in \mathcal{I},\ \forall (t,x) \in (0,T] \times \mathcal{K},\ (\theta,\zeta) \in \partial_V \varphi_i(t,x), \tag{4.38}$$

where $\varphi_i = \varphi$ over $\overline{\mathcal{M}}_i$ and $+\infty$ elsewhere.

Proof. Notice first that if $\operatorname{dom} F_i = \emptyset$ for some $i \in \mathcal{I}$, then (4.38) does not provide any information and holds trivially.

The implication ($\Rightarrow$) is proven as follows. Take $i \in \mathcal{I}$ so that $\operatorname{dom} F_i \neq \emptyset$, then ($H_4^4$) implies that F_i is locally Lipschitz continuous on $\mathcal{M}_i$. Consequently, it is lower semicontinuous and its images are nonempty compact convex sets of $\mathbb{R}^N$. Take $(t,x) \in (0,T] \times \mathcal{M}_i$ and $v \in F_i(x)$ fixed but arbitrary. By the Michael's Selection Theorem (Proposition 2.2.4), there exists a continuous selection f_i of F_i that verifies $f_i(x) = v$. By the Nagumo's Theorem (Proposition 2.4.2), there exist $\delta > 0$ with $t - \delta \geq 0$ and a continuously differentiable trajectory of the control system $y \in \mathbb{S}_{t-\delta}^{t+\delta}(x)$ that verifies $\dot{y}(s) = f_i(y(s))$ for any $s \in (t-\delta, t+\delta)$.

Suppose $\partial_V \varphi(t,x) \neq \emptyset$, otherwise, (4.38) is immediately satisfied. Thus, $\vartheta(t,x) \in \mathbb{R}$ and so $\mathbb{S}_t^T(x) \neq \emptyset$. Take $\bar{y} \in \mathbb{S}_t^T(x)$ and remark that $\tilde{y} = y\mathbb{1}_{[t-\delta,t)} + \bar{y}\mathbb{1}_{[t,T]} \in \mathbb{S}_{t-\delta}^T(y(t-\delta))$. Therefore, if φ is strongly increasing we have $\varphi(t,x) \geq \varphi(s, y(s))$ for any $s \in [t-\delta, t]$.

Let $(\theta, \zeta) \in \partial_P \varphi(t,x)$, the proximal inequality and the monotone property yield to

$$\theta + \left\langle \frac{x - y(s)}{t-s}, \zeta \right\rangle \geq \sigma \left[(s-t) + |y(s) - x| \left| \frac{x-y(s)}{t-s} \right| \right], \tag{4.39}$$

for some $\sigma > 0$ and for any $s \in (t-\delta, t)$ close to t. Notice that

$$\frac{x - y(s)}{t-s} = \int_0^1 f_i(y(\lambda s_n + (1-\lambda)t)) d\lambda \to v, \quad \text{if } s \to t \text{ with } s < t.$$

Hence, letting $s \to t$ and noticing that $v \in F_i(x)$ is arbitrary we get (4.38) for any proximal subgradient. The extension to viscosity subgradients is a consequence of Proposition 2.3.10.

The sufficiency of (4.38) follows the same arguments as the proof of Proposition 4.2.7, so we will skip some of the details and we will focus mainly on the inductive procedure (Step 2 in the aforesaid proof). We divide the rest of the proof in several claims.

> **Claim A:** If $i \in \mathcal{I}$ with $\operatorname{dom} F_i \neq \emptyset$, then for each $(t,x) \in \operatorname{dom} \vartheta$, $\tau \in (t,T]$ and $y \in \mathbb{S}_t^\tau(x)$ for which $y(s) \in \mathcal{M}_i$ for all $s \in (t,\tau)$, we have $\varphi(\tau, y(\tau)) \geq \varphi(t,x)$.

Proof of Claim A. Set $\Gamma_i(x) = \{-1\} \times -F_i(x) \times \{0\}$ for any $x \in \mathcal{M}_i$, $\mathcal{S}_i = \operatorname{epi}(\varphi_i)$ and $M_i = \mathbb{R} \times \mathcal{M}_i \times \mathbb{R}$. As done for Lemma 4.2.3, with the help of Proposition 2.4.6 we can prove for any $\gamma : [t,\tau] \to \overline{M}_i$ verifying

$$\dot{\gamma} \in \Gamma_i(\gamma), \quad \text{a.e. on } [t,\tau], \quad \gamma(s) \in M_i, \quad \forall s \in (t,\tau), \quad \text{and } \gamma(t) \in \mathcal{S}_i,$$

that $\gamma(s) \in \mathcal{S}_i$ for any $s \in (t,\tau]$. Hence, if y is as in claim A,

$$\gamma_y(s) = (\tau + t - s, y(\tau + t - s), \varphi(\tau, y(\tau))), \quad \forall s \in [t,\tau]$$

fulfills the required conditions, because $\gamma_y(t) = (\tau, y(\tau), \varphi(\tau, y(\tau))) \in \mathcal{S}_i$ and so $\gamma_y(\tau) \in \mathcal{S}_i$, which leads to $\varphi(t,x) = \varphi(t, y(t)) \leq \varphi(\tau, y(\tau))$. $\square$

> **Claim B:** For any $(t,x) \in \operatorname{dom} \vartheta$ and $y \in \mathbb{S}_t^T(x)$ we have $\varphi(t,x) \leq \varphi(s, y(s))$ for any $s \in [t,T]$ provided that there is a partition $\{t = t_0 < t_1 < \ldots < t_n < t_{n+1} = T\}$, so that for any $l \in \{0, \ldots, n\}$ we can find $i \in \mathcal{I}$ such that $y(s) \in \mathcal{M}_i$ on (t_l, t_{l+1}).

Proof of Claim B. This is a direct consequence of Claim A. □

The next step is the most technical and difficult argument of the section. This statement is the corresponding version of Step 2 in the proof of Proposition 4.2.7 and the idea of the proof is similar. However, in this case we are dealing with a possible discontinuous final cost and a HJB equation that only applies for trajectories defined on an interval of time contained in $[0,T]$. Recall that the approximating curve may by defined on a slightly larger interval of time. Therefore, once constructed the approximate trajectory we have to fixed the final time and take a possible different initial time so that the curve is defined on an interval of time of length at most T. Having a different but close initial time is a difficulty that can be overcome by means of the lower semicontinuity of the function φ.

Notice too that the emphasis is put on constructing a trajectory rather than on a admissible control. For all these reasons, we mainly explain how to use the controllability assumption (H_5^4) to construct a regular trajectory.

> **Claim C:** If $(t,x) \in \operatorname{dom}\vartheta$ and $y \in \mathbb{S}_t^T(x)$ are given, then for any $\varepsilon > 0$ and $\tau \in [t,T]$ we can find $x_\varepsilon \in \mathbb{B}(x,\varepsilon) \cap \mathcal{K}$, $t_\varepsilon \in (t-\varepsilon, t+\varepsilon) \cap [0,\tau]$ and $y_\varepsilon \in \mathbb{S}_{t_\varepsilon}^{\tau}(x_\varepsilon)$ that verifies the conditions of Claim B and also $y_\varepsilon(\tau) = y(\tau)$.

Before proving the foregoing claim, let us see how the conclusion can be reached.

Let $(t,x) \in [0,T] \times \mathcal{K}$, $s \in [t,T]$ and $y \in \mathbb{S}_t^T(x)$, take a sequence $\{\varepsilon_n\} \subseteq (0,1)$ with $\varepsilon_n \to 0$. Let $x_n \in \mathcal{K}$, $t_n \in [0,T]$ and $y_n \in \mathbb{S}_{t_n}^s(x_n)$ given by claim C for $\varepsilon = \varepsilon_n$.

In the light of Claim B, we have $\varphi(t_n, x_n) \leq \varphi(s, y(s))$. Therefore, due to $x_n \to x$, $t_n \to t$ and φ is lower semicontinuous, the strongly increasing inequality holds and so the conclusion.

So, to finish the proof we only need to prove Claim C.

Proof of Claim C. We exclusively do the case $\tau = T$, any other situation is analogous.

Let us assume there exists $i \in \mathcal{I}$ so that $J_i = \{s \in [t,T] \mid y(s) \in \mathcal{M}_i\}$ contains infinitely many disjoint open intervals, otherwise the triple (x,t,y) satisfies the conclusion. Since the stratification is locally finite and the strata of $\mathcal{K}$ are disjoint, we might assume that $\mathcal{M}_i$ is unique and of minimal dimension; it may be, as the matter of fact, a finite union of strata of the same dimension.

Because of the minimality of the dimension of $\mathcal{M}_i$, $J := (t,T) \setminus J_i$ is open and, $a = \min J_i$ and $b = \max J_i$ are well-defined. So, for any $\varepsilon > 0$ we can construct a partition of $[a,b]$

$$b_0 := a \leq a_1 < b_1 \leq a_2 < b_2 \leq \ldots \leq a_n < b_n \leq b =: a_{n+1} \quad \text{with} \quad \operatorname{meas}\left(J \setminus \bigcup_{l=1}^{n}(a_l, b_l)\right) \leq \varepsilon,$$

$y(a_l), y(b_l) \in J_i$ and $(a_l, b_l) \subseteq J$ for any $l = 1, \ldots, n$. In addition,

$$\bigcup_{l=0}^{n}[b_l, a_{l+1}] \setminus J_i = J \setminus \bigcup_{l=1}^{n}(a_l, b_l).$$

Hence, if we set $J^l := [b_l, a_{l+1}] \setminus J_i$ and $\varepsilon_l = \operatorname{meas}(J^l)$, we have $\sum_{l=0}^{n} \varepsilon_l \leq \varepsilon$.

On the other hand, there must be some $l \in \{0, \ldots, n\}$ for which there is a countable family of intervals $(\alpha_p, \beta_p) \subseteq [b_l, a_{l+1}]$, pairwise disjoint that verifies $\varepsilon_l = \sum_{p\in\mathbb{N}}(\beta_p - \alpha_p)$, $y(t) \in \mathcal{M}$

for any $t \in (\alpha_p, \beta_p)$ and $y(\alpha_p), y(\beta_p) \in \mathcal{M}_i$. Without loss of generality we might assume that each $l \in \{0, \dots, n\}$ for which $b_l < a_{l+1}$, verifies this property.

Since there are infinitely many $\{\alpha_p\}$ and $[t, T]$ is compact, it has an accumulation point, say $\alpha \in [t, T]$. The same argument used in the necessity part of Proposition 4.3.2 allows us to show that any accumulation point of $v_p := \frac{1}{\alpha_p - \alpha}(y(\alpha_p) - y(\alpha))$ belongs to $F(y(\alpha))$. From where $\operatorname{dom} F_i \neq \emptyset$.

Let $r > 0$ so that $y(s) \in \mathbb{B}(0, r)$ for any $s \in [t, T]$. Consider as well $\varepsilon_i > 0$ and $\Delta_i > 0$ the constant given by (H_5^4), and suppose $\varepsilon \leq \varepsilon_i$. So, for any $p \in \mathbb{N}$, if we set $\tau_p = \alpha_p + \Delta_i(\beta_p - \alpha_p)$, we can pick $y_p \in \mathbb{S}_{\alpha_p}^{\tau_p}(y(\alpha_p))$ and $t_p \in (\alpha_p, \tau_p]$ such that

$$y_p(s) \in \mathcal{M}_i, \ \forall s \in [\alpha_p, t_p), \quad y_p(\alpha_p) = y(\alpha_p), \quad \text{and} \quad y_p(t_p) = y(\beta_p).$$

Let $J_i^l := [b_l, a_{l+1}] \cap J_i$ and the measurable function $\omega : [b_l, a_{l+1}] \to \mathbb{R}$

$$\omega(s) = \mathbb{1}_{J_i^l}(s) + \sum_{p \in \mathbb{N}} \frac{t_p - \alpha_p}{\beta_p - \alpha_p} \mathbb{1}_{(\alpha_p, \beta_p)}(s) > 0, \quad \forall s \in [b_l, a_{l+1}].$$

Accordingly with the proof of Proposition 4.2.7, the map $s \mapsto \nu(s) = b_l + \int_{b_l}^s \omega(\tau) d\tau$ defined on $[b_l, a_{l+1}]$ is a homeomorphism from $[b_l, a_{l+1}]$ into $[b_l, c_{l+1}]$. Moreover, $(t_p - \alpha_p) \leq \Delta_i(\beta_p - \alpha_p)$ which leads to

$$c_{l+1} - a_{l+1} = \operatorname{meas}(J_i^l) - (a_{l+1} - b_l) + \sum_{p \in \mathbb{N}} (t_p - \alpha_p) \leq \Delta_i \varepsilon_l. \tag{4.40}$$

Consider the measurable function $v_l : [b_l, c_{l+1}] \to \mathbb{R}^N$ given by

$$v_l(s) = \dot{y}(\nu^{-1}(s)) \mathbb{1}_{J_i^l}(\nu^{-1}(s)) + \sum_{p \in \mathbb{N}} \dot{y}_p(s) \mathbb{1}_{(\alpha_p, \beta_p)}(\nu^{-1}(s)), \quad \text{for a.e. } s \in [b_l, c_{l+1}].$$

Let $\tilde{y}_l : [b_l, c_{l+1}] \to \mathbb{R}^N$ be defined via

$$y_l(s) = y(b_l) + \int_{b_l}^s v_l(\tau) d\tau, \quad \forall s \in [b_l, a_{l+1}].$$

By construction $y_l(\nu(t)) = y(t)$ for any $t \in J_i^l$ and $y_l(t) \in \mathcal{M}_i$ for any $t \in [b_l, c_{l+1}]$. In particular, $y_l(c_{l+1}) = y(a_{l+1})$.

If $l \in \{0, \dots, n\}$ is so that $b_l = a_{l+1}$, we set $c_{l+1} = b_l$ and $y_l(c_{l+1}) = y(a_{l+1})$.

Therefore, doing the same procedure for each $l \in \{0, \dots, n\}$, we can construct inductively an absolutely continuous curve y_ε in the following way:

- Set first

$$\begin{aligned} y_\varepsilon(s) &= y(s), \quad s \in [t, t_0], & t_0 &= b_0 \\ y_\varepsilon(s) &= y_0(s), \quad s \in [t_0, t_1], & t_1 &= c_1. \end{aligned}$$

- Then for any $l \in \{1, \dots, n\}$

$$\begin{aligned} y_\varepsilon(s) &= y(a_l - t_{2l-1} + s), \quad s \in [t_{2l-1}, t_{2l}], & t_{2l} &= t_{2l-1} + b_l - a_l \\ y_\varepsilon(s) &= y_l(b_l - t_{2l} + s), \quad s \in [t_{2l}, t_{2l+1}], & t_{2l+1} &= t_{2l} + c_{l+1} - b_l. \end{aligned}$$

- Finally, $y_\varepsilon(s) = y(a_{n+1} - t_{2n+1} + s)$ for $s \in [t_{2n+1}, T_\varepsilon]$ with $T_\varepsilon = t_{2n+1} + T - a_{n+1}$.

Notice that $[b_l, a_{l+1}] = J_i^l \cup J^l$, so $c_{l+1} - b_l \geq \text{meas}(J_i^l) = a_{l+1} - b_l - \varepsilon_l$. Hence, after a few algebraic steps we obtain, by virtue of (4.40),

$$T_\varepsilon = T + \sum_{l=0}^{n}(c_{l+1} - a_{l+1}) \in [T - \varepsilon, T + \Delta_i\varepsilon].$$

To summarize, we have constructed a trajectory of the control systems for which the set $\{s \in [t, T_\varepsilon] \mid y_\varepsilon(s) \in \mathcal{M}_i\}$ can be decomposed into a finite number of intervals. Furthermore, this trajectory verifies $y_\varepsilon(t) = x$ and $y_\varepsilon(T_\varepsilon) = y(T)$. Notice that process described above can also be applied to y_ε but in this case the manifold that plays the role of $\mathcal{M}_i$ has dimension strictly larger than $\mathcal{M}_i$. We can then repeat procedure one more time for the resulting trajectory and once again the dimension of the manifold playing the role of $\mathcal{M}_i$ is strictly larger than the preceding one. Thus, it is clear that this scheme finishes in a finite number of steps (there are only N possible choices for the dimension of $\mathcal{M}_i$), and the resulting trajectory verifies the conditions of Claim B.

Since $\varepsilon > 0$ is arbitrary and $\Delta_i > 0$ does not depends upon ε, we may assume that $T_\varepsilon \in (T - \varepsilon, T + \varepsilon)$; using $\min\left\{\varepsilon, \frac{1}{\Delta_i}\varepsilon\right\}$ instead of ε for instance.

Finally, re-scaling ε if necessary, we can assume that $y_\varepsilon(t + T_\varepsilon - T) \in \mathbb{B}(x, \varepsilon)$. Therefore, to complete the proof it is enough to take $t_\varepsilon = T_\varepsilon$ and $x_\varepsilon = x$ if $T_\varepsilon \leq T$ or $t_\varepsilon = T$ and $x_\varepsilon = y_\varepsilon(t + T_\varepsilon - T)$. □

□

4.4 Discussion and perspectives.

We conclude this chapter with a discussion about the results we have obtained as well as the assumptions considered.

4.4.1 Contributions of the chapter.

The main contributions of Theorem 4.2.1 and Theorem 4.3.1 is the characterization of the Value Function in situations where the set $\mathcal{K}$ is not necessarily the closure of its interior, and the Value Function is not necessarily continuous. As already mentioned in the introduction, several contributions have been devoted to the case where Inward Pointing conditions (IPC) are satisfied and the interior of $\mathcal{K}$ is not empty; see the pioneering work of Soner [120], the more recent works of Clarke-Stern [42], Frankowska-Mazzola [52] and the references therein.

When the IPC is not satisfied, the idea of characterizing the Value Function by a system of HJB equations on whole the domain $\mathcal{K}$, including its boundary, appears already in the work of Ishii-Koike [78]. However, in that paper the set $\mathbb{U}(x)$ is assumed nonempty everywhere on $\mathcal{K}$, requiring in particular that the viable set is whole the set $\mathcal{K}$. Moreover, the result in [78] assume some restrictive hypothesis on the structure of $\mathcal{K}$ and on the set-valued map $\mathbb{U}(\cdot)$.

Let us also mention the work of Bokanowski-Forcadel-Zidani [23] where it was shown that the HJB equation should be completed by additional *information* on the increasing property of the solution along trajectories lying on $\partial\mathcal{K}$. In the present work, we explicitly express the additional information in terms of HJB inequalities on each strata. The regularity assumptions on the set $\mathcal{K}$ are quite general and allow several situations that are not covered by the known literature. However, Theorem 4.2.1 and Theorem 4.3.1 require a new controllability assumption that is needed only on the strata where some chattering behavior may occur.

We point out that there is an increasingly interest in control problems in stratified domains; see the contributions of Bressan-Hong [31], Barnard-Wolenski [17], Barles-Briani-Chasseigne [15, 16] and Zidani and her coauthors [106, 105]. In those papers, the control problem is formulated in the whole space $\mathbb{R}^N$ with a given stratification, and in [15, 16, 106, 105] a strong controllability assumption is imposed in order to guarantee the continuity of the Value Function which provides an appropriate framework for analyzing the transmission conditions. In the this Chapter, the stratification is used in a completely different way for characterizing the lower semicontinuous Value Function of state-constrained control problems.

On another hand, several papers have been devoted to control problems on networks; cf. Achdou et al. [1, 2] and Imbert et al. [75, 74]. The framework in the quoted works is also different from the one considered in Theorem 4.2.2. Indeed, in the aforementioned papers, the dynamics is not Lipschitz continuous in the whole network. Our attention here has been focused on the state-constrained setting, nonetheless the general result we have obtained indicates that in the particular case of networks, it is possible to avoid the controllability assumption usually considered in the literature at the junction points. The characterization of the Value Function could be then considered in the bilateral viscosity sense. Moreover, the arguments exposed in this chapter can be adapted to more general control problems on networks (with discontinuous dynamics and also in higher dimensions). This study will be developed in Chapter 9 where we also treat the case in which the dynamics is given separately in each branch.

4.4.2 Optimality principles

The analysis we have proposed in this chapter is based on monotone principles over trajectories of the control system instead of purely viscosity arguments. The reason of this choice is that this approach allows us to understand the interplay between how trajectories can be approximated and the corresponding notion of solution to the HJB equation.

To clarify this affirmation, we evoke that the fundamental tool required for ours analysis is the Dynamic Programing Principle, which for the Mayer problem reads as follows:

$$\vartheta(t,x) = \inf\left\{\vartheta(s,y(s)) \mid \quad y \in \mathbb{S}_t^T(x)\right\}, \quad \forall x \in \mathcal{K}, \quad \forall 0 \le t \le s \le T.$$

The two important aspects of the Dynamic Programming Principle are that the Value Function is constant along optimal trajectories and is not decreasing along non-optimal ones. This remark, as done earlier on the chapter, motivates the next definition.

Definition 4.4.1. *Let $\mathfrak{S}^T = \{\mathfrak{S}_t^T\}_{t\in[0,T]}$ be a collection of set-valued maps defined on $\mathcal{K}$, $\mathfrak{S}_t^T : \mathcal{K} \rightrightarrows \mathcal{AC}([t,T];\mathbb{R}^N)$ for each $t \in [0,T]$. A function $\varphi : [0,T]\times\mathcal{K} \to \mathbb{R}\cup\{+\infty\}$ is said:*

i) weakly decreasing along trajectories of $\mathfrak{S}^T$ provided for all $(t,x) \in \operatorname{dom}\varphi$ we can find a curve $y \in \mathfrak{S}_t^T(x)$ so that $\varphi(s,y(s)) \le \varphi(t,x)$ for all $t \le s \le T$.

ii) strongly increasing along trajectories of $\mathfrak{S}$ if for each $(t,x) \in [0,T]\times\mathcal{K}$ and each $y \in \mathfrak{S}_t^T(x)$ satisfies $\varphi(s,y(s)) \ge \varphi(t,x)$ for all $t \le s \le T$.

The notions above-stated are closely related to those of Definition 4.2.2. However, in the present case we have used only a family of admissible trajectories $\mathfrak{S}^T$; in Definition 4.2.2 it was written for *all* the trajectories of the control system. The reason is, as we are going to show shortly, that there is no real need of working with all the arcs produced by the control systems but only with some of them. For this reason we introduce the following concept.

Definition 4.4.2. *Let $\mathfrak{S}^T = \{\mathfrak{S}^T_t\}_{t\in[0,T]}$ be a collection of set-valued maps defined on $\mathcal{K}$, where $\mathfrak{S}^T_t : \mathcal{K} \rightrightarrows \mathcal{AC}([t,T];\mathbb{R}^N)$ for each $t \in [0,T]$. We say that $\mathfrak{S}^T$ is suboptimal for the Mayer problem*

$$\vartheta(t,x) := \inf\left\{\psi(y(T)) \mid \quad y \in \mathbb{S}^T_t(x)\right\}, \quad \forall (t,x) \in [0,T]\times\mathcal{K},$$

if the following conditions are verified:

1. *$\mathfrak{S}^T_t(x) \subseteq \mathbb{S}^T_t(x)$ for each $x \in \mathcal{K}$.*
2. *For any $(t,x) \in dom\,\vartheta$ and $\varepsilon > 0$ we can find $x_\varepsilon \in \mathbb{B}(x,\varepsilon)\cap\mathcal{K}$, $t_\varepsilon \in (t-\varepsilon,t+\varepsilon)\cap[0,T]$ and $y_\varepsilon \in \mathfrak{S}^T_{t_\varepsilon}(x_\varepsilon)$ so that $\psi(y_\varepsilon(T)) \leq \vartheta(t,x)+\varepsilon$.*

In this case, we just refer to $\mathfrak{S}^T$ as a suboptimal collection of trajectories.

It is clear that if we take the collection $\mathfrak{S}^T = \{\mathbb{S}^T_t(\cdot)\}_{t\in[0,T]}$ to be the set of all trajectories of the control systems, then $\mathfrak{S}^T$ is suboptimal. We can also go to the other extreme and set $\mathfrak{S}^T$ as the set of minimizers. Clearly, in one case, there are too many trajectories, some of them with a wild structure difficult to handle, and on the other case, there are too few, which makes the HJB approach useless. Hence, it will be convenient to work with a suboptimal collection of trajectories that lies in between the situations described earlier, meaning that it is large enough so that it is not too complicated to construct it, and that it is sufficiently small so that it contains only tame trajectories.

In any case, the utility of the preceding notion will be clear in the next statement.

Lemma 4.4.1. *Let $\varphi : [0,T]\times\mathcal{K} \to \mathbb{R}\cup\{+\infty\}$ be a lower semicontinuous function satisfying $\varphi(T,x) = \psi(x)$ for all $x \in \mathcal{K}$. Consider as well $\mathfrak{S}^T$ a suboptimal collection of trajectories.*

1. *If φ is weakly decreasing along trajectories of $\mathfrak{S}^T$, then $\vartheta(t,x) \leq \varphi(t,x)$ on $[0,T]\times\mathcal{K}$.*
2. *If φ is strongly increasing along trajectories of $\mathfrak{S}^T$, then $\vartheta(t,x) \geq \varphi(t,x)$ on $[0,T]\times\mathcal{K}$.*

Proof.

1. The case $\varphi(t,x) = +\infty$ is trivial, so assume $\varphi(t,x) < \infty$. By definition, there exists a trajectory $y \in \mathfrak{S}^T_t(x) \subseteq \mathbb{S}^T_t(x)$ such that $\varphi(t,x) \geq \varphi(T,y(T)) = \varphi(y(T)) \geq \vartheta(t,x)$, the last equality being a consequence of the definition of the Value Function.

2. Let $(t,x) \in [0,T]\times\mathcal{K}$, if $\mathbb{S}^T_t(x) = \emptyset$ then $\vartheta(t,x) = +\infty$ and the conclusion follows easily. Otherwise, take $\{\varepsilon_n\} \subseteq (0,1)$ so that $\varepsilon_n \to 0$, thanks to Definition 4.4.2, we can pick $x_n \in \mathcal{K}$, $t_n \in [0,T]$ and $y_n \in \mathbb{S}^T_{t_n}(x_x)$ with $x_n \to x$, $t_n \to t$ and $\psi(y_n(T)) \leq \vartheta(t,x)+\varepsilon_n$.

   The strongly increasing property yields to

   $$\varphi(t_n,x_n) \leq \varphi(T,y_n(T)) = \psi(y_n(T)) \leq \vartheta(t,x)+\varepsilon_n, \quad \forall n \in \mathbb{N}.$$

   Since φ is lower semicontinuous, by taking the inferior limit on the foregoing inequality we conclude the proof.

□

The last lemma leads to assert that, as in Proposition 4.2.4, the Value Function is the unique function that is weakly decreasing and strongly increasing along trajectories of $\mathfrak{S}^T = \{\mathbb{S}_t^T(\cdot)\}_{t\in[0,T]}$ at the same time. But, this result is even stronger because it tells us that we do not need all the trajectories of the control system to characterize the Value Function, just some of them that are almost-optimal. This remark yields to an explanation why the NFT approach as well as our strategy work and why in each case a different notion of solution has been considered.

In the NFT approach it is proven that any trajectory can be approximated by one staying in the interior of the state-constraints. Consequently, this approach deal with the suboptimal collection of trajectories that remain on $\mathrm{int}(\mathcal{K})$. This is the reason why the subsolution principle can be characterized using exclusively the information on the interior of $\mathcal{K}$.

In the setting of this chapter, as clearly reflected in Claim C in the proof of Proposition 4.3.3, we make our analysis with the suboptimal collection of trajectories that do not chatter between contiguous strata, that is, each suboptimal trajectory stay on a single strata during intervals of times whose length is bounded from below. This fact explains why in our approach we need the information on the boundary.

In the methodology we have proposed, we accomplish the construction of the non-chattering trajectories by means of the controllability assumption (H_5^4). However, any other hypothesis that guarantees this type of approximation could also be considered. As a matter of fact, we have conjectured the following

> **Conjecture:** Given a control system with sufficiently regular dynamics. For any trajectory of the control system $y(\cdot)$ with initial condition $y(t) = x$ and any $T > t$ we can find $\tilde{T} > t$ as close as wanted of T and a non-chattering curve of the control systems $\tilde{y}(\cdot)$ with initial condition $\tilde{y}(t) = x$ that verifies $y(T) = \tilde{y}(\tilde{T})$.

We want to stress that in the context of the chapter non-chattering means with respect to the stratification, and so, the conjecture basically says that any chattering arc of the control systems can be approximated by a sequence of curves that switch from stratum to stratum only a finite number of times.

We also mention that the idea described above motivates the theory we present in Chapter 5. In that case, we show that the subsolution principle can be characterized with only the information on the interior of the state-constraints provided that $\mathcal{K}$ is convex and the dynamics are linear-like. This is because the convex-linear structure and the Accessibility Lemma allow us to construct suboptimal curves that remain on the relative interior of $\mathcal{K}$.

4.4.3 Lipschitz-like hypothesis

We close the discussion with a few words on the Lipschitz continuity assumptions (H_2^4) and (H_4^4). In Section 3.2.2 we have discussed about continuity properties of $x \mapsto \mathcal{T}_{\mathcal{M}}(x)$, where $\mathcal{M}$ is an embedded manifold of $\mathbb{R}^N$. We evoke from Proposition 3.2.4 that this map is always lower semicontinuous on $\mathcal{M}$ and from Proposition 3.2.5 that for any $r > 0$, the cut map $x \mapsto \mathcal{T}_{\mathcal{M}}(x) \cap \mathbb{B}(0,r)$ is locally Lipschitz continuous on $\mathcal{M}$ provided $\mathcal{M}$ is at least $\mathcal{C}^2$.

Let $\Gamma : \mathcal{M} \rightrightarrows \mathbb{R}^N$ be the multivalued function that stands for

$$f(x,\mathcal{U}) \cap \mathcal{T}_{\mathcal{M}}(x) \quad \text{or} \quad F(x) \cap \mathcal{T}_{\mathcal{M}}(x), \quad \forall x \in \mathcal{M},$$

where $f : \mathbb{R}^N \times \mathcal{U} \to \mathbb{R}^N$ is a parametrized vector field and $F : \mathbb{R}^N \rightrightarrows \mathbb{R}^N$ is a multifunction, both being considered as the dynamics of the optimal processes studied in Section 4.2 and Section 4.3, respectively. The manifold $\mathcal{M}$ should be thought as one of the stratum of $\mathcal{K}$.

We now address our attention in the important issue of how to provide simpler criterions for (H_2^4) or (H_4^4) to be satisfied. To do so, we can analyze the structure of the multifunction Γ from two points of view:

(A) Apply some available criterion for the Lipschitz continuity of the intersection of two locally Lipschitz set-valued maps.

(B) Construct a stratification of $\mathcal{M}$ so that the condition(H_2^4) (respectively (H_4^4)) is verified for that stratification on $\mathcal{M}$.

Notice that, in general, it is not obvious that the intersection of two lower semicontinuous set-valued maps verifies the same property, never mind the locally Lipschitz continuous case. The usual criterion that ensures the lower semicontinuity of Γ on $\mathcal{M}$ (as intersection of lower semicontinuous maps) require Γ to have nonempty interior which is never the case if $\dim(\mathcal{M}) < N$; see for instance Lechicki-Spakowski [84] and Penot [100]. For the Lipschitz continuity of the intersection that defines Γ, we can derive a necessary condition from [113, Theorem 4.12] which holds under a rather strong qualification condition. This can be done by means of a weaker notion of Lipschitz-like continuity called *pseudo-Lipschitz continuity*, also referred by some authors as the *Aubin property*. This is a localized version (on the graph of Γ) of the Lipschitz continuity; see for instance [12, Definition 1.4.5] and [114, Definition 9.36], respectively. Consequently, if we work in full generality, that is, we do not impose any further structural condition on the data of the problem, the first approach proposed above is suitable, but in rather restrictive situations.

Nonetheless, if we do assume some additional structural condition over the data of the problem, then simpler criterions can be stablished. For instance, if the graph of Γ is a polyhedral convex set, that is, for some $\alpha_1, \ldots, \alpha_p \in \mathbb{R}$, $\xi_1, \ldots, \xi_p, \eta_1, \ldots, \eta_p \in \mathbb{R}^N$ we have

$$\mathrm{gr}(\Gamma) = \left\{ (x, v) \in \mathcal{M} \times \mathbb{R}^N \;\middle|\; \begin{array}{ll} \langle \xi_n, x\rangle + \langle \eta_n, v\rangle = \alpha_n, & n = 1, \ldots, l, \\ \langle \xi_n, x\rangle + \langle \eta_n, v\rangle \leq \alpha_n, & n = l+1, \ldots, p \end{array} \right\},$$

then [47, Theorem 3C.3] yields to the Lipschitz continuity of Γ on its domain. Consequently, if $\operatorname{dom}\Gamma = \mathcal{M}$, then ($H_2^4$) or ($H_4^4$) holds (according to the problem at hand).

A notable case where the foregoing situation happens is whenever $\mathcal{T}_{\mathcal{M}}(x) = \ker(P)$ for any $x \in \mathcal{M}$, with $P \in \mathbb{M}_{d\times N}(\mathbb{R})$ a full rank matrix, and the dynamics is linear.

The second point (B) however, seems to be less restrictive and more likely to occur than the (A) point, at least for a large class of state-constraints. Apparently, in this situation it is enough to verify for each manifold of the stratification that the condition $\operatorname{dom}\Gamma = \mathcal{M}$ holds.

In the light of a result reported by Daniilidis-Pang in [46], if the problems are posed in the framework of *tame optimization* (cf. [76]) it is possible to establish the existence of a stratification of $\mathcal{K}$ for which the condition (H_2^4) (respectively (H_4^4)) is verified. To do so, the following is a suitable scheme:

(i) Construct a stratification of $\mathcal{K}$, take one of its stratum $\mathcal{M}$ and Γ as above-defined so that $\operatorname{dom}\Gamma = \mathcal{M}$.

(ii) Verify that the graph of Γ and $\mathcal{M}$ are semialgebraic sets (see Section 3.3.2).

(iii) Find a semialgebraic set $\widetilde{\mathcal{M}} \subseteq \mathcal{M}$ with $\dim(\widetilde{\mathcal{M}}) + 1 = \dim(\mathcal{M})$ so that Γ is locally Lipschitz on $\mathcal{M} \setminus \widetilde{\mathcal{M}}$.

(iv) Take a stratification of $\widetilde{\mathcal{M}}$ and repeat the process for $\widetilde{\mathcal{M}}$ in place of $\mathcal{K}$.

In this scheme, the point (iii) is guaranteed by [46, Theorem 28][1]. Additionally, given that $\widetilde{\mathcal{M}}$ is a semialgebraic set, we can find a stratification for it (we refer to the discussion in Section 3.3.2), so step (iv) is also guaranteed.

Notice that, since the dimension of the role-playing manifold $\widetilde{\mathcal{M}}$ decreases in each iteration of the scheme, it is clear that the procedure finishes in a finite number of steps. Also, as claimed by the authors in [46], the scheme works as well if the word *semialgebraic* is replaced by *definable sets on an o-minimal structure* (see Section 3.3.2).

Hence, the only thing we need to check each time is that we can find a stratification of $\mathcal{K}$ so that each stratum of it verifies the condition $\operatorname{dom}\Gamma = \mathcal{M}$ (so that point (i) is verified) and afterwards check that point (ii) is also fulfilled. In any case, it is reduced to verify some algebraic criterions.

[1] It is enough to note that the notion of continuity used in that statement agrees with the locally Lipschitz continuity of Γ if the dynamics has linear growth (cf. [114, Theorem 9.30]) and that the residual set on [46, Theorem 28] is also semialgebraic (see for instance [76, Theorem 4.1]).

CHAPTER 5

Convex State-Constraints I: Linear-like Dynamics

Abstract. In this chapter we revisit the standard notion of constrained viscosity solution associated with the Hamilton-Jacobi-Bellman equations for the special framework of convex state-constraints and linear-like dynamics. We show that, under these circumstances, the Value Function of the optimal control process can be identified as the unique lower semicontinuous function that is a viscosity supersolution on $\mathcal{K}$ and a viscosity subsolution on the relative interior of the state-constraints in the bilateral sense.

5.1 Introduction

In the discussion about the optimality principles in Chapter 4 we have pointed out that, for the Hamilton-Jacobi-Bellman (HJB) approach, a suitable definition of viscosity solution has to be intrinsically related to the fashion in which controlled trajectories can be approximated. In particular, we have shown that if $\{\mathcal{M}_i\}_{i\in\mathcal{I}}$ is a stratification of $\mathcal{K}$ and if any admissible curve can be approximated by a sequence of *non-chattering* trajectories, then the Value Function is the unique map that verifies:

$$\vartheta(\cdot) \text{ is supersolution on } \mathcal{K} \text{ and for each } i \in \mathcal{I},\ \vartheta|_{\mathcal{M}_i}(\cdot) \text{ is a subsolution on } \mathcal{M}_i.$$

In the classical approach found in the current literature, the methodology usually consists in approximating any feasible curve of the control systems by a sequence of trajectories lying in $\mathrm{int}(\mathcal{K})$; see for example [120, 94, 78, 95, 54, 123, 42, 52]. This approach leads to assert that the Value Function is the unique constrained viscosity solution, that is,

$$\vartheta(\cdot) \text{ is supersolution on } \mathcal{K} \text{ and a subsolution on } \mathrm{int}(\mathcal{K}).$$

In this chapter we show that if the state is constrained to remain in a convex set and the dynamics are linear-like, then the Value Function is the unique constrained viscosity solution (in the bilateral sense). Furthermore, the setting allows us also to treat the case in which $\mathrm{int}(\mathcal{K})$ is replace with the relative interior of $\mathcal{K}$ which is denoted by $\mathrm{ri}(\mathcal{K})$. Therefore in this chapter we show that $\vartheta(\cdot)$ is the only constrained viscosity solution in the following sense:

$$\vartheta(\cdot) \text{ is supersolution on } \mathcal{K} \text{ and a subsolution on } \mathrm{ri}(\mathcal{K}).$$

The novelty of this exposition is that we do not require any further compatibility assumption between dynamics and state-constraints such as the pointing conditions (classical NFT approach) or the Lipschitz character of the tangential dynamics at the boundary of $\mathcal{K}$ (as in the setting of Chapter 4).

The advantage of considering a convex-linear structure lies in the Accessibility Lemma (Proposition 2.3.1). Indeed, this statement provides a simple way to approximate any feasible trajectory by curves lying on the relative interior of $\mathcal{K}$, as long as the dynamics has a linear-like structure. We present the analysis in two different contexts, namely for control-dependent cost-to-go functionals and independent ones; in both cases, the techniques are essentially the same with the difference that, in the first case the dynamics is a linear vector field (jointly in the state and control) and, in the second one, it is more general, so the dynamics is a set-valued map with convex graph. The study we exhibit is supported on [69].

Another possible way to deal with convex state-constrained optimal control problems is going to be considered in the next chapter. In that case, we make an analysis based on a suitable class of penalization functions through a Riemannian metric technique.

We also mention that fully convex optimal control processes has been studied in the literature but mainly for problems without state-constraints; see for instance the works of Rockafellar-Wolenski [115, 116], Rockafellar-Goebel [61], Goebel [58, 59, 60] and the references therein. Under these circumstances, the final cost is presupposed to be a proper lower semi-continuous convex function as well as the running cost (convex jointly with respect to state and control). Indeed, the running cost may have unbounded values but the commonly done assumption

$$\inf\{\ell(x,u) \mid u \in \mathcal{U}\} < +\infty, \quad \forall x \in \mathbb{R}^N$$

precludes the existence of state-constraints implicitly included in the running cost.

Consequently, in the fully convex setting the Value Function turns out to be quite regular, as in most of the problems with unrestricted state-space. In this case, it is locally Lipschitz continuous on its domain, this is because the Value Function is itself a convex map.

Finally, we want to stress that here the structural hypotheses are mainly imposed over the state-constraints and the dynamics (convexity and linear-like, respectively). For the purpose of the chapter, the final cost can be any arbitrary continuous function. The running cost is also a continuous maps but, in order to ensure the lower semicontinuity of the Value Function, it is assumed in Theorem 5.2.1 to be convex with respect to the control; the latter does not mean that $\ell : \mathbb{R}^N \times \mathcal{U} \to \mathbb{R}$ is a convex function jointly with respect to (x, u).

5.1.1 Convex sets

We assume all through the chapter that the state variable is constrained to remain in a convex set of $\mathbb{R}^N$ whose interior may be empty, that is,

$$(H_0^5) \qquad \mathcal{K} \text{ is a closed and } \textit{convex} \text{ subset of } \mathbb{R}^N.$$

We recall that $\mathrm{ri}(\mathcal{K})$, the relative interior of $\mathcal{K}$, is always a nonempty set and coincides with $\mathrm{int}(\mathcal{K})$ whenever the interior of $\mathcal{K}$ is nonempty. Furthermore, this set is always an embedded manifold of $\mathbb{R}^N$.

Proposition 5.1.1. *Suppose that (H_0^5) holds, then $ri(\mathcal{K})$ is a $\mathcal{C}^\infty$-embedded manifold of $\mathbb{R}^N$.*

Proof. Let $\mathrm{aff}(\mathcal{K})$ stands for the affine hull of $\mathcal{K}$, then there exists $v_1, \ldots, v_p \in \mathbb{R}^N$ linearly independent that are orthogonal to $\mathrm{aff}(\mathcal{K}) - x_0$ for any $x_0 \in \mathcal{K}$. We set $h_i(x) = \langle v_i, x - x_0 \rangle$ for any $x \in \mathbb{R}^N$ then the function $h(x) = (h_1(x), \ldots, h_p(x))$ is a $\mathcal{C}^\infty$ submersion on $\mathbb{R}^N$. Moreover, for every $x \in \mathrm{ri}(\mathcal{K})$ there exists an open set $\mathcal{O} \subseteq \mathbb{R}^N$ containing x such that

$$\mathrm{ri}(\mathcal{K}) \cap \mathcal{O} = \mathrm{aff}\, \mathcal{K} \cap \mathcal{O} = \{x \in \mathbb{R}^N \mid h(x) = 0\} \cap \mathcal{O}.$$

In particular, we have that $\mathrm{ri}(\mathcal{K})$ is a $\mathcal{C}^\infty$-embedded manifold of $\mathbb{R}^N$. □

On the other hand, we evoke also that $x \mapsto \mathrm{proj}_{\mathcal{K}}(x)$, the projection onto $\mathcal{K}$, is a single-valued Lipschitz continuous map. Consequently, the dynamics needs only to be defined on $\mathcal{K}$, because it can always be extended to a multifunction, whose domain is the whole space, by means of the projection map

$$\widetilde{F}(x) = F(\mathrm{proj}_{\mathcal{K}}(x)), \quad \forall x \in \mathbb{R}^N.$$

Furthermore, as in Proposition 3.2.9, $\widetilde{F}$ inherites several of the properties verified by F. For instance, if F is Lipschitz continuous on $\mathcal{K}$, then $\widetilde{F}$ is Lipschitz continuous on $\mathbb{R}^N$.

5.2 Infinite horizon problem and linear systems

In section 4.2 we have seen that if the state-constraints verifies a stratification assumption in addition to some compatibility requirements, the Value Function of the infinite horizon problem

$$\vartheta(x) := \inf \left\{ \int_0^\infty e^{-\lambda t} \ell(y_x^u(t), u(t)) dt \;\middle|\; u \in \mathbb{U}(x) \right\}, \quad \forall x \in \mathcal{K},$$

can be identified as the sole lower semicontinuous function which solves a system of Hamilton-Jacobi-Bellman equations. Recall that in this case $\mathbb{U}(x)$ stands for the set of admissible controls related to the dynamical constraint

$$\dot{y} = f(y, u), \quad \text{a.e. } t \geq 0 \quad y(0) = x, \quad u : [0, +\infty) \to \mathcal{U} \text{ measurable}, \quad y(t) \in \mathcal{K}, \forall t \geq 0.$$

This section aims to draw the attention that if the state-constraints are convex, then $\vartheta(\cdot)$ for linear dynamics, is the unique constrained bilateral viscosity solution of the HJB equation

$$\lambda \vartheta(x) + H(x, \nabla \vartheta(x)) = 0, \quad \forall x \in \mathcal{K}.$$

5.2.1 Linear systems

All along this section we assume that the dynamics and the control space verify the following

$$(H_f^5) \qquad \begin{cases} i) \;\; \mathcal{U} \subseteq \mathbb{R}^m \text{ is nonempty, convex and compact.} \\ ii) \;\; \exists A \in \mathbb{M}_{N \times N}(\mathbb{R}),\ \exists B \in \mathbb{M}_{N \times m}(\mathbb{R}) \text{ for which} \\ \qquad f(x, u) = Ax + Bu \text{ for any } x \in \mathbb{R}^N,\ u \in \mathcal{U}. \\ iii) \; \exists x_0 \in \mathrm{ri}(\mathcal{K}),\ \exists u_0 \in \mathcal{U} \text{ so that } f(x_0, u_0) = 0. \end{cases}$$

Example 5.2.1. *We evoke the soft-landing dynamics with state-constraints*

$$\begin{pmatrix}\dot y_1\\ \dot y_2\end{pmatrix} = \begin{pmatrix} y_2\\ u\end{pmatrix}, \quad u(t)\in\mathcal{U} := [-1,1] \text{ a.e. } t\geq 0, \quad y_1(t), y_2(t)\in[-r,r],\ \forall t\geq 0.$$

It is clear in this example that (H_0^5) *and* (H_f^5) *are fulfilled. Indeed, the state-constraints is* $\mathcal{K}=[-r,r]\times[-r,r]$, *the dynamics is given by*

$$f(x,u) = \begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix} + \begin{pmatrix}0\\ 1\end{pmatrix}u, \quad \forall x\in\mathbb{R}^2,\ u\in[-1,1],$$

and we can take $x_0=(0,0)$ *and* $u_0=0$ *for instance.*

Furthermore, neither the IPC nor the OPC condition is satisfied. However, as we will see later on (Theorem 5.2.1 and 5.3.1), the Value Function associated with this dynamical system is the unique constrained viscosity solution of the HJB equation.

On the other hand, we easily see that, by setting

$$c_f := \max\left\{|A|, \max_{u\in\mathcal{U}}|Bu|\right\},$$

the Gronwall's Lemma (Proposition 2.4.1) yields to

$$|y_x^u(t)-x| \leq (1+|x|)(e^{c_f t}-1), \quad \forall x\in\mathcal{K}, \forall u\in\mathbb{U}(x), \forall t\geq 0. \tag{5.1}$$

In this section the running cost $\ell:\mathbb{R}^N\times\mathcal{U}\to\mathbb{R}$ is supposed to satisfy the same conditions as in Chapter 4, that is,

$$(H_\ell^5)\qquad \begin{cases} (i) & \ell(\cdot,\cdot) \text{ is continuous on } \mathbb{R}^N\times\mathcal{U}.\\ (ii) & \ell(\cdot,\cdot) \text{ is locally Lipschitz continuous on } \mathcal{K}\times\mathcal{U}.\\ (iii) & \exists c_\ell>0,\ \lambda_\ell\geq 1 \text{ such that } \forall (x,u)\in\mathcal{K}\times\mathcal{U}:\\ & \qquad 0\leq \ell(x,u)\leq c_\ell(1+|x|^{\lambda_\ell}).\end{cases}$$

Moreover, under these circumstances, we have that the minimal cost can be approximated by a sequence of trajectories remaining most of times on the relative interior of the state-constraints. To show this we do not require any further compatibility assumption.

Proposition 5.2.1. *Assume* (H_0^5) *holds together with* (H_f^5) *and* (H_ℓ^5), *and take* $\lambda>\lambda_\ell c_f$. *Then, for every* $x\in\operatorname{dom}\vartheta$ *and* $\varepsilon>0$ *we can find* $x_\varepsilon\in ri(\mathcal{K})\cap\mathbb{B}(x,\varepsilon)$ *and* $u_\varepsilon\in\mathbb{U}(x_\varepsilon)$ *so that*

$$\vartheta(x)+\varepsilon \geq \int_0^\infty e^{-\lambda t}\ell(y_{x_\varepsilon}^{u_\varepsilon}(t), u_\varepsilon(t))dt \quad \text{and} \quad y_{x_\varepsilon}^{u_\varepsilon}(s)\in ri(\mathcal{K}),\ \forall t\in[0,+\infty).$$

Proof. First of all note that (H_f^5) implies that $F(x)=\{Ax+Bu \mid\ u\in\mathcal{U}\}$ has convex graph on $\mathcal{K}$. This is essentially due to the linearity of the dynamics and convexity of $\mathcal{U}$.

Fix $x\in\operatorname{dom}\vartheta$ and, let $\bar u\in\mathbb{U}(x)$ be a $\frac{\varepsilon}{3}$-suboptimal control and $\bar y(\cdot)$ be its corresponding trajectory with initial condition $\bar y(0)=x$. Pick $T>0$ large enough, but finite, such that the next inequality is guaranteed

$$0<\frac{1}{\lambda-c_f\lambda_\ell}c_\ell(1+|x|+\varepsilon)^{\lambda_\ell}e^{-(\lambda-c_f\lambda_\ell)T}\leq\frac{\varepsilon}{3}.$$

Let $r > 0$ so that $x_0, \bar{y}(t) \in \mathbb{B}(0,r)$ for any $t \in [0,T]$, and write $\mu = \max_{u\in\mathcal{U}} |u|$. We set $L_\ell > 0$ as the Lipschitz constant of ℓ on $\mathbb{B}(0,r) \times \mathcal{U}$ and take $\tilde{\varepsilon} \in (0, \min\{\varepsilon, 1\}]$ satisfying

$$0 < \tilde{\varepsilon} L_\ell \left(\frac{1}{\lambda}(|x - x_0| + 2\mu) + \frac{1}{\lambda - c_f}(1 + |x|) \right) \leq \frac{\varepsilon}{3}.$$

Recall also that $\lambda > c_f \lambda_\ell$ implies that $\lambda > c_f$, so $\tilde{\varepsilon}$ is well-defined.

Consider $y_\varepsilon = \tilde{\varepsilon} x_0 + (1 - \tilde{\varepsilon})\bar{y}$, this is an absolutely continuous function that remains on $\mathrm{ri}(\mathcal{K})$ because of the Accessibility Lemma. Note that, since $Ax_0 + Bu_0 = 0$, we have

$$\dot{y}_\varepsilon = (1 - \tilde{\varepsilon})\dot{\bar{y}} = \tilde{\varepsilon}(Ax_0 + Bu_0) + (1 - \tilde{\varepsilon})(A\bar{y} + B\bar{u}) = Ay_\varepsilon + B(\tilde{\varepsilon} u_0 + (1 - \tilde{\varepsilon})\bar{u}).$$

By virtue of the convexity of $\mathcal{U}$, $u_\varepsilon := \tilde{\varepsilon} u_0 + (1 - \tilde{\varepsilon})\bar{u} \in \mathcal{U}$ a.e. on $[0, +\infty)$, and consequently, y_ε is a trajectory of the control system. In particular, $u_\varepsilon \in \mathbb{U}(x_\varepsilon)$ with $x_\varepsilon = \tilde{\varepsilon} x_0 + (1 - \tilde{\varepsilon})x$. Notice as well that, by convexity, $y_\varepsilon \in \mathbb{B}(0,r)$, so for any $t > 0$

$$\begin{aligned} |\ell(y^*(t), \bar{u}(t)) - \ell(y_\varepsilon(t), u_\varepsilon(t))| &\leq L_\ell(|y^*(t) - y_\varepsilon(t)| + |\bar{u}(t) - u_\varepsilon(t)|) \\ &\leq \tilde{\varepsilon} L_\ell(|x - x_0| + (1 + |x|)(e^{c_f t} - 1) + 2\mu \end{aligned}$$

where the last step comes from the definition of y_ε and (5.1). Multiplying the inequality by $e^{-\lambda t}$ and integrating between $t = 0$ and $t = T$ we get

$$\int_0^T e^{-\lambda t} |\ell(y^*, \bar{u}) - \ell(y_\varepsilon, u_\varepsilon)| dt \leq \tilde{\varepsilon} L_\ell \left(\frac{1}{\lambda}(|x - x_0| + 2\mu) + \frac{1}{\lambda - c_f}(1 + |x|) \right).$$

Remark that, by the way how we have taken $\tilde{\varepsilon} \in (0, 1]$, the righthand side in the last inequality is smaller than $\frac{\varepsilon}{3}$. Furthermore, by the choice of T we have too

$$\int_T^\infty e^{-\lambda t} \ell(y_\varepsilon(t), u_\varepsilon(t)) dt \leq \frac{\varepsilon}{3},$$

this is due to (5.1), to the superlinear growth of ℓ, to the fact that $x_\varepsilon \in \mathbb{B}(x, \tilde{\varepsilon})$ and to $\tilde{\varepsilon} \leq \varepsilon$. Hence, gathering all the information, we get the desired result with u_ε and x_ε as described earlier, because $\ell \geq 0$ and

$$\begin{aligned} \frac{\varepsilon}{3} + \vartheta(x) \geq \int_0^\infty e^{-\lambda t} \ell(y^*, \bar{u}) dt &\geq \int_0^\infty e^{-\lambda t} \ell(y_\varepsilon, u_\varepsilon) dt \\ &+ \int_0^T e^{-\lambda t} (\ell(y^*, \bar{u}) - \ell(y_\varepsilon, u_\varepsilon)) dt - \int_T^\infty e^{-\lambda t} \ell(y_\varepsilon, u_\varepsilon) dt \end{aligned}$$

□

5.2.2 Characterization of the Value Function

In light of the property established in the preceding part for the optimal cost map, we can state the following theorem which provides a characterization of the Value Function in the framework adopted in the section. Recall that $\mathrm{ri}(\mathcal{K})$ is an embedded manifold of $\mathbb{R}^N$ and so, several of the conclusions of Section 4.2 suit on the setting. For this purpose, we introduce the set of tangent maps to the relative interior of $\mathcal{K}$

$$\mathcal{U}(x) = \{u \in \mathcal{U} \mid \; f(x,u) \in \mathcal{T}_{\mathrm{ri}(\mathcal{K})}(x)\}, \quad \forall x \in \mathrm{ri}(\mathcal{K}).$$

Recall that the affine hull of $\mathcal{K}$ can be identified with $x_0 + \ker(P)$, where P is a surjective matrix. So, $\mathcal{T}_{\mathrm{ri}(\mathcal{K})}(x) = \ker(P)$ for every $x \in \mathrm{ri}(\mathcal{K})$ and thereby

$$\mathcal{U}(x) = \{u \in \mathcal{U} \mid \; P(Ax + Bu) = 0\}, \quad \forall x \in \mathrm{ri}(\mathcal{K}).$$

Hence, as well as done in Section 4.2, we assume that the tangent controls is a regular map in the following sense:

(H_1^5) $\quad$ $\mathcal{U}(\cdot)$ is locally Lipschitz continuous on $\mathrm{ri}(\mathcal{K})$ w.r.t. the Hausdorff distance.

In case that $\mathrm{int}(\mathcal{K}) \neq \emptyset$, we can take $P = 0$ and (H_1^5) holds trivially with $\mathcal{U}(x) \equiv \mathcal{U}$ all along $\mathrm{ri}(\mathcal{K}) = \mathrm{int}(\mathcal{K})$. In addition, as aforesaid in Remark 4.2.2, $\mathcal{U}(\cdot)$ can be extended up to $\mathcal{K}$ in a locally Lipschitz continuous way, which we might denote by $x \mapsto \overline{\mathcal{U}}(x)$. Notice that, in general, we only have

$$\{u \in \mathcal{U} \mid \; f(x,u) \in \mathcal{T}_{\mathcal{K}}^B(x)\} \subseteq \overline{\mathcal{U}}(x), \quad \forall x \in \mathcal{K}.$$

Accordingly, the tangential Hamiltonian to $\mathrm{ri}(\mathcal{K})$ is defined via:

$$H_{\mathrm{ri}}(x,\zeta) = \max_{u \in \mathcal{U}(x)} \{-\langle Ax + Bu, \zeta \rangle - \ell(x,u)\}, \quad \forall x \in \mathcal{K}, \zeta \in \mathbb{R}^N.$$

By the reasoning exposed in Proposition 4.2.2, this Hamiltonian is locally Lipschitz continuous provided (H_1^5) holds. Additionally, in agreement with the earlier discussion, if $\mathrm{int}(\mathcal{K}) \neq \emptyset$, then H_{ri} coincides with the usual Hamiltonian H.

We are now in position to provide and prove the principal statement of the section.

Theorem 5.2.1. *Suppose that (H_0^5) and (H_1^5) hold in addition to (H_f^5) and (H_ℓ^5). Assume also that $\lambda > \lambda_\ell c_f$ and*

$$u \mapsto \ell(x,u) \text{ is a convex function from } \mathcal{U} \text{ into } \mathbb{R} \text{ for any } x \in \mathcal{K} \text{ fixed.}$$

Then the value function $\vartheta(\cdot)$ of the infinite horizon problem is the only lower semicontinuous function with λ_ℓ-superlinear growth which is $+\infty$ on $\mathbb{R}^N \setminus \mathcal{K}$ and that satisfies:

$$\lambda\vartheta(x) + H_{ri}(x,\zeta) = 0 \qquad \forall x \in ri(\mathcal{K}),\ \forall \zeta \in \partial_V \vartheta(x), \tag{5.2}$$

$$\lambda\vartheta(x) + H_{ri}(x,\zeta) \geq 0 \qquad \forall x \in rbd(\mathcal{K}),\ \forall \zeta \in \partial_V \vartheta(x). \tag{5.3}$$

Proof. Notice first that if $\mathrm{int}(\mathcal{K}) = \mathrm{ri}(\mathcal{K})$, then is rather standard that $\vartheta(\cdot)$ verifies (5.2) and (5.3), this is because of $H = H_{\mathrm{ri}}$. Let us focus on the case $\mathrm{int}(\mathcal{K}) = \emptyset$ and $\mathrm{ri}(\mathcal{K}) \neq \emptyset$

Remark that since $f(x_0, u_0) = 0$ then $\mathcal{U}(x_0) \neq \emptyset$, and given that $\mathcal{U}(\cdot)$ is locally Lipschitz continuous, it has nonempty images all along $\mathrm{ri}(\mathcal{K})$. Furthermore, by Remark 4.2.1 we have that the augmented dynamics of Section 4.2 defined by

$$G(\tau, x) = \left\{ \begin{pmatrix} f(x,u) \\ e^{-\lambda\tau}(\ell(x,u) + r) \end{pmatrix} \;\middle|\; \begin{matrix} u \in \mathcal{U}, \\ 0 \leq r \leq \beta(x,u) \end{matrix} \right\}, \quad \forall (\tau, x) \in \mathbb{R} \times \mathbb{R}^N,$$

has convex images. This is due to the fact that $\mathcal{U}$ is convex, f is linear and $u \mapsto \ell(x,u)$ is a convex function for any $x \in \mathcal{K}$. Notice that (H_f^4) and (H_ℓ^4) are also satisfies thanks to (H_f^5) and (H_ℓ^5), respectively. Hence, we can apply many of the results proved in Chapter 4.

In particular, given that $\mathrm{ri}(\mathcal{K})$ can be treated as a embedded manifold (Proposition 5.1.1), adapting the arguments of Section 4.2, it is not difficult to see that $\vartheta(\cdot)$ is lower semicontinuous, has λ_ℓ-superlinear and verifies (5.2) and (5.3).

Additionally, by Proposition 4.2.5, Lemma 4.2.1 and Remarks 4.2.7 we have that any lower semicontinuous function $\varphi : \mathcal{K} \to \mathbb{R} \cup \{+\infty\}$ having λ_ℓ-superlinear growth that solves (5.2)-(5.3) must satisfy $\varphi \geq \vartheta$ over $\mathcal{K}$. This is because $H(\cdot) \geq H_{\mathrm{ri}}(\cdot)$ on $\mathbb{R}^N \times \mathbb{R}^N$ which implies that φ verifies

$$\lambda\varphi(x) + H(x,\zeta) \geq 0 \qquad \forall x \in \mathcal{K},\ \forall \zeta \in \partial_V \varphi(x).$$

On the other hand, let $x \in \mathcal{K}$ and $u \in \mathbb{U}(x)$, and suppose φ solves (5.2). Due to the inclusion, $\partial_P\varphi(\cdot) \subseteq \partial_V\varphi(\cdot)$ we have that

$$\lambda\varphi(\tilde{x}) + H_{\mathrm{ri}}(\tilde{x},\zeta) \leq 0 \qquad \forall \tilde{x} \in \mathrm{ri}(\mathcal{K}),\ \forall \zeta \in \partial_P \varphi(\tilde{x}).$$

Notice that the proof given for Lemma 4.2.3 can be suited for this framework, so that we can show, for any $t > 0$, that if $y_x^u(s) \in \mathrm{ri}(\mathcal{K})$ for every $s \in (0,t)$ then

$$\varphi(x) \leq e^{-\lambda t}\varphi(y_x^u(t)) + \int_0^t e^{-\lambda s}\ell(y,u)ds.$$

Let $\varepsilon_n > 0$ with $\varepsilon_n \to 0$. We take $x_n \in \mathrm{ri}(\mathcal{K}) \cap \mathbb{B}(x,\varepsilon_n)$ and $u_n \in \mathbb{U}(x_n)$ the point and control given by Proposition 5.2.1. Thereupon, we write y_n for $y_{x_n}^{u_n}$, and thus, in light of the preceding remark, for any $t > 0$, we have

$$\varphi(x_n) \leq e^{-\lambda t}\varphi(y_n(t)) + \int_0^t e^{-\lambda s}\ell(y_n,u_n)ds \leq e^{-\lambda t}\varphi(y_n(t)) + \vartheta(x) + \varepsilon_n.$$

By virtue of the superlinear growth of φ, letting $t \to +\infty$, we obtain $\varphi(x_n) \leq \vartheta(x) + \varepsilon_n$. Accordingly, since $x_n \to x$ and φ is lower semicontinuous, making $n \to +\infty$ we find out that $\varphi(x) \leq \vartheta(x)$. So, $\varphi = \vartheta$ over $\mathcal{K}$ which ends the proof.

$\square$

5.3 Mayer problem and convex dynamics

The technique used for the infinite horizon problem with linear dynamics can be extended to the case in which the cost to be minimized does not depends directly upon the control, as for instance, in the Mayer problem. We evoke that the Value Function for this class of processes is

$$\vartheta(t,x) := \inf_{y \in \mathbb{S}_t^T(x)} \psi(y(T)), \quad \forall x \in \mathcal{K}, t \in [0,T],$$

where $\mathbb{S}_t^T(x)$ stands for the set of admissible curves of the control system

$$\dot{y} \in F(y) \quad \text{a.e. on } [t,T], \quad y(t) = x, \quad y(s) \in \mathcal{K},\ \forall s \in [t,T].$$

The study of the section considers only the case in which the final cost $\psi : \mathbb{R}^N \to \mathbb{R}$ satisfies:

(H_ψ^5) $\qquad$ $\psi(\cdot)$ is continuous on $\mathcal{K}$.

Remark 5.3.1. *In the setting of the problem, the continuity of ψ is the minimal requirement we are compelled to do over the final cost. This precludes the implicit incorporation of an end-point constraint in the definition of ψ because our analysis can not be directly extended to a lower semicontinuous framework. To do so, it might be necessary to consider some further assumptions; we refer to the discussion at the end of the chapter for more details.*

5.3.1 Interior approximations

At present, the linearity of the dynamics is replaced by requiring that the graph of the dynamics multifunction is convex. In addition, we suppose the following hypothesis in all the rest of the section.

$$(H_F^5)\quad \begin{cases} i) \quad F(x) \text{ is nonempty compact and convex on a neighborhood of } \mathcal{K}. \\ ii) \quad \mathrm{gr}(F) = \{(x,v)\in\mathbb{R}^N\times\mathbb{R}^N \mid \ v\in F(x)\} \text{ is convex and closed on } \mathbb{R}^N\times\mathbb{R}^N. \\ iii)\ \exists c_F>0 \text{ such that } \max\{|v| \mid \ v\in F(x)\}\le c_F(1+|x|),\ \forall x\in\mathcal{K}. \end{cases}$$

Under the above-state assumption, trajectories of the dynamical system can be approximated by arcs lying in $\mathrm{ri}(\mathcal{K})$. We summed up this idea in the contiguous statement.

Proposition 5.3.1. *Let $t,T\in\mathbb{R}$ with $t<T$. Suppose (H_0^5) and (H_F^5) hold, and let the dynamics map $F:[t,T]\times\mathcal{K}\to\mathbb{R}^N$ verifies in addition*

$$(H_2^5)\qquad \exists x_F\in\mathcal{K},\ \exists y_F\in\mathbb{S}_t^T(x_F) \ \textit{s.t.}\ y_F(s)\in \mathit{ri}(\mathcal{K}),\ \forall s\in(t,T).$$

Then, for any $x\in\mathcal{K}$, $y\in\mathbb{S}_t^T(x)$ and $\varepsilon>0$ we can find $x_\varepsilon\in\mathcal{K}\cap\mathbb{B}(x,\varepsilon)$ and $y_\varepsilon\in\mathbb{S}_t^T(x_\varepsilon)$ such that

$$\sup_{s\in[t,T]}|y(s)-y_\varepsilon(s)|\le\varepsilon \quad \textit{and} \quad y_\varepsilon(s)\in \mathit{ri}(\mathcal{K}),\ \forall s\in(t,T).$$

Remark 5.3.2. *The hypothesis (H_2^5) is a strengthen version of the assumption done in the preceding section ($\exists(x_0,u_0)\in \mathit{ri}(\mathcal{K})\times\mathcal{U}$ so that $f(x_0,u_0)=0$). Indeed, if there exists a critical point $x_0\in \mathit{ri}(\mathcal{K})$ of F, i.e. $0\in F(x_0)$, then (H_2^5) is straightforward.*

Proof of Proposition 5.3.1. Take $y\in\mathbb{S}_t^T(x)\setminus\{y_F\}$ and set $\rho=\sup_{s\in[t,\tau]}|y_F(s)-y(s)|>0$. If $\rho\le\varepsilon$ we put $y_\varepsilon=y_F$ and the result follows. Otherwise, we set $\lambda=\varepsilon/\rho\in(0,1)$. We define $y_\varepsilon:[t,T]\to\mathcal{K}$ via

$$y_\varepsilon(s)=[1-\lambda]y(s)+\lambda y_F(s),\quad \forall s\in[t,T].$$

Notice that y_ε is an absolutely continuous function and, by (H_2^5) and the Accessibility Lemma (Proposition 2.3.1), $y_\varepsilon(s)\in\mathrm{ri}(\mathcal{K})$ for any $s\in(t,T)$. Furthermore, by construction,

$$|y_\varepsilon(s)-y(s)|\le\lambda|y_F(s)-y(s)|\le\varepsilon,\quad \forall s\in[t,T].$$

To conclude the proof it is required to show that $y_\varepsilon\in\mathbb{S}_t^T(x_\varepsilon)$ where $x_\varepsilon=\lambda x_F+(1-\lambda)x$; notice that $x_\varepsilon\in\mathbb{B}(x,\varepsilon)\cap\mathcal{K}$ by convexity. By the definition of y_ε we get

$$\dot y_\varepsilon(s)=(1-\lambda)\dot y(s)+\lambda\dot y_F(s),\qquad \text{for a.e. } s\in[t,T].$$

Since F has convex graph, then

$$\lambda w+(1-\lambda)v\in F(\lambda y_F(s)+(1-\lambda)y(s)),\quad \forall s\in[t,T],\forall w\in F(y_F(s)),\forall v\in F(y(s)).$$

Finally, due to $\dot y_F(s)\in F(y_F(s))$ and $\dot y(s)\in F(y(s))$ a.e. on (t,T) the proof is complete. □

In view of the foregoing result, we can provide a statement, which is analogous to Proposition 5.2.1, adapted for the Mayer problem.

Proposition 5.3.2. *Suppose* (H_0^5) *and* (H_2^5) *hold together with* (H_ψ^5) *and* (H_F^5). *Then, for every* $(t, x) \in dom\,\vartheta$ *and* $\varepsilon > 0$ *there exist* $x_\varepsilon \in \mathcal{K} \cap \mathbb{B}(x, \varepsilon)$ *and* $y_\varepsilon \in \mathbb{S}_t^T(x_\varepsilon)$ *such that*

$$\vartheta(t, x) + \varepsilon \geq \psi(y_\varepsilon(T)) \quad \textit{and} \quad y_\varepsilon(s) \in ri(\mathcal{K}),\ \forall s \in (t, T).$$

Proof. Let $\bar{y} \in \mathbb{S}_t^T(x)$ be an $\frac{\varepsilon}{2}$-suboptimal trajectory. By continuity, there exists $\delta > 0$ so that if $\tilde{x} \in \mathbb{B}(\bar{y}(T), \delta)$ then $|\psi(\bar{y}(T)) - \psi(\tilde{x})| \leq \frac{\varepsilon}{2}$. Let x_ε and y_ε be the point and curve given by Proposition 5.3.1 with approximation parameter δ (instead of ε in that statement). Conclusively, the result follows from noticing $y_\varepsilon(T) \in \mathbb{B}(\bar{y}(T), \delta)$ and that

$$\vartheta(t, x) + \frac{\varepsilon}{2} \geq \psi(\bar{y}(T) \geq \psi(y_\varepsilon(T)) - |\psi(\bar{y}(T) - \psi(y_\varepsilon(T))|.$$

□

5.3.2 The Value Function

The aim of this section is to prove that the Value Function of the Mayer problem is the only constrained viscosity solution of the HJB equation

$$-\partial\vartheta(t, x) + H(x, \nabla\vartheta(t, x)) = 0, \quad \forall (t, x) \in [0, T] \times \mathcal{K}.$$

The main result of this section requires also a Lipschitz-like assumption over the admissible velocities on the relative interior of $\mathcal{K}$. For this purpose we recall that in Section 4.3 we introduced the set-valued map $F^\sharp(x) = F(x) \cap \mathcal{T}_\mathcal{K}^B(x)$ for any $x \in \mathcal{K}$, where $\mathcal{T}_\mathcal{K}^B(\cdot)$ stands for the Bouligand tangent cone. Hence the main result of this section is as follows.

Theorem 5.3.1. *Suppose* (H_0^5) *and* (H_2^5) *hold together with* (H_ψ^5) *and* (H_F^5). *Assume in addition that*

(H_3^5) $\qquad F^\sharp(\cdot)$ *is locally Lipschitz continuous on* $ri(\mathcal{K})$.

Then ϑ *is the unique lower semicontinuous function on* $\mathcal{K}$ *that is* $+\infty$ *elsewhere, that satisfies* $\vartheta(T, x) = \psi(x)$ *for any* $x \in \mathcal{K}$ *and that verifies*

$$-\theta + \max_{v \in F^\sharp(x)} \{-\langle v, \zeta\rangle\} \leq 0, \quad \forall (t, x) \in (0, T] \times ri(\mathcal{K}),\ \forall (\theta, \zeta) \in \partial_V\vartheta(t, x), \tag{5.4}$$

$$-\theta + \max_{v \in F^\sharp(x)} \{-\langle v, \zeta\rangle\} \geq 0, \quad \forall (t, x) \in [0, T) \times \mathcal{K},\ \forall (\theta, \zeta) \in \partial_V\vartheta(t, x). \tag{5.5}$$

The proof of the foregoing theorem follows the same scheme as that of Theorem 5.2.1. However, before going further we discuss a few points about this statement.

Remark 5.3.3. *It is worthy to note the similarity between this statement and Theorem 4.3.1. Indeed, in this case* $ri(\mathcal{K})$ *plays exactly the same role as one of the strata of* $\mathcal{K}$ *(in the case it is stratifiable). This is the reason why the Lipschitz continuity of* $F^\sharp$ *is only required on* $ri(\mathcal{K})$ *and not in the whole set* $\mathcal{K}$.

Furthermore, as aforesaid, if $int(\mathcal{K}) \neq \emptyset$ *then the* (H_3^5) *is immediately satisfied.*

Proof of Theorem 5.3.1. First of all, note that (H_ψ^5) and (H_F^5) implies that (H_ψ^4) and (H_F^4) are verified. Hence, by Proposition 4.3.1 the Value Function of the Mayer problem is lower semicontinuous on $\mathcal{K}$.

Thanks to Proposition 4.3.2 we have that the Value function verifies (5.5) and that any other lower semicontinuous function φ that satisfies the end-point condition $\varphi(T,x) = \psi(x)$ on $\mathcal{K}$ and (5.5) must be larger or equal to ϑ on $[0,T]\times\mathcal{K}$.

On the other hand, a slight modification of Proposition 4.3.3 leads to assert that the Value function satisfies (5.4). Moreover, the same arguments used to prove Claim A in the proof of Proposition 4.3.3 yields to the following affirmation

Claim D: For each $(t,x)\in\operatorname{dom}\vartheta$ and $y\in\mathbb{S}_t^T(x)$ for which $y(s)\in\operatorname{ri}(\mathcal{K})$ for all $s\in(t,T)$, we have $\psi(y(T))\geq\varphi(t,x)$.

Finally, take a sequence $\varepsilon_n>0$ so that $\varepsilon_n\to 0$ as $n\to+\infty$ and $(t,x)\in\operatorname{dom}\vartheta$. By Proposition 5.3.2 we can construct a sequence $\{x_n\}\subseteq\operatorname{ri}(\mathcal{K})$ converging to x and pick for any $n\in\mathbb{N}$, a curve of the control system $y_n\in\mathbb{S}_t^T(x_n)$ that verifies

$$\vartheta(t,x)+\varepsilon_n\geq\psi(y_n(T))\quad\text{and}\quad y_n(s)\in\operatorname{ri}(\mathcal{K}),\ \forall s\in(t,T).$$

By Claim D, we get $\vartheta(t,x)+\varepsilon_n\geq\varphi(t_n,x_n)$ for any $n\in\mathbb{N}$. consequently, letting $n\to+\infty$ and using the lower semicontinuity of φ we conclude the proof.

□

5.4 Discussion and perspectives

We conclude this chapter comparing the results we have stated so far and showing some possible adaptation to further general cases.

5.4.1 Contributions of the chapter

In this chapter we have pointed out a well-structured class of problems where the Value Function is the unique constrained viscosity solution of the HJB equation, that is, $\vartheta(\cdot)$ is a viscosity supersolution on $\mathcal{K}$ and a subsolution on $\operatorname{ri}(\mathcal{K})$. This has been achieved without the necessity of introducing any further compatibility assumptions between dynamics and state-constraints, which is the main novelty of the presented work.

Let us stress that the exposition of the chapter strongly relies on the linear-like structure of the dynamics and in the existence of an arbitrary interior feasible trajectory. Therefore, the analysis we have done is hardly applicable to purely nonlinear systems because in that case the graph of the dynamics is seldom convex. For instance, a control-affine system for which

$$\left\{(x,v)\in\mathcal{K}\times\mathbb{R}^N \mid\ v=f_0(x)+\sum u_i f_i(x),\ u\in\mathcal{U}\right\}$$

is convex, is likely to be a linear system.

Notably, if the condition $\mathcal{K}=\overline{\operatorname{ri}(\mathcal{K})}$ is not meet (even for linear systems), then the stratified methodology of Chapter 4 enhances its importance.

In synthesis, the contribution of this chapter lies in the identification of a class of problems of wide concern for which no additional compatibility hypotheses are required in order to make the HJB approach well-posed.

5.4.2 End-point constraints and star-shaped sets

Remark that for the Mayer problem we have only considered the case with continuous final cost which precludes possible end-point constraints such as $y(T) \in \Theta$. Nevertheless, an extension to this case can be proposed with the help of an additional controllability assumption.

A small-time controllability hypothesis (see for instance [13, Chapter 4]) suffices if the target Θ lies on the relative interior of $\mathcal{K}$. Otherwise, the assumption has to be strengthen to small-time controllability from the relative interior. Roughly speaking, this means that for any point close enough of the target Θ it is possible to reach the target without passing for $\mathcal{K} \setminus \mathrm{ri}(\mathcal{K})$, the relative boundary of $\mathcal{K}$, and hitting directly the target. In such cases, the statement of Proposition 5.3.1 has to be replaced with next one.

Proposition 5.4.1. *Let $t, T \in \mathbb{R}$ with $0 \geq t < T$. Suppose that (H_0^5), (H_F^5) and (H_2^5) hold along with the small-time controllability hypothesis described above. Then, for any $x \in \mathcal{K}$, $y \in \mathbb{S}_t^T(x)$ and $\varepsilon > 0$ we can find $x_\varepsilon \in \mathcal{K} \cap \mathbb{B}(x, \varepsilon)$, $t_\varepsilon \in [0, T] \cap [t - \varepsilon, t + \varepsilon]$ and $y_\varepsilon \in \mathbb{S}_{t_\varepsilon}^T(x_\varepsilon)$ such that*

$$|y(T) - y_\varepsilon(T)| \leq \varepsilon, \quad y_\varepsilon(s) \in ri(\mathcal{K}),\ \forall s \in (t_\varepsilon, T) \quad \text{and} \quad y_\varepsilon(T) \in \Theta.$$

The foregoing statement is enough to prove Proposition 5.3.2 and so Theorem 5.3.1.

On the other hand, notice that the convexity of the state-constraints can be slightly relaxed without altering the veracity of the results stated along the chapter. This is principally because the fact that $\mathcal{K}$ is convex is used to ensure the following property:

> **Claim:** There exist $x_0 \in \mathcal{K}$ and $y_0 \in \mathbb{S}_t^T(x_0)$ with $y_0(s) \in \mathrm{ri}(\mathcal{K})$ for every $s \in (t, T)$, so that for any $x \in \mathcal{K}$ and $y \in \mathbb{S}_t^T(x)$
>
> $$\alpha y(s) + (1 - \alpha) y_0(s) \in \mathrm{ri}(\mathcal{K}), \quad \forall s \in (t, T), \forall \alpha \in [0, 1).$$

If $\mathcal{K}$ is convex, then the Accessibility Lemma guarantees the preceding claim (for dynamics with convex graph). However, the property holds as well in others context. For instance if $\mathcal{K}$ is a star-shaped set whose center is an equilibrium point, that is, there exists $x_0 \in \mathrm{ri}(\mathcal{K})$ verifying

- $\alpha x + (1 - \alpha) x_0 \in \mathcal{K}$ for each $x \in \mathcal{K}$ and any $\alpha \in [0, 1]$.
- Either $0 \in F(x_0)$ or $\exists u_0 \in \mathcal{U}$ such that $f(x_0, u_0) = 0$.

In this situation, the above-stated claim is fulfilled with the curve defined via $y_0(s) = x_0$ for each $s \in [t, T]$, provided the dynamics has convex graph.

CHAPTER 6

Convex State-Constraints II: Absorbing Dynamics

Abstract. In this chapter we study a special type of nonlinear optimal control problems with convex state-constraints. We exhibit a technique based on a class of penalization maps called the Legendre functions. We provide a characterization of the Value Function by means of the Hamilton-Jacobi-Bellman approach.

6.1 Introduction

In the preceding chapter we have investigated optimal control processes with a convex-linear structure. At present, we explore another way of dealing with convex state-constraints but from a penalization point of view. This approach allows us to relate the problem at hand to another without state-constraints, and so, to apply the available results for the unrestricted case to the original one through an appropriate auxiliary problem.

The fact that the original optimal control problem with convex state-constraints is equivalent to an unconstrained one is due to a suitable change of variables which turns out to be an isometry between $\mathrm{ri}(\mathcal{K})$ and an Euclidean space; the existence of this isometry was already remarked by Álvarez-Bolte-Brahic in [5]. Under this framework we can characterize the Value Function of the Mayer problem as the unique (continuous) viscosity solution of the Hamilton-Jacobi-Bellman (HJB) equation

$$-\partial_t \vartheta(t,x) + H(x, \nabla\vartheta(t,x)) = 0, \quad \forall t \in [0,T] \times \mathrm{ri}(\mathcal{K}).$$

The class of problems we consider in the chapter might verify an absorbing property at the relative boundary of $\mathcal{K}$, namely

$$F(x) = \{0\}, \quad \forall x \in \mathcal{K} \setminus \mathrm{ri}(\mathcal{K}).$$

It is recognizable that the class of systems taken into account do not satisfy any pointing qualification condition. Furthermore, a stratified approach as in Chapter 4 might be suitable for treating this case. Nonetheless, the exposition does not take in consideration Lipschitz continuous dynamics as in the standard sense, but in another one, more appropriate for the penalization technique; see (H_2^6) for more details. This allows us to deal with cases not covered by either the current literature, Chapter 4 or Chapter 5.

The contents of the present chapter has been included in [67] where a broader analysis has been carried out. That study encompasses some applications to mathematical programming as well as the study about the HJB approach announced earlier.

6.2 Characterization for the Mayer problem

Recall that a function $g : \mathbb{R}^N \to \mathbb{R} \cup \{+\infty\}$ is called of Legendre type if it is a lower semi-continuous convex and proper that is in addition essentially smooth and essentially strictly convex on its domain, meaning that g verifies:

- $\mathrm{int}(\mathrm{dom}\, g) \neq \emptyset$ and g is differentiable on $\mathrm{int}(\mathrm{dom}\, g)$.
- $|\nabla g(x_n)| \to +\infty$ whenever $\{x_n\} \subseteq \mathrm{int}(\mathrm{dom}\, g)$ so that $x_n \to \bar{x}$ for some $\bar{x} \in \mathrm{bdry}(\mathrm{dom}\, g)$.
- g is strictly convex on every convex subset of $\mathrm{dom}\, \partial g$.

We also refer to Section 2.3.1 for further details and its extension to arbitrary finite dimensional vectorial spaces. All through this chapter we write g^* for the Legendre-Fenchel conjugate of g and we denote by ∇g the gradient of g. If no confusion can arise, we write

$$G(x) := \nabla^2 g(x), \quad \forall x \in \mathrm{int}(\mathrm{dom}\, g).$$

Everywhere on the chapter we use the notation $\Omega = \mathrm{ri}(\mathcal{K})$ and $\partial\Omega = \mathcal{K} \setminus \mathrm{ri}(\mathcal{K})$. Without loss of generality we assume that $0 \in \mathcal{K}$. Hence, in virtue of the convexity of $\mathcal{K}$, there exists a vectorial subspace X of $\mathbb{R}^N$ so that $\mathcal{K} \subseteq X$ and $\mathrm{ri}(\mathcal{K})$ is an open set relatively to X. For the rest of the exposition, we assume that $\mathcal{K}$, Ω and X are given and fixed.

Let us consider the following definition that is based in the notion of Legendre function.

Definition 6.2.1. *Let $g : X \to \mathbb{R} \cup \{+\infty\}$ be a convex, proper and lower semicontinuous function and let $\Omega \subseteq X$ be a nonempty open convex set. We say that the pair (Ω, g) is Legendre zone consistent if g is a Legendre function with int(dom g) $= \Omega$.*

In general, Ω is given and we look for a Legendre function such that the pair (Ω, g) is Legendre zone consistent. The existence and construction of such a function for an arbitrary open convex set is not a trivial task, however, as shown by Borwein-Vanderwerff in [27], for many important cases it is possible to give a positive answer; see also [28, Chapter 7.4].

The key assumption of the chapter is the following

$$(H_0^6) \qquad \begin{cases} \text{There exists a Legendre function } g : X \to \mathbb{R} \cup \{+\infty\} \text{ so that:} \\ i) \quad (\Omega, g) \text{ is Legendre zone consistent with } \mathrm{dom}\, g^* = X. \\ ii) \quad g \in \mathcal{C}^2(\Omega) \text{ and } G(x) \text{ is positive-definite for each } x \in \Omega. \end{cases}$$

This hypothesis yields to assert that ∇g is a diffeomorphism from Ω into X. Consequently, ∇g is a suitable change of coordinates.

Lemma 6.2.1. *If (H_0^6) holds, then $\nabla g : \Omega \longrightarrow X$ is a diffeomorphism with $\nabla g^{-1} = \nabla g^*$.*

Proof. Since g is a Legendre function with nonsingular Hessian, the conclusion follows from the Inverse Function Theorem and Proposition 2.3.3. ☐

The main theorem we study is about the characterization of the Value Function with convex state-constraints for a system where no pointing qualification assumption holds. The analysis is focalized on the Mayer problem

$$\vartheta(t,x) := \inf_{y\in\mathbb{S}_t^T(x)} \psi(y(T)), \quad \forall x \in \mathcal{K}, t\in[0,T],$$

where $\mathbb{S}_t^T(x)$, as usual, stands for the set of admissible curves of the dynamical system

$$\dot{y}(s) \in F(y(s)) \quad \text{a.e. on } [t,T], \quad y(t)=x, \quad y(s)\in\mathcal{K},\ \forall s\in[t,T].$$

We recall that given $\mathcal{O}\subseteq X$ open (relative to X), a *viscosity solution* of the Hamilton-Jacobi (HJ) equation (see for instance [13, Chapter 2])

$$\mathbb{H}(x,\nabla\omega(t,x))=0, \quad \forall (t,x)\in(0,T)\times\mathcal{O}$$

is a continuous function which is a supersolution and subsolution in the following sense:

- A lower semicontinuous function ω is a *viscosity supersolution* of the HJ equation if
$$\mathbb{H}(x,(\theta,\zeta))\geq 0, \quad \forall (t,x)\in(0,T)\times\mathcal{O},\ (\theta,\zeta)\in\partial_V\omega(t,x).$$
- An upper semicontinuous function ω is a *viscosity subsolution* of the HJ equation if
$$\mathbb{H}(x,(\theta,\zeta))\leq 0, \quad \forall (t,x)\in(0,T)\times\mathcal{O},\ (-\theta,-\zeta)\in\partial_V(-\omega)(t,x).$$

The framework of the problem is posed under rather nonstandard hypotheses, which are anyhow, suitable for the penalization technique; the underlying reason is that (H_0^6) allows us to endow Ω with the structure of Riemannian manifold. These type of assumptions are not covered by the current literature, which is mainly done for Lipschitz continuous dynamics. Then the main theorem of the chapter reads as follows.

Theorem 6.2.1. *Suppose (H_0^6) holds. Let $F:\mathcal{K}\rightrightarrows X$ be a set-valued maps having nonempty compact convex images. Assume that ψ is uniformly continuous and that the dynamics is absorbing at $\partial\Omega$ in the sense that*

$$(H_1^6) \qquad \exists c>0 \text{ so that } \max_{v\in F(x)}|v| \leq c\left(\frac{1+|\nabla g(x)|}{|G(x)|}\right), \quad \forall x\in\Omega.$$

Consider as well that F satisfies a Lipschitz-like estimate, that is, for each $r>0$, there is $L>0$ for which

$$(H_2^6) \qquad \nabla^2 g(x)F(x)\subseteq \nabla^2 g(\tilde{x})F(\tilde{x}) + L|\nabla g(x)-\nabla g(\tilde{x})|\mathbb{B}, \quad \forall x,\tilde{x}\in\Omega\cap\mathbb{B}(0,r).$$

Then ϑ is the unique uniformly continuous function on Ω that satisfies $\vartheta(T,x)=\psi(x)$ for any $x\in\Omega$ and it is a viscosity solution of

$$-\partial_t\omega(t,x)+H(x,\nabla_x\omega(t,x))=0, \quad (t,x)\in(0,T)\times\Omega.$$

Remark 6.2.1. *We say that (H_1^6) is an absorbing condition at the boundary of Ω because we can extend the function $\omega(x)=\dfrac{1+|\nabla g(x)|}{|G(x)|}$ in a continuous way up to $\mathcal{K}$ by setting $\omega(x)=0$ whenever $x\in\partial\Omega$. This implies in particular that $F(x)=\{0\}$ on $\partial\Omega$. The extension is justified by the Fundamental Theorem of Calculus.*

6.3 Dual optimal control problem

The proof of Theorem 6.2.1 is based on the Legendre change of coordinates $\nabla g : \Omega \to X$. As a first stage, we study an auxiliary problem which is intrinsically related to the original Mayer problem. We begin with the definition of an auxiliary differential inclusion.

Take $(t, x) \in [0, T] \times \Omega$ and let $y \in \mathbb{S}_t^T(x)$ satisfying $y(s) \in \Omega$ for any $s \in [t, T]$. Then the arc $s \mapsto p(s) := \nabla g(y(s))$ is a solution of the differential inclusion

$$\dot{p}(s) \in \Phi_g(p(s)) \quad \text{a.e. } s \in [t, T], \quad p(s) = \nabla g(x), \tag{6.1}$$

where $\Phi_g(q) = [\nabla^2 g^*(q)]^{-1} F(\nabla g^*(q))$ for any $q \in X$.

Remark 6.3.1. *Let Ω be given and suppose that there exists a Legendre function on X, together with two matrices A and B of dimensions $n \times n$ and $n \times m$, respectively, for which*

$$\nabla^2 g(x) F(x) = \{A \nabla g(x) + Bu : \ u \in [-1, 1]^m\}, \quad \forall x \in \Omega.$$

Under these circumstances, the dual differential equation (6.1) is a linear system because

$$\Phi_g(q) = \{Aq + Bu : \ u \in [-1, 1]^m\}, \quad \forall q \in X.$$

Furthermore, if there exist two vector fields $f_1, f_2 : X \to X$ so that

$$\nabla^2 g(x) F(x) = \{f_1(\nabla g(x)) + f_2(\nabla g(x))u : \ u \in [-1, 1]\}, \quad \forall x \in \Omega,$$

the dual differential equation (6.1) is a control-affine system.

Notice that since $\operatorname{dom} g^* = X$, then $p(\cdot)$ remains in X, and so we can associate to (6.1) an unconstrained Mayer process. More precisely, let $\mathsf{S}_t^T(q\ ; g)$ indicate the set of absolutely continuous curves satisfying (6.1) with initial condition $p(t) = q$. The auxiliary problem at issue is:

$$\varpi_g(t, q) := \inf \left\{ \psi_g(p(T)) \mid \ p \in \mathsf{S}_t^T(q\ ; g) \right\}, \ \forall (t, q) \in [0, T] \times X,$$

where $\psi_g : X \to \mathbb{R}$ is the *auxiliary final cost* defined via

$$\psi_g(q) = \psi(\nabla g^*(q)), \qquad \forall q \in X.$$

The maps $\varpi_g : [0, T] \times X \to \mathbb{R} \cup \{+\infty\}$ is the Value Function of the auxiliary Mayer problem.

On the other hand, let $p \in \mathsf{S}_t^T(q\ ; g)$, then by means of the Legendre change of coordinates, $\nabla g^*(p) \in \mathbb{S}_t^T(\nabla g^*(q))$ and so, there is a one-to-one correspondence between $\mathsf{S}_t^T(\nabla g(x)\ ; g)$ and the trajectories of $\mathbb{S}_t^T(x)$ that live in the relative interior of the state-constraints. Consequently, in any case we have

$$\vartheta(t, x) \leq \varpi_g(t, \nabla g(x)), \quad \forall (t, x) \in [0, T] \times \Omega.$$

Without additional hypotheses the equally may not hold. However, under an interior approximation hypothesis the equality is reached.

Proposition 6.3.1. *Suppose (H_0^6) holds. Assume that ψ is uniformly continuous of modulus $\omega_\psi(\cdot)$ and that the following holds:*

$$(H_3^6) \qquad \begin{cases} \forall \varepsilon > 0,\ \forall t \in [0,T],\ \forall x \in \Omega,\ \forall y \in \mathbb{S}_t^T(x),\ \exists y_\varepsilon \in \mathbb{S}_t^T(x) \\ \text{such that: } y_\varepsilon(s) \in \Omega,\ \forall s \in [t,T] \text{ and } |y(T) - y_\varepsilon(T)| \leq \varepsilon. \end{cases}$$

Then

$$\varpi_g(t, \nabla g(x)) = \vartheta(t,x), \quad \forall (t,x) \in [0,T] \times \Omega.$$

Proof. If $\vartheta(t,x) = +\infty$, we get that $\varpi_g(t, \nabla g(x)) = +\infty$ as well, so under these circumstances, there is nothing to be proved.

Assume that $\vartheta(t,x) < +\infty$ and take $\{\varepsilon_n\} \subseteq (0,1)$ a sequence such that $\varepsilon_n \to 0$ as $n \to +\infty$. Thus, for any $n \in \mathbb{N}$, there exists $y_n \in \mathbb{S}_t^T(x)$ for which $\psi(y_n(T)) \leq \vartheta(t,x) + \varepsilon_n$. Besides, by ($H_3^6$), for any $n \in \mathbb{N}$ we can find $\tilde{y}_n \in \mathbb{S}_t^T(x)$ such that

$$\tilde{y}_n(s) \in \Omega,\ \forall s \in [t,T] \quad \text{and} \quad |\tilde{y}_n(T) - y_n(T)| \leq \varepsilon_n.$$

By gathering the last inequalities we get

$$\psi\left(\tilde{y}_n(T)\right) \leq \psi(y_n(T)) + \omega_\psi\left(\varepsilon_n\right) = \vartheta(t,x) + \varepsilon_n + \omega_\psi\left(\varepsilon_n\right).$$

Moreover, since $p_n := \nabla g\left(\tilde{y}_n\right) \in \mathsf{S}_t^T(\nabla g(x)\ ; g)$

$$\psi\left(\tilde{y}_n(T)\right) = \psi_g\left(p_n(T)\right) \geq \varpi_g(t, \nabla g(x)) \geq \vartheta(t,x).$$

So, letting $n \to +\infty$, the conclusion follows. □

The importance of the previous statement lies in the fact that for optimal control problems without state-constraints, the Value Function can be characterized by means of the HJB approach. Indeed, under the assumptions of Theorem 6.2.1 we can identify ϖ_g as the unique viscosity solution of a HJB equation.

Proposition 6.3.2. *Suppose that (H_0^6), (H_1^6) and (H_2^6) hold. Assume that ψ is continuous and F has nonempty convex compact images on $\mathcal{K}$. Then ϖ_g is continuous, satisfies $\varpi_g(T,q) = \psi_g(q)$ for any $q \in X$ and is the unique viscosity solution of*

$$-\partial_t \omega(t,q) + \widetilde{H}(q, \nabla_q \omega(t,q)) = 0, \quad (t,q) \in (0,T) \times X.$$

where $\widetilde{H}(q,\xi) = \sup\{-\langle v, \xi\rangle \mid\ v \in \Phi_g(q)\}$ for any $q, \xi \in X$.

Proof. Since F has nonempty convex compact images on $\mathcal{K}$, we can easily check that $\Phi_g(\cdot)$ has nonempty convex compact images on X. By the absorbing property (H_1^6), we have that $\Phi_g(\cdot)$ has linear growth on X and due to (H_2^6) it is also locally Lipschitz continuous on X. Moreover, since ψ is continuous so does ψ_g.

Finally, since auxiliary problem has no state-constraints, it is a classical result that under these circumstances the Value Function is the unique continuous viscosity solution of the HJB equation on the statement; see for instance [13, Theorem 3.7], [41, Proposition 4.7.10] or [131, Theorem 12.3.7]. Therefore, the result follows. □

Remark 6.3.2. *$\widetilde{H}$ can be written in terms of the original dynamics F as follows*

$$\widetilde{H}(q,\xi) := \sup\left\{\left\langle \nabla^2 g(\nabla g^*(q)) v, \xi\right\rangle \mid\ v \in F(\nabla g^*(q))\right\}, \quad \forall q, \xi \in X.$$

6.4 Auxiliary problem and final proof

Before presenting the proof of Theorem 6.2.1 we require some intermediate results to link the viscosity subgradients and supergradients of ϑ with those of ϖ. For this reason, in this section we investigate in a general way constrained mathematical programs over sets for which there exists a Legendre function so that (H_0^6) is verified.

6.4.1 Properties of the auxiliary problem

Let $\alpha : [0,T] \times X \to \mathbb{R} \cup \{+\infty\}$ be a given function. We consider the mathematical program

$$\text{(P)} \qquad \inf\{\alpha(t,x) \mid (t,x) \in [0,T] \times \Omega\}.$$

Define $\beta(t,q) = \alpha(t, \nabla g^*(q))$ for any $(t,q) \in [0,T] \times X$. By virtue of the Legendre coordinate transformation $q = \nabla g(x)$ we have

$$\text{(6.2)} \qquad \forall x \in \Omega,\ \exists q \in X \text{ such that } \alpha(t,x) = \beta(t,q).$$

$$\text{(6.3)} \qquad \forall q \in X,\ \exists x \in \Omega \text{ such that } \beta(t,q) = \alpha(t,x).$$

Therefore, we can associate to (P) an unconstrained auxiliary optimization problem defined on X which plays the role of *dual-auxiliary problem*

$$\text{(D)} \qquad \inf\{\beta(t,q) \mid (t,q) \in [0,T] \times X\}.$$

Furthermore, we can easily deduce the following statement.

Proposition 6.4.1. *Assume (H_0^6) holds and let $\alpha : [0,T] \times X \to \mathbb{R} \cup \{+\infty\}$ be a given function and $\beta : [0,T] \times X \to \mathbb{R} \cup \{+\infty\}$ be defined as above. Suppose $\{(t_n,x_n)\} \subseteq [0,T] \times \Omega$, then it is a minimizing sequence for (P) if and only if $\{(t_n, \nabla g(x_n))\}$ is a minimizing sequence for (D). Besides, we also have that $(t,x) \in [0,T] \times \Omega$ is a local minimizer (maximizer) of α if and only if $(t, \nabla g(x))$ is a local minimizer (maximizer) of β. Moreover, if α is continuous on its domain and* val(P)$\in \mathbb{R}$*, then* val(P)=val(D).

Proof. Note that by (6.2) and (6.3) the affirmation for the minimizing sequence holds true as well as the one about the local optimizers. Suppose that val(P)$\in \mathbb{R}$, thereupon, there exists a minimizing sequence $\{(t_n,x_n)\}$ such that $\alpha(t_n,x_n) \to$val(P). For every $n \in \mathbb{N}$ we can find $\tilde{x}_n \in \Omega$ so that $\alpha(t_n,\tilde{x}_n) \leq \alpha(t_n,x_n) + \frac{1}{n}$. Hence, noticing that $\alpha(t_n,\tilde{x}_n) \to$val(P) and using that $\{(t_n, \nabla g(\tilde{x}_n))\}$ is minimizing sequence for (D) we get the result. $\square$

Remark 6.4.1. *Define the function*

$$\alpha(t,x_1,x_2) := \begin{cases} \frac{1}{2}\log^2 x_1 + \exp\left(\tan x_2 \sec^2 x_2\right) & \textit{if } x_1 > 0,\ x_2 \in (-\frac{\pi}{2}, \frac{\pi}{2}) \\ \frac{1}{2}\log^2 x_1 & \textit{if } x_1 > 0,\ x_2 = -\frac{\pi}{2} \\ +\infty & \textit{otherwise.} \end{cases}$$

Let $\Omega = (0,+\infty) \times (-\frac{\pi}{2}, \frac{\pi}{2})$ and take $g(x_1,x_2) = x_1 \log x_1 - x_1 + \frac{1}{2}\tan^2 x_2$. We see that (Ω, g) is a Legendre zone consistent pair satisfying (6.8). Let us consider the mathematical program

$$(P_1) \quad \min\left\{\alpha(t,x_1,x_2) \mid 0 < x_1, -\frac{\pi}{2} \leq x_2 < \frac{\pi}{2}\right\}.$$

The dual-auxiliary problem is

$$(D_1) \quad \min\left\{\frac{1}{2}y_1^2 + \exp y_2 \mid y_1, y_2 \in \mathbb{R}\right\}.$$

Notice that a solution of (P_1) *is* $(t, 1, -\frac{\pi}{2}) \in [0,T] \times \partial\Omega$ *and no solution of* (D_1) *exists, but* $\{(t, 0, -n)\}_n$ *is a minimizing sequences for the dual problem and* $\mathrm{val}(\mathrm{P}_1) = \mathrm{val}(\mathrm{D}_1)$.

When two function α and β are defined as above, one may expect that differentiability properties on the primal function are inherited by the dual function, and vice versa. This is summarized in the upcoming proposition.

Proposition 6.4.2. *Suppose* (H_0^6) *holds and consider two extended-real valued functions* $\alpha, \beta : [0,T] \times X \to \mathbb{R} \cup \{+\infty\}$ *that satisfy*

$$\alpha(t, x) = \beta(t, \nabla g(x)), \quad \forall (t, x) \in [0,T] \times \Omega.$$

Let $t \in [0,T]$ *arbitrary. Then,* $x \mapsto \alpha(t, x)$ *is continuously differentiable at* $x \in \Omega$ *if and only if* $q \mapsto \beta(t, q)$ *is continuously differentiable at* $q = \nabla g(x)$*. In any case, we have*

$$\nabla_q \beta(t, q) = G(x)^{-1} \nabla_x \alpha(t, x).$$

Proof. It is enough to use the chain rule and the regularity of g and g^*. □

Now, this type of result can also be asserted for non differentiable functions. Indeed, it is possible to characterize $\partial_V \beta(\cdot)$ when β is merely lower semicontinuous.

Recall that, if $\alpha : [0,T] \times X \to \mathbb{R} \cup \{+\infty\}$ is lower semicontinuous, $\partial_V \alpha(t, x)$, its viscosity subdifferential at $(t, x) \in (0,T) \times \Omega$, is the collection of all $(\theta, \zeta) \in \mathbb{R} \times X$ for which there exists a continuous function $\varphi : \mathbb{R} \times X$ which is continuously differentiable at (t, x) so that

$$\partial_t \varphi(t, x) = \theta, \quad \nabla_x \varphi(t, x) = \zeta \text{ and } \alpha - \varphi \text{ attains a local minimum at } (t, x).$$

Proposition 6.4.3. *Assume* (H_0^6) *holds and consider two extended-real valued functions* $\alpha, \beta : [0,T] \times X \to \mathbb{R} \cup \{+\infty\}$ *that satisfy*

$$\alpha(t, x) = \beta(t, \nabla g(x)), \quad \forall (t, x) \in [0,T] \times \Omega.$$

Suppose that α *is lower semicontinuous, then for any* $(t, x) \in (0,T) \times \Omega$

$$(\theta, \zeta) \in \partial_V \alpha(t, x) \quad \Leftrightarrow (\theta, G(x)\zeta) \in \partial_V \beta(t, \nabla g(x)).$$

Proof. Let $(t, x) \in (0,T) \times \Omega$, $q = \nabla g(x)$ and $(\theta, \varrho) \in \partial_V \beta(t, q)$, then there exist $\delta > 0$ and a continuous function φ, continuously differentiable at (t, q) such that $\nabla \varphi(t, q) = (\theta, \varrho)$ and

$$\beta\left(\tilde{t}, \tilde{q}\right) - \varphi\left(\tilde{t}, \tilde{q}\right) \geq \beta(t, q) - \varphi(t, q), \quad \forall \tilde{t} \in (t - \delta, t + \delta), \forall \tilde{q} \in \mathbb{B}_X(q, \delta).$$

Set $\mathcal{O} = \nabla g^*(\mathbb{B}_X(q, \delta))$ and consider $\varphi_g : (t - \delta, t + \delta) \times \mathcal{O} \to \mathbb{R}$ given by

$$\varphi_g(t, x) = \varphi(t, \nabla g(x)), \quad \forall x \in \Omega.$$

By virtue of Proposition 6.4.2, this function is continuous on $(t - \delta, t + \delta) \times \mathcal{O}$ and continuously differentiable at (t, x) with $\nabla_x \varphi(t, x) = G(x)^{-1} \nabla_x \varphi_g(t, x)$. Additionally, by Proposition 6.4.1, (t, x) is a local minimizer of $\alpha - \psi_g$. Hence ψ_g is an admissible test function and $\nabla \varphi_g(t, x) \in \partial_V \alpha(t, x)$. So we have proven the sufficient condition. For the necessity the demonstration is analogous, so the conclusion follows. □

6.4.2 Proof of Theorem 6.2.1

Now with all this tools at hand we are in position to provide the proof of the principal statement of chapter.

Proof of Theorem 6.2.1. First of all, note that (H_3^6) holds trivially because for any $x \in \Omega$, $t \in [0,T]$ and $y \in \mathbb{S}_t^T(x)$, we have $y(s) \in \Omega$ as long as $s \in [t,T]$. Indeed, by shifting t if necessary, assume that $T = \inf\{s \in [t,T] \mid \ y(t) \notin \Omega\}$. Due to the Gronwall's Lemma, for any $\tau \in [t,T]$, each $p \in \mathsf{S}_t^\tau(\nabla g(x)\ ; g)$ verifies

$$|p(\tau)| \leq (1 + |\nabla g(x)|)e^{c(\tau-t)}.$$

Notice that for any $\tau \in [0,T)$, $\nabla g(y) \in \mathsf{S}_t^\tau(\nabla g(x)\ ; g)$ because $y(s) \in \Omega$ on $[t,T)$. Therefore,

$$|\nabla g(y(\tau))| \leq (1 + |\nabla g(x)|)e^{c(\tau-t)}, \quad \forall \tau \in [t,T).$$

Since $\operatorname{dist}_{\partial\Omega}(y(\tau)) \to 0$ as $\tau \to T$ we get a contradiction with the fact that g is essentially smooth on Ω, which implies that (H_3^6) holds.

On the other hand, By Proposition 6.3.2 we have ϖ_g is the unique continuous function that satisfies

$$-\theta + \widetilde{H}(q,\varrho) \geq 0, \quad \forall (t,q) \in (0,T) \times X,\ \forall (\theta,\varrho) \in \partial_V \omega(t,q), \tag{6.4}$$

$$-\theta + \widetilde{H}(q,\varrho) \leq 0, \quad \forall (t,q) \in (0,T) \times X,\ \forall (-\theta,-\varrho) \in \partial_V (-\omega)(t,q), \tag{6.5}$$

$$\omega(T,q) = \psi_g(q), \quad \forall q \in X. \tag{6.6}$$

Therefore, by Proposition 6.3.1 implies that $\vartheta(t,x) = \varpi_g(t,\nabla g(x))$ on $[0,T] \times \Omega$. By Proposition 6.4.3 we get that $(\theta, \nabla g(x)\zeta) \in \partial_V \vartheta(t,x)$ if and only if $(\theta,\zeta) \in \partial_V \varpi_g(t,\nabla g(x))$. A similar relation also holds for $-\vartheta$ and $-\varpi_g$. Hence, the Value Function ϑ satisfies

$$\begin{aligned}
&-\theta + H(x,\zeta) \geq 0, \quad \forall (t,x) \in (0,T) \times \Omega,\ \forall (\theta,\zeta) \in \partial_V \vartheta(t,x),\\
&-\theta + H(x,\zeta) \leq 0, \quad \forall (t,x) \in (0,T) \times \Omega,\ \forall (\theta,\zeta) \in \partial_V (-\vartheta)(t,x),\\
&\ \vartheta(T,x) = \psi(x), \quad \forall x \in \Omega.
\end{aligned}$$

The previous arguments shows that the Value Function ϑ is a viscosity solution of the HJB equation on $(0,T) \times \Omega$. The uniqueness comes from the fact that, if $\alpha : [0,T] \times \Omega \to \mathbb{R}$ is a viscosity solution of the preceding HJB equation, Proposition 6.4.3 implies that $\beta(t,q) = \alpha(t,\nabla g^*(q))$ defined on $[0,T] \times X$ is a viscosity solution of (6.4)-(6.5)-(6.6), so by Proposition 6.3.2 we obtain that $\beta = \varpi_g$ and wherefore $\alpha = \vartheta$. □

6.5 Discussion and perspectives

In this chapter we have investigated the HJB approach for problems with state-constraints from a point of view that seems to be quite new for optimal control. The methodology used has been widely studied in mathematical programming theory in the so-called *interior-point methods* where a suitable barrier function is introduced in order to construct algorithms whose iterations are strictly feasible on the interior of the constraints; see for instance the books of Nocedal-Wright [98] or Renegar [107] among many others. The technique has also been employed to study continuous versions of numerical methods, usually referred as central path methods; we point out the works of Bayer-Lagarias [19], Fiacco [48], McCormick [92], Iusem-Svaiter-Da Cruz Neto [80], Álvarez-Bolte-Brahic [5], Bolte-Teboulle [25] and Attouch-Bolte-Redont-Teboulle [10], for mentioning some authors.

At the present chapter we have taken the idea of penalizing the state-constraints and make use of it in order to study the Value Function of an optimal control problem with convex state-constraints. The outcome is that we have identified a class of problems, neither covered by the current literature nor by the preceding chapters, for which the characterization of the Value Function as unique viscosity solution of a HJB equation is possible.

6.5.1 A Riemannian manifolds interpretation

We finish the chapter with an interpretation of our results in terms of Riemannian geometry.

We evoke from Chapter 3 that the tangent space to a smooth manifold is finite dimensional vectorial space. This fact allows to endow each $\mathcal{T}_{\mathcal{M}}(x)$ with an inner product $(\cdot,\cdot)_x$ so that it has the structure of Euclidean space. Roughly speaking, when this procedure can be done in such a way the inner dot depends on x in an appropriate smooth manner, the manifold is called a Riemannian manifold.

Formally, a *Riemannian metric* on a $\mathcal{C}^k$-smooth manifold $\mathcal{M}$ ($k \geq 2$) is a map that associates each $x \in \mathcal{M}$ with an inner product $(\cdot,\cdot)_x$ defined on the tangent space $\mathcal{T}_{\mathcal{M}}(x)$, which in addition is smooth in the following sense:

$$x \mapsto (\Psi_1(x), \Psi_2(x))_x \text{ belongs to } \mathcal{C}^{k-1}(\mathcal{M}), \quad \forall \Psi_1, \Psi_2 \in \mathfrak{X}(\mathcal{M}).$$

Here $\mathfrak{X}(\mathcal{M})$ stands for the collection of *smooth vector fields* on $\mathcal{M}$, that is, the $\mathcal{C}^{k-1}$ maps $\Psi : \mathcal{M} \to \cup_{x\in\mathcal{M}} \mathcal{T}_{\mathcal{M}}(x)$ satisfying $\Psi(x) \in \mathcal{T}_{\mathcal{M}}(x)$ for any $x \in \mathcal{M}$.

Remark 6.5.1. *The set $\cup_{x\in\mathcal{M}} \mathcal{T}_{\mathcal{M}}(x)$ is called the tangent bundle of $\mathcal{M}$ and it is a $\mathcal{C}^{k-1}$-smooth manifold by itself; see for instance [55, Theorem 1.30]. Consequently, the set $\mathfrak{X}(\mathcal{M})$ is well-defined as set of maps between manifolds.*

The main purpose of introducing the concept of Legendre zone consistent pairs lies in the possibility of defining a Riemannian metric on a given open convex set. Let (Ω, g) be a Legendre zone consistent pair with $g \in \mathcal{C}^2(\Omega)$. For any $x \in \Omega$ consider the bilinear mapping defined on $X \times X$

$$\rho_x(u, v) := \langle \nabla^2 g(x)u, \nabla^2 g(x)v \rangle, \qquad \forall u, v \in X. \tag{6.7}$$

It turns out that the preceding bilinear maps can define a Riemannian metric if the hypothesis (H_0^6) is slightly strengthen.

Lemma 6.5.1. *Let (Ω, g) be a Legendre zone consistent pair with $g \in \mathcal{C}^2(\Omega)$. Then, the family $\{\rho_x\}_{x\in\Omega}$ is a Riemannian metric on Ω provided*

$$g \in \mathcal{C}^3(\Omega) \text{ and } \nabla^2 g(x) \text{ is positive-definite } \forall x \in \Omega. \tag{6.8}$$

Proof. By (6.8), $[\nabla^2 g(x)]^2$ is symmetric and positive-definite for any $x \in \Omega$, so, ρ_x is an inner product on X. Besides, since Ω is an open set, the tangent space to Ω at x can be identified with X for any $x \in \Omega$, furthermore, given that g is thrice continuously differentiable on Ω, for any pair of differentiable vector fields $\Psi_1, \Psi_2 : \Omega \to X$, the map $x \mapsto \rho_x(\Psi_1(x), \Psi_2(x))$ is differentiable. Hence, $\{\rho_x\}_{x\in\Omega}$ is a Riemannian metric on Ω. □

Definition 6.5.1. *Let (Ω, g) be a Legendre zone consistent pair that satisfies (6.8). The family of inner dots given by (6.7) is called the squared Hessian Riemannian metric on Ω induced by the Legendre function g, and it is denoted by $(\cdot,\cdot)_x^{G^2}$, that is, $\forall x \in \Omega$,*

$$(u, v)_x^{G^2} := \langle G(x)^2 u, v\rangle = \langle G(x)u, G(x)v\rangle, \qquad \forall u, v \in X.$$

Under these circumstances, we say that (Ω, G^2) is a squared Hessian Riemannian manifold.

Example 6.5.1. *Take $\Omega = \mathbb{B}_X$ in X, some suitable choices of Legendre functions that make (Ω, G^2) a squared Hessian Riemannian manifolds are*

$$g_1(x) = \begin{cases} -\log(1-|x|^2) & |x| < 1 \\ +\infty & |x| \le 1 \end{cases} \text{ and } g_2(x) = \begin{cases} -\log\left(\cos\left(\frac{\pi}{2}|x|\right)\right) & |x| < 1 \\ +\infty & |x| \le 1. \end{cases}$$

In the light of the foregoing definition we can see that all along this chapter we have basically studied a Mayer problem on a squared Hessian Riemannian manifold. This explain the form of the hypotheses we have required. Actually, the assumption $\operatorname{dom} g^* = X$ in nothing else that the completeness of the Riemannian manifold as a metric space endowed with the distance

$$\mathrm{d}_{G^2}(x_0, x_1) = |\nabla g(x_0) - \nabla g(x_1)|, \qquad \forall x_0, x_1 \in \Omega.$$

For a proof of this affirmation and further properties about squared Hessian Riemannian manifolds we refer to [67].

On the other hand, in principle, the approach we have presented can be applied to any flat Riemannian manifold $\mathcal{M}$, which means that there exists an isometry between $\mathcal{M}$ and an Euclidean space X. In that case, the role of ∇g would be played by the current isometry. However, the advantage of the exhibited case, is that constructing a Legendre function, as required here, can be done for a large family of convex sets; we refer for example to the aforementioned works of Borwein-Vanderwerff [27, 28].

We finally point out that squared Hessian Riemannian metrics can also be applied in other contexts such as mathematical programming. In [67] we have introduced a discrete scheme for solving nonlinear programs with convex constraints. The algorithm proposed can be classified as a geodesic-search method, and we have shown that, under suitable assumptions it is well-posed (the value of the cost decreases at each iteration). We also have provided a convergence analysis.

PART III

ANALYSIS OF OPTIMAL FEEDBACK LAWS

Abstract. In this part we investigate some issues related to discontinuous feedback laws. We are principally concerned with control strategies that have a stratified set of singularities on the state-space. For this purpose. we first provide a well-posed theory for dealing with stratified closed-loop systems and, afterwards, we study the possibility to construct continuous suboptimal feedbacks laws from discontinuous optimal policies.

Resumé. Dans cette partie nous étudions quelques problèmes liés aux contrôles en boucle fermé qui sont aussi discontinus. Nous sommes principalement intéressés par les lois de feedback dont les ensembles de points singuliers ont une structure stratifiée par rapport à l'espace d'état. À cet effet nous commençons par une étude sur les équations différentielles ordinaries dont le champ vectoriel est continu dans un sense stratifié. Ensuite, nous étudions la possibilité de construire des contrôles en boucle fermé sous-optimaux mais continues à partir de l'information fourni par les lois de feedback optimaux.

CHAPTER 7

Stratified ordinary differential equations

Abstract. This chapter is concerned with state-constrained ordinary differential equations for which the corresponding vector field has a set of singularities that forms a stratification of the state-space. We study the existence of solutions and the robustness with respect to external perturbations of the righthand. We stress that in this chapter the notion of relative wedgedness plays an important role.

7.1 Introduction

One important issue in control theory of ordinary differential equations is the feedback synthesis, that is, for a control system on $\mathbb{R}^N$

$$\dot{x}(t) = f(x(t), u(t)), \quad u(t) \in \mathcal{U}, \quad \text{a.e. } t \in [0,T], \quad \text{with } x(t) \in \mathcal{K} \subseteq \mathbb{R}^N, \quad \forall t \in [0,T],$$

construct a function $U : \mathcal{K} \to \mathcal{U}$ in such a way that all the trajectories of the vector field $x \mapsto f(x, U(x))$ belong to a certain class of curves of the control system; typically, (sub)minimizers of a given cost function.

It is an accepted fact that optimal feedback laws are in general discontinuous functions of the state. For instance, the first order necessary conditions of optimality (Pontryagin maximum principle) show that even for linear systems it is likely to occur.

We have seen in Section 1.2.2 that it is suitable to assume that optimal feedback strategies may exhibit a stratified structure, meaning that there exists a partition of the state-space $\mathcal{K}$ into a disjoint family of sets $\{\mathcal{M}_i\}_{i\in\mathcal{I}}$, such that

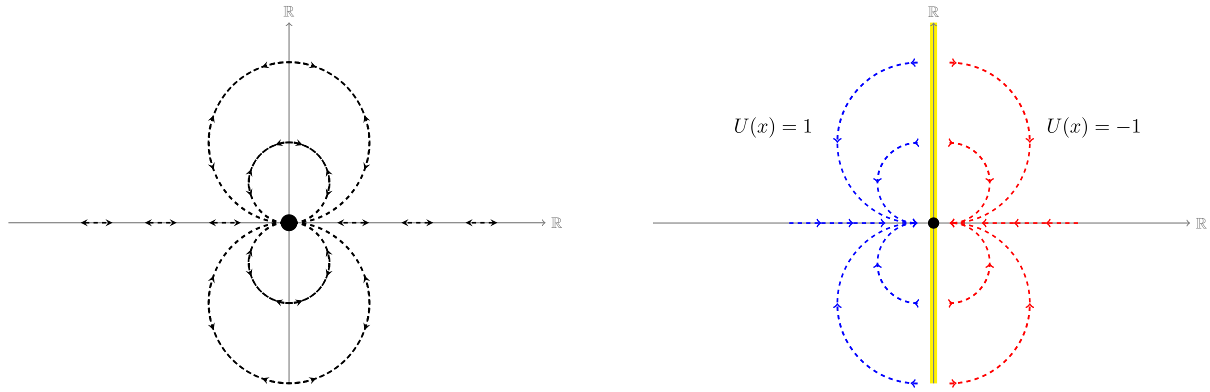
$$U(x) = U_i(x), \quad \text{whenever } x \in \mathcal{M}_i,$$

and in some of these strata, the ordinary differential equation

$$\dot{y} = f(y, U_i(y)), \quad \text{a.e. on } [0,T],$$

admits arcs remaining on $\mathcal{M}_i$ for at least a small interval of time. In fact, it is likely to have a subfamily of strata where no trajectory of the system can stay for a set of times with positive measure. In this way, as for the Artstein's circles synthesis (See Figure 7.1), trajectories may pass from one stratum to another of bigger dimension straightaway; in the quoted example the trajectories starting at the x_2-axis pass immediately to $\{x_1 > 0\}$ or well to $\{x_2 > 0\}$.

The aim of this chapter is to study ordinary differential equations arising in this context. For this purpose the concept of *stratified vector fields (SVF)* is introduced. The main issues are the existence of solutions and the robustness of trajectories with respect to external perturbations



(a) Dynamics. (b) Optimal synthesis.

Figure 7.1: The Artstein's Circles.

on the velocity. Particular emphasis is put on the interplay between regularity conditions on the sets $\mathcal{M}_i$ and pointwise conditions on the vector field in order to ensure the existence of solutions. In particular, ordinary differential equations on closed sets with empty interior are covered. A key tool for this is the notion of *relatively wedgedness* introduced in Section 3.4.

Let us mention that the notions of SVF was brought to light in the notes of Mather [91]. This was introduced in order to prove properties about Whitney stratifications. The definition used in that notes is slightly different from the adopted in this work. The principal and most important difference between both definitions is that here a SVF does not need to be defined for all the strata. In fact, as described above, it is likely that this situation occurs. For example the discontinuous dynamical system illustrated in Figure 7.2 does not fit in the setting of [91].

The essential tool for studying the stability of solutions to stratified ordinary differential equations is the *modulus outward-pointing*. The main feature of this function is that it measures the maximum size of the perturbations and describes the class of singularities allowed in order to make the system stable in the sense described earlier.

The exposition of the chapter is based on the results reported in [68].

7.2 Stratified vector fields

Roughly speaking, a stratified vector field is a piecewise continuous mapping opportunely determined on some of the strata of the partition. Consequently, a selection of index where the vector fields are well-defined need to be involved in the definition.

Definition 7.2.1. *Let $\mathcal{K} \subseteq \mathbb{R}^N$ be a stratifiable closed set whose stratification is denoted by $\{\mathcal{M}_i\}_{i\in\mathcal{I}}$. Let $\mathcal{I}_0 \subseteq \mathcal{I}$ be a subset of index such that $\{\mathcal{M}_i\}_{i\in\mathcal{I}_0}$ is dense in $\mathcal{K}$. Then a stratified vector field (SVF) is a family of vector fields $G = \{g_i : \mathcal{M}_i \to \mathcal{T}_{\mathcal{M}_i}\}_{i\in\mathcal{I}_0}$ such that*

$$g_i(x) \in \mathcal{T}_{\mathcal{M}_i}(x), \quad \forall i \in \mathcal{I}_0,\ \forall x \in \mathcal{M}_i, \tag{7.1}$$

A SVF is said to be regular if for each $i \in \mathcal{I}_0$, g_i is continuous on $\mathcal{M}_i$ and it can be continuously extended up to $\overline{\mathcal{M}}_i$. In addition, a SVF has linear growth if there is a constant $c > 0$ such that

$$|g_i(x)| \le c(1 + |x|), \quad \forall i \in \mathcal{I}_0, x \in \mathcal{M}_i.$$

By (7.1), the subset of index $\mathcal{I}_0$ is in fact the selection of strata where it is allowed to slide for. Therefore, the manifold corresponding to the index $\mathcal{I}_0$ are usually called *sliding strata* and the others, *bifurcation strata*.

Figure 7.2 shows a SVF defined on $\mathcal{K} = \mathbb{R}^2$. In this case, the stratification of the space consists in $\mathcal{M}_0 = \{(0,0)\}$, $\mathcal{M}_1, \ldots, \mathcal{M}_4$ the positive and negative semi-axis and $\mathcal{M}_5, \ldots, \mathcal{M}_8$ the quadrants of $\mathbb{R}^2$. Note that in this situation $\mathcal{I} = \{0, \ldots, 8\}$ and the vector fields are defined on all the strata except on $\mathcal{M}_1$ and $\mathcal{M}_2$. Hence, $\mathcal{I}_0 = \{0, 3, 4, 5, 6, 7, 8\}$ and, $\mathcal{M}_1$ and $\mathcal{M}_2$ are the bifurcation strata of this SVF.

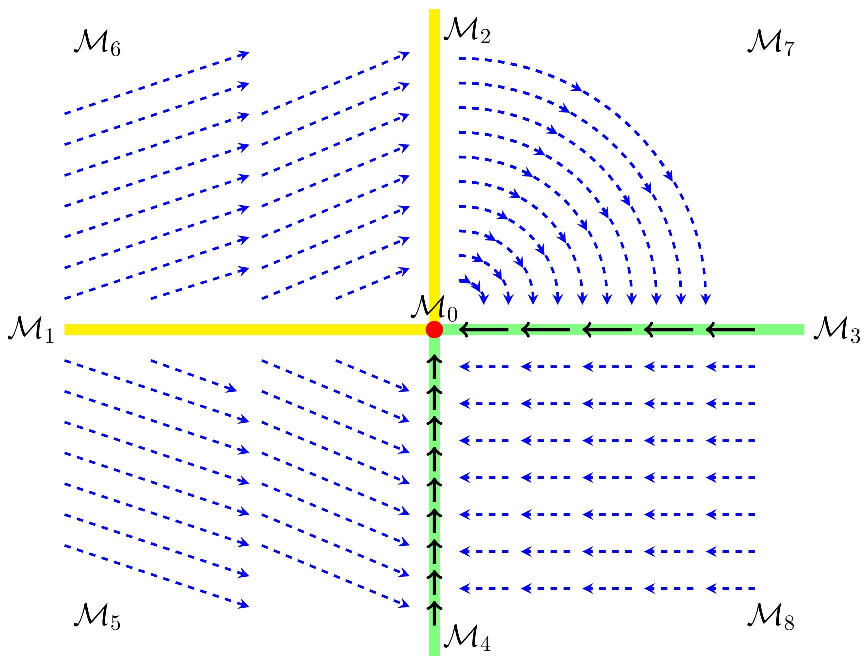


Figure 7.2: Example of stratified vector field on $\mathcal{K} = \mathbb{R}^2$

On the other hand, notice that G induces a discontinuous map with well determined singularities. For this reason, it is useful to introduce some notation to identify the surrounding strata where the vector field is defined. Recall that the notation $j \succeq i$ means that $M_i \subseteq \overline{\mathcal{M}}_j$. Then the index of the surrounding strata is given by

$$\mathcal{I}_0(i) := \{j \in \mathcal{I}_0 : \ j \succeq i\} \qquad \forall i \in \mathcal{I}.$$

We allude to the fact that given a stratifiable set, the index map $\imath : \mathcal{K} \to \mathcal{I}$ is the function that links any $x \in \mathcal{K}$ with the index $i \in \mathcal{I}$ for which $x \in \mathcal{M}_i$.

7.2.1 Stratified ordinary differential equations

Once the notion of SVF is well stablished, it is possible to formalize what is the central object of investigation in this chapter, namely, the discontinuous ordinary differential equation engendered by this mapping.

$$(\mathbf{D}) \qquad \begin{cases} \dot{y} = g_i(y) & \text{a.e. whenever } y \in \mathcal{M}_i \\ y(0) = x. \end{cases}$$

For all the purposes, the (**D**) is called the *stratified ordinary differential equation* associated with the stratification $\{\mathcal{M}_i\}_{i\in\mathcal{I}}$ and with the dynamics $G = \{g_i : \mathcal{M}_i \to \mathcal{T}_{\mathcal{M}_i}\}_{i\in\mathcal{I}_0}$. The attention is focus on Carathéodory solutions to (**D**), that is, absolutely continuous arcs satisfying

$$y(t) = x + \sum_{i\in\mathcal{I}_0} \int_{J_i(t)} g_i(y(s))ds, \quad \forall t \in [0,T), \tag{7.2}$$

where $J_i(t) = \{s \in [0,t) : \; y(s) \in \mathcal{M}_i\}$. In particular, this implies that each $J_i(T)$ is a negligible set whenever $i \notin \mathcal{I}_0$. Any curve satisfying the equation (7.2) is referred to as a *stratified solution* to (**D**).

Remark 7.2.1. *Broadly speaking, the set of solutions defined in this way is not closed. In fact, since $\{\mathcal{M}_i\}_{i\in\mathcal{I}_0}$ is dense in $\mathcal{K}$, the set $\mathcal{I}_0(i) \neq \emptyset$ for any $i \in \mathcal{I}$ and so, it may be possible to create an absolutely continuous arc that is the limit of trajectories,*

$$\dot{y}_n = g_j(y_n), \quad y_n(0) = x_n \in \mathcal{M}_j,$$

with $x_n \to x \in \mathcal{M}_i$ with $i \preceq j$. It could happen that the limiting trajectory lies on $\mathcal{M}_i$ for a set of times of positive measure but it is not a stratified solution. In Figure 7.3 this is exactly the case with $i = 1$ or $i = 2$ and $j = 4$. Notice that this issue does not relie upon whether i belongs or not to $\mathcal{I}_0$. Furthermore, to avoid these situations, some types of singularities have to be ruled out; this will be discussed more deeply in Section 7.4.

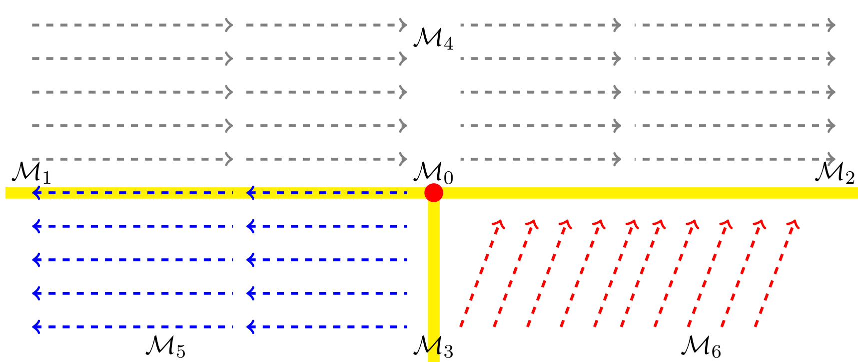


Figure 7.3: Example of stratified solutions

Remark 7.2.2. *If y is a stratified solution with $G = \{g_i : \mathcal{M}_i \to \mathcal{T}_{\mathcal{M}_i}\}_{i\in\mathcal{I}_0}$ having linear growth, then*

$$\dot{y}(t) \leq c(1 + |y(t)|), \quad a.e. \; t \in [0,T).$$

and so, the Gronwall's estimation holds (Proposition 2.4.1),

$$|y(t) - y(s)| \leq (e^{c(t-s)} - 1)\,(|y(s)| + 1)\,, \quad \forall 0 \leq s < t < T.$$

In particular, $\sup\{|y(s)| \mid \; s \in [0,T)\} < +\infty$.

7.3 Existence of stratified solutions

We evoke once again that a SVF is not necessarily defined everywhere on the state space. For this reason, the analysis is divided into two eventualities, namely, when the SVF is prescribed everywhere ($\mathcal{I}_0 = \mathcal{I}$) and when it is not ($\mathcal{I}_0 \neq \mathcal{I}$). The first case is simpler than the other and it merely requires the definition of SVF. The other one, is more delicate to treat and some extra hypotheses are required.

The analysis of the existence of stratified solutions strongly relies in the celebrated Nagumo's Theorem (Proposition 2.4.2).

7.3.1 Case $\mathcal{I}_0 = \mathcal{I}$

In this situation the existence of local solutions is ensured by the continuity and the tangentiality of the vector fields on each stratum as we show in the next statement.

Theorem 7.3.1. *Let $\mathcal{K} \subseteq \mathbb{R}^N$ be a closed stratifiable set and let $\{M_i\}_{i\in\mathcal{I}}$ be its strata. Consider a regular SVF denoted by $G = \{g_i : \mathcal{M}_i \to \mathcal{T}_{\mathcal{M}_i}\}_{i\in\mathcal{I}_0}$ and suppose that $\mathcal{I}_0 = \mathcal{I}$. Then for every $x \in \mathcal{K}$ there exist $T > 0$ and a stratified solution to (**D**) defined on the interval $[0, T)$. Moreover, if the SVF has linear growth, then $T = +\infty$.*

Proof. Let $x \in \mathcal{K}$ and set $i = \imath(x)$. Since $i \in \mathcal{I}_0$, the following ordinary differential equation is well defined

$$(\mathbf{D^i_0}) \qquad \dot{y} = g_i(y), \quad y(0) = x.$$

Note that $\mathcal{M}_i$ is locally compact and by Proposition 3.2.2, the Bouligand cone agrees with the tangent space to $\mathcal{M}_i$. Since g_i is continuous and satisfies condition (7.1), the Nagumo's Theorem implies that ($\mathbf{D^i_0}$) has at least a solution remaining in $\mathcal{M}_i$ on an interval $[0, T)$, for some $T > 0$. we readily check that (7.2) holds and thereby it is a stratified solution to (**D**).

On the other hand, suppose that G has linear growth on $\mathcal{K}$. Let y be a maximal solution (**D**) defined on $[0, T)$ and assume that $T < +\infty$. By Remark 7.2.2 there exists a constant $C = C(x) > 0$ such that

$$|y(t) - y(s)| \leq C(e^{L|t-s|} - 1), \quad \forall s, t \in [0, T).$$

This means that for any $t_n \nearrow T$, the sequence $\{y(t_n)\}$ satisfies the Cauchy condition, and consequently, the limit is well defined. Using the same inequality it is possible to prove that the limit does not depend upon the sequence taken. Therefore, we can set

$$y(T) := \lim_{t \to T^-} y(t)$$

and get $y(T) \in \mathcal{K}$, which contradicts the maximality of T. So, T can not be bounded. $\square$

7.3.2 Case $\mathcal{I}_0 \neq \mathcal{I}$

In presence of bifurcation strata ($\mathcal{M}_i$ with $i \notin \mathcal{I}_0$) the existence of solutions to a stratified ordinary differential equation requires additional hypotheses. For example, consider $\mathcal{K} = \mathbb{R}$, the stratification

$$\mathcal{M}_0 = \{0\}, \quad \mathcal{M}_1 = (-\infty, 0), \quad \text{ and } \mathcal{M}_2 = (0, +\infty),$$

and the SVF

$$G = \{(\mathcal{M}_i, g_i)\}_{i \in \{1,2\}}, \qquad \text{with } g_1(x) = 1 \text{ and } g_2(x) = -1.$$

Clearly in this case, no solution starting from $x = 0$ exists.

The problem in these circumstances is reduced to study the existence of solutions of an ordinary differential equation on $\mathcal{M}$, an embedded manifold of $\mathbb{R}^N$, whose initial condition lives in $\overline{\mathcal{M}} \setminus \mathcal{M}$.

Theorem 7.3.2. *Let $\mathcal{K} \subseteq \mathbb{R}^N$ be a closed stratifiable set and let $\{\mathcal{M}_i\}_{i\in\mathcal{I}}$ be its strata. Consider a regular SVF denoted by $G = \{g_i : \mathcal{M}_i \to \mathcal{T}_{\mathcal{M}_i}\}_{i\in\mathcal{I}_0}$ and suppose*

$$(\mathbf{H_0^7}) \qquad \begin{cases} \forall x \in \mathcal{K} \textit{ with } \imath(x) \notin \mathcal{I}_0, \exists j \in \mathcal{I}_0(\imath(x)) \textit{ and } \exists r > 0 \textit{ such that:} \\ i) \;\; x + (0, r]\mathbb{B}(g_j(x), r) \cap \overline{\mathcal{M}}_j \subseteq \mathcal{M}_j. \\ ii) \;\; g_j(y) \in \mathcal{T}^B_{\overline{\mathcal{M}}_j}(y) \setminus \{0\}, \forall y \in \mathbb{B}(x, r) \cap \overline{\mathcal{M}}_j. \end{cases}$$

Then for every $x \in \mathcal{K}$ there exist $T > 0$ and a solution to (**D**) *defined on $[0, T)$. Additionally, if the SVF has linear growth, then $T = +\infty$.*

Proof. Let $x \in \mathcal{K}$ and set $i = \imath(x)$. In view of Theorem 7.3.1, it is only necessary to consider the case $i \notin \mathcal{I}_0$.

Let $j \in \mathcal{I}_0(i)$ and $r > 0$ given by ($\mathbf{H_0^7}$). Notice that $\mathbb{B}(x, r) \cap \overline{\mathcal{M}}_j$ is locally compact, then by condition ($\mathbf{H_0^7}$) part (ii) and the Nagumo's Theorem, there exists a curve y associated with the vector field g_j starting from x lying in $\overline{\mathcal{M}}_j \cap \mathbb{B}(x, r)$ on an interval of time $[0, T)$. Moreover, $g_j(x) \neq 0$ and since g_j is continuous on $\overline{\mathcal{M}}_j$, by reducing T if necessary,

$$y(t) \in x + (0, r]\mathbb{B}(g(x), r), \quad \forall t \in [0, T).$$

Thus, by ($\mathbf{H_0^7}$) part (i), $y(t) \in \mathcal{M}_j$ on $[0, T)$ and so, the arc y is a stratified solution of (**D**).

Finally, if G has linear growth, the same arguments as in the proof of the preceding theorem are valid and the conclusion follows. □

We now turn our attention into the search of rather simple conditions in order to ensure ($\mathbf{H_0^7}$) to hold. In this respect, the notion of wedgedness introduced in Section 3.4 starts playing an important role.

We recall from Definition 3.4.1 that given an embedded manifold $\mathcal{M}$ of $\mathbb{R}^N$, its closure $\overline{\mathcal{M}}$ is said to be relatively wedged at $x \in \overline{\mathcal{M}}$ if there exists a unique decomposition of $\mathcal{N}^C_{\overline{\mathcal{M}}}(x)$ as the direct sum of a pointed cone $\mathcal{N}_x$ on X and a vectorial subspace $X_{\mathcal{N}}(x)$ whose dimension matches the codimension of $\mathcal{M}$, that is,

$$\mathcal{N}^C_{\overline{\mathcal{M}}}(x) = \mathcal{N}_x \oplus X_{\mathcal{N}}(x).$$

We may also remind that $\pi_x(\eta)$ stands for the projection of $\eta \in \mathcal{N}^C_{\overline{\mathcal{M}}}(x)$ over $\mathcal{N}_x$.

Besides, from Corollary 3.3.1 we have that if the stratification satisfies the Whitney (a)-condition, $x \mapsto \mathcal{T}^C_{\overline{\mathcal{M}}_j}(x)$ is lower semicontinuous for each index $j \in \mathcal{I}$, and so, in view of Proposition 3.4.5, the next statement is a direct consequence of the quoted results.

Proposition 7.3.1. *Let $\mathcal{K} \subseteq \mathbb{R}^N$ be a (W_a)-stratifiable set and let $\{\mathcal{M}_i\}_{i\in\mathcal{I}}$ be its strata. Consider $G = \{g_i : \mathcal{M}_i \to \mathcal{T}_{\mathcal{M}_i}\}_{i\in\mathcal{I}_0}$, a regular SVF. Then, for any $j \in \mathcal{I}_0$ such that $\overline{\mathcal{M}}_j$ is relatively wedged at some $x \in \overline{\mathcal{M}}_j$ with $g_j(x) \in ri\left(\mathcal{T}^C_{\overline{\mathcal{M}}_j}(x)\right)$, there exists $r > 0$ so that*

$$g_j(\tilde{x}) \in \mathcal{T}^C_{\overline{\mathcal{M}}_j}(\tilde{x}) \setminus \{0\}, \quad \forall \tilde{x} \in \mathbb{B}(x,r) \cap \overline{\mathcal{M}}_j.$$

Therefore, under the framework of the preceding proposition, we have a sufficient condition for (ii) in $(\mathbf{H_0^7})$. Furthermore, if the dimension of $\mathcal{M}_j$ in $(\mathbf{H_0^7})$ happens to agree with the dimension of the ambient space, that is N, relatively wedgedness implies that $\overline{\mathcal{M}}_j$ is epi-Lipschitz around x (it is also referred to as wedged sets); c.f [114, Section 9.F] and [41, Section 3.6]. We summed up this issue in the ensuing proposition.

Proposition 7.3.2. *Let $\mathcal{K} \subseteq \mathbb{R}^N$ be a stratifiable set and let $\{\mathcal{M}_i\}_{i\in\mathcal{I}}$ be its strata. Consider $G = \{g_i : \mathcal{M}_i \to \mathcal{T}_{\mathcal{M}_i}\}_{i\in\mathcal{I}_0}$, a regular SVF. If there exists $j \in \mathcal{I}_0$ with $\dim(\mathcal{M}_j) = N$ and $g_j(x) \in int\left(\mathcal{T}^C_{\overline{\mathcal{M}}_j}(x)\right)$, then $(\mathbf{H_0^7})$ holds.*

Proof. The condition (i) is a direct consequence of [41, Theorem 6.4]. Moreover, by [41, Proposition 6.7] we have that $x \mapsto \mathcal{T}^C_{\overline{\mathcal{M}}_j}(x)$ is lower semicontinuous. Combining this with Proposition 3.4.5 we get the condition (ii), which finishes the proof. □

On the other hand, if $\dim(\mathcal{M}_j) < N$ we need to add some further assumptions because [41, Theorem 6.4] does not apply if we change the interior with the relative interior. To do this, we evoke, from Section 3.2.2, that a manifold is said to have bounded curvature if there exists $\kappa_0 > 0$ so that

$$\sup\left\{\frac{2\langle \eta, \tilde{x} - x\rangle}{|\tilde{x} - x|^2} : \eta \in \mathcal{N}_{\mathcal{M}}(x),\ |\eta| = 1,\ \tilde{x} \in \mathcal{M} \setminus \{x\}\right\} \leq \kappa_0, \quad \forall x \in \mathcal{M}.$$

Proposition 7.3.3. *Let $\mathcal{K}$ be a stratifiable set and let $\mathcal{M}$ be one of the strata. Suppose that $\overline{\mathcal{M}}$ is relatively wedged at $x \in \overline{\mathcal{M}}$ and for any $i \succeq \imath(x)$, $\mathcal{M}_i \neq \mathcal{M}$ has bounded curvature. Then, for any $v \in ri\left(\mathcal{T}^C_{\overline{\mathcal{M}}}(x)\right)$ there exists $r > 0$ so that*

$$(x + (0,r]\mathbb{B}(v,r)) \cap \overline{\mathcal{M}} \subseteq \mathcal{M}.$$

Proof. We only consider the case that $\dim(M) < N$ otherwise it holds by Proposition 7.3.2. By contradiction, suppose there exist two sequences $\{t_n\} \subseteq (0,+\infty)$ and $\{v_n\} \subseteq \mathbb{R}^N$ with $t_n \to 0$ and $v_n \to v$ such that $x_n := x + t_n v_n \in \overline{\mathcal{M}} \setminus \mathcal{M}$. Take $\eta_n \in \mathcal{N}^L_{\overline{\mathcal{M}}}(x_n)$ with $\pi_{x_n}(\eta_n) = \eta_n$ and $|\eta_n| = 1$. Since $\mathcal{K}$ is stratifiable, there exists a subsequence totally contained in some stratum $\mathcal{M}_i$ with $i \succeq \imath(x)$. Passing into subsequences, which we avoid to relabel, $\{x_n\} \subseteq \mathcal{M}_i$ and $\eta_n \to \eta$ with $\eta \in \mathcal{N}_x$ and $|\pi_x(\eta)| = 1$

By Proposition 3.3.7, $\eta_n \in \mathcal{N}_{\mathcal{M}_i}(x_n)$ and since $\mathcal{M}_i$ has constant curvature, there exists $\kappa_0 > 0$ such that

$$\frac{\kappa_0}{2}|x_n - \tilde{x}|^2 \geq \langle \eta_n, \tilde{x} - x_n \rangle \qquad \forall \tilde{x} \in \overline{\mathcal{M}}_i, \forall n \in \mathbb{N}.$$

Evaluating at $\tilde{x} = x$ we get

$$\langle v_n, \eta_n \rangle \geq -\frac{\kappa_0 t_n}{2}|v_n|^2 \qquad \forall n \in \mathbb{N}.$$

Letting $n \to +\infty$ we finally obtain $\langle v, \eta \rangle \geq 0$. However, due to $|\pi_x(\eta)| = 1$, we obtain a contradiction with Proposition 3.4.3. Thereby, the conclusion follows. $\square$

In view of Proposition 7.3.1 and 7.3.3, combined with Theorem 7.3.2 , we get the following theorem about the existence of stratified solutions.

Theorem 7.3.3. *Let $\mathcal{K} \subseteq \mathbb{R}^N$ be a closed (W_a)-stratifiable set. Consider a regular SVF denoted by $G = \{g_i : \mathcal{M}_i \to \mathcal{T}_{\mathcal{M}_i}\}_{i \in \mathcal{I}_0}$ and suppose that*

$$(\mathbf{H_1^7}) \qquad \begin{cases} \forall x \in \mathcal{K} \textit{ with } \imath(x) \notin \mathcal{I}_0, \exists j \in \mathcal{I}_0(\imath(x)) \textit{ such that:} \\ i) \;\; \overline{\mathcal{M}}_j \textit{ is relatively wedged at } x \textit{ with } g_j(x) \in ri\left(\mathcal{T}^C_{\mathcal{M}_j}(x)\right). \\ ii) \; \forall l \succeq \imath(x), l \neq j, \; \mathcal{M}_l \textit{ has bounded curvature.} \end{cases}$$

Then for every $x \in \mathcal{K}$ there exist $T > 0$ and a solution to (**D**) *defined on $[0, T)$. Moreover, if the SVF has linear growth, then $T = +\infty$.*

7.4 Necessary condition for Robustness

This final section has as purpose to study the robustness of a stratified ordinary differential equation under external perturbations. The principal issue is to identify some conditions in order to ensure that the corresponding solutions to that type of discontinuous equations are still stratified solutions of the original system, even if the velocities are slightly perturbed.

Hereinafter, the stratification associated with the set $\mathcal{K}$ is supposed to be relatively wedged. Accordingly, we assume the following hypothesis for the rest of the chapter.

$$(\mathbf{H_2^7}) \qquad \begin{cases} i) \;\;\; \mathcal{K} \subseteq \mathbb{R}^N \text{ is a closed and } (W_a)\text{-stratifiable set.} \\ ii) \;\; \text{Any stratum } \mathcal{M} \text{ of } \mathcal{K} \text{ is } \mathcal{C}^2 \text{ and } \overline{\mathcal{M}} \text{ is relatively wedged.} \\ iii) \; G \text{ is a regular SVF defined on } \mathcal{K}. \end{cases}$$

In any case, $\{\mathcal{M}_i\}_{i \in \mathcal{I}}$ stands for the strata of $\mathcal{K}$ and for some $\mathcal{I}_0 \subseteq \mathcal{I}$, $G = \{g_i : \mathcal{M}_i \to \mathcal{T}_{\mathcal{M}_i}\}_{i \in \mathcal{I}_0}$ denotes a SVF subordinated to $\{\mathcal{M}_i\}_{i \in \mathcal{I}}$ with $\mathcal{I}_0$ being the selection of index.

7.4.1 Robustness with respect to external perturbations

In section 1.2.1 we have reviewed some notions of robustness for closed-loop systems which we proceed to evoke anew accordingly to the stratified context we have posed the problem. Let $\mathcal{S}$ be a closed set and $g : \mathcal{S} \to \mathbb{R}^N$ a given vector field. Let $x \in \mathcal{S}$ and consider $\{x_n\} \subseteq \mathcal{S}$ and $\{\xi_n\} \subseteq L^1 := L^1([0,T];\mathbb{R}^N)$ such that $x_n \to x$ and $\xi_n \to 0$ in L^1. Suppose that for each $n \in \mathbb{N}$, there exists $y_n : [0,T] \to \mathbb{R}^N$ a Carathéodory solution to the perturbed system:

$$\dot{y} = g(x) + \xi_n \quad \text{a.e. on } (0,T), \quad y(t) \in \mathcal{S} \quad \text{for any } t \in [0,T], \quad y(0) = x_n.$$

Let y be an accumulation point of $\{y_n\}$ in the topology of the uniform convergence on $[0,T]$ and suppose that this is a solution to

$$\dot{y} = g(x) \quad \text{on } (0,T), \quad y(t) \in \mathcal{S} \quad \text{for any } t \in [0,T], \quad y(0) = x.$$

In this case, the map g is said to be *robust with respect to external perturbations*. We readily see that if g is continuous (which is not the most suitable framework for a SVF), then the property holds. However, thank to tameness of the singular set of a stratified dynamics, gathering the continuity on each strata, it is possible to study the robustness of the equation by ruling some types singularities out.

7.4.2 Outward-pointing modulus

To investigate the way how the continuous part of a SVF interact between each other, we need to introduce the *outward-pointing modulus* of a vector field.

Definition 7.4.1. *Let $\mathcal{M}$ be an embedded manifold of $\mathbb{R}^N$ and $g : \overline{\mathcal{M}} \to \mathbb{R}^N$. Suppose that $\overline{\mathcal{M}}$ is relatively wedged. For any $x \in \overline{\mathcal{M}} \setminus \mathcal{M}$, the outward-pointing modulus of g at x is given by:*

$$\alpha_g(x) = \max\left\{\langle g(x), \eta\rangle : \ \eta \in \mathcal{N}^C_{\overline{\mathcal{M}}}(x) \ s.t. \ |\pi_x(\eta)| = 1\right\}.$$

The reason why we have imposed the condition $|\pi_x(\eta)| = 1$ in the preceding definition is important for our purposes. Indeed, if we relax it to only $|\eta| = 1$, we would get $\alpha_g(x) = 0$ whenever $\dim(\mathcal{M}) < N$ and $g(x) \in \mathcal{T}^C_{\overline{\mathcal{M}}}(x)$, whereas, if $\dim(\mathcal{M}) = N$ it can be strictly negative. Indeed, $\alpha_g(x) < 0$ if and only if $g(x) \in \operatorname{int}\left(\mathcal{T}^C_{\overline{\mathcal{M}}}(x)\right)$.

The main characteristic of this function is that provides a generalization to lower-dimensional embedded manifolds of the latest fact noted above, where the role of the interior is played by the relative interior of the Clarke tangent cone. This is recapitulated in the next proposition.

Proposition 7.4.1. *Let $\mathcal{M}$ be an embedded manifold of $\mathbb{R}^N$ for which $\overline{\mathcal{M}}$ is relatively wedged at $x \in \overline{\mathcal{M}} \setminus \mathcal{M}$ and let $g : \overline{\mathcal{M}} \to \mathbb{R}^N$ be a vector field. Then*

$$\alpha_g(x) < 0 \quad \text{if and only if} \quad g(x) \in ri\left(\mathcal{T}^C_{\overline{\mathcal{M}}}(x)\right).$$

In addition, if g is continuous at x and satisfies $g(y) \in \mathcal{T}_{\mathcal{M}}(y)$, $\forall y \in \mathcal{M}$, then

1. *$\alpha_g(x) = 0$ if and only if $g(x) \in rbd\left(\mathcal{T}^C_{\overline{\mathcal{M}}}(x)\right)$.*
2. *$\alpha_g(x) > 0$ if and only if $g(x) \notin \mathcal{T}^C_{\overline{\mathcal{M}}}(x)$.*

Proof. By Proposition 3.4.3, $g(x) \in \mathrm{ri}\left(\mathcal{T}^C_{\mathcal{M}}(x)\right)$ if and only if $\exists \sigma > 0$ such that

$$\langle g(x), \eta \rangle \leq -\sigma |\pi_x(\eta)| \quad \forall \eta \in \mathcal{N}^C_{\mathcal{M}}(x).$$

The first equivalence comes from the definition of the modulus. Moreover, by the polar relation between $\mathcal{T}^C_{\mathcal{M}}(\cdot)$ and $\mathcal{N}^C_{\mathcal{M}}(\cdot)$ (Proposition 2.3.6),

$$g(x) \in \mathrm{rbd}\left(\mathcal{T}^C_{\mathcal{M}}(x)\right) \Rightarrow \alpha_g(x) = 0, \quad g(x) \notin \mathcal{T}^C_{\mathcal{M}}(x) \Rightarrow \alpha_g(x) \geq 0.$$

We also have that if $\alpha_g(x) > 0$ necessarily $g(x) \notin \mathcal{T}^C_{\mathcal{M}}(x)$. Note that if $g(x) \notin \mathcal{T}^C_{\mathcal{M}}(x)$ and $\alpha_g(x) = 0$, then

$$\langle g(x), \eta \rangle \leq 0, \quad \forall \eta \in \mathcal{N}^C_{\mathcal{M}}(x) \text{ s.t. } |\pi_x(\eta)| \neq 0.$$

On the other hand, since $\langle g(\tilde{x}), \eta \rangle = 0$ for any $\eta \in \mathcal{N}_{\mathcal{M}}(\tilde{x})$ if $g(\tilde{x}) \in \mathcal{T}_{\mathcal{M}}(\tilde{x})$, if in addition g is continuous on $\overline{\mathcal{M}}$, one gets

$$\langle g(x), \eta \rangle = 0, \quad \eta \in \mathcal{N}^C_{\mathcal{M}}(x) \text{ with } \pi_x(\eta) = 0.$$

Hence,

$$\langle g(x), \eta \rangle \leq 0, \quad \eta \in \mathcal{N}^C_{\mathcal{M}}(x),$$

meaning that $g(x) \in \mathcal{T}^C_{\mathcal{M}}(x)$ which is not possible. So, if $g(x) \notin \mathcal{T}^C_{\mathcal{M}}(x)$ we get $\alpha_g(x) > 0$, and the proof is completed. ☐

Further properties of the outward-pointing modulus of a vector field can be stated. Thanks to the relatively wedgedness of the closure of the manifold, it can be shown that this is upper semicontinuous provided the $\mathcal{T}^C_{\mathcal{M}}(\cdot)$ is lower semicontinuous (If the stratification satisfies the Whitney (a)-condition it is the case). Continuity along a submanifold contained in the frontier of the manifold is also achieved.

Proposition 7.4.2. *Let $\mathcal{M}$ be an embedded manifold of $\mathbb{R}^N$ with $\overline{\mathcal{M}}$ relatively wedged and $\mathcal{T}^C_{\mathcal{M}}(\cdot)$ is lower semicontinuous. Consider $g : \overline{\mathcal{M}} \to \mathbb{R}^N$ continuous on $\overline{\mathcal{M}} \setminus \mathcal{M}$, then:*

1. *α_g upper semi-continuous on $\overline{\mathcal{M}} \setminus \mathcal{M}$.*
2. *Let $\mathcal{M}_b$ be another embedded manifold of $\mathbb{R}^N$ such that $\mathcal{M}_b \subseteq \overline{\mathcal{M}} \setminus \mathcal{M}$.*
   (a) *Suppose $\mathcal{N}^C_{\mathcal{M}}(\cdot)$ is lower semicontinuous restricted to $\mathcal{M}_b$, then α_g is continuous restricted to $\mathcal{M}_b$.*
   (b) *Suppose g and $\mathcal{N}^C_{\mathcal{M}}(\cdot) \cap \overline{\mathbb{B}}$, both restricted to $\mathcal{M}_b$, are locally Lipschitz continuous, then α_g is locally Lipschitz continuous on $\mathcal{M}_b$.*

Proof. For sake of simplicity assume that $\mathcal{M}$ is an open set (i.e. an embedded manifold of dimension N), otherwise it is enough to replace η with $\pi_x(\eta)$ whenever appropriate.

1. Let $x \in \overline{\mathcal{M}} \setminus \mathcal{M}$ and take $\{x_n\} \subseteq \overline{\mathcal{M}} \setminus \mathcal{M}$ such that $x_n \to x$. Then, by compactness, there exists $\eta_n \in \mathcal{N}^C_{\mathcal{M}}(x_n)$ with $|\eta_n| = 1$ such that $\alpha_g(x_n) = \langle g(x_n), \eta_n \rangle$.

   Without lose of generality, assume that $\eta_n \to \eta$, with $|\eta| = 1$. We know that the multifunction $x \mapsto \mathcal{N}^C_{\mathcal{M}}(x)$ has closed graph on $\overline{\mathcal{M}}$ because $\mathcal{T}^C_{\mathcal{M}}(\cdot)$ is lowersemicontinous. So $\eta \in \mathcal{N}^C_{\mathcal{M}}(x)$. Thus,

$$\lim_{n\to\infty} \alpha_g(x_n) = \langle g(x), \eta \rangle \leq \alpha_g(x).$$

2. It is enough to show that $\alpha_g(\cdot)$ restricted to $\mathcal{M}_b$ is lower semi-continuous. So, let $x \in \mathcal{M}_b$ and $\{x_n\} \subseteq \mathcal{M}_b$ such that $x_n \to x$. Since, $\mathcal{N}^C_{\mathcal{M}}(\cdot)$ is lsc restricted to $\mathcal{M}_b$, for any $\eta \in \mathcal{N}^C_{\mathcal{M}}(x)$, we can assume that, passing to a subsequence if necessary, there exists a sequence $\eta_n \in \mathcal{N}^C_{\mathcal{M}}(x_n)$ such that $\eta_n \to \eta$. Let $\bar{\eta} \in \mathcal{N}^C_{\mathcal{M}}(x)$ satisfying $\alpha_g(x) = \langle g(x), \bar{\eta}\rangle$, thus for any $n \in \mathbb{N}$

$$\alpha_g(x_n) \geq \langle g(x_n), \eta_n\rangle \geq \langle g(x_n), \bar{\eta}\rangle - |g(x_n)||\eta_n - \bar{\eta}|.$$

   Taking the limit inferior in the last inequality, the proof is completed.

3. Let $\bar{x} \in \mathcal{M}_b$ and $r > 0$, take $x, \tilde{x} \in \mathbb{B}(\bar{x}, r) \cap \mathcal{M}_b$. Let L_r be the Lipschitz constant of $g(\cdot)$ and $\mathcal{N}^C_{\mathcal{M}}(\cdot) \cap \overline{\mathbb{B}}$ on $\mathbb{B}(\bar{x}, r) \cap \mathcal{M}_b$. Consider too $C_r > 0$ an upper bound for the norm of g on $\mathbb{B}(\bar{x}, r) \cap \mathcal{M}_b$. By compactness, take $\eta_x \in \mathcal{N}^C_{\mathcal{M}}(x) \cap \overline{\mathbb{B}}$ such that $\alpha_g(x) = \langle g(x), \eta_x\rangle$, thereupon, for any $\eta_{\tilde{x}} \in \mathcal{N}^C_{\mathcal{M}}(\tilde{x}) \cap \overline{\mathbb{B}}$

$$\begin{aligned}\alpha_g(x) - \alpha_g(\tilde{x}) &\leq \langle g(x), \eta_x\rangle - \langle g(\tilde{x}), \eta_{\tilde{x}}\rangle,\\ &\leq \langle g(x) - g(\tilde{x}), \eta_x\rangle + \langle g(\tilde{x}), \eta_x - \eta_{\tilde{x}}\rangle,\\ &\leq L_r|x - \tilde{x}| + C_r|\eta_x - \eta_{\tilde{x}}|.\end{aligned}$$

   Therefore, taking the infimum over all $\eta \in \mathcal{N}^C_{\mathcal{M}}(\tilde{x})$ with $|\eta| = 1$

$$\alpha_g(x) - \alpha_g(\tilde{x}) \leq L_r|x - \tilde{x}| + C_r \mathrm{dist}_{\mathcal{N}^C_{\mathcal{M}}(\tilde{x})\cap\overline{\mathbb{B}}}(\eta_x),$$

   but,

$$\mathrm{dist}_{\mathcal{N}^C_{\mathcal{M}}(\tilde{x})\cap\overline{\mathbb{B}}}(\eta_x) \leq \mathcal{D}\left(\mathcal{N}^C_{\mathcal{M}}(x) \cap \overline{\mathbb{B}}, \mathcal{N}^C_{\mathcal{M}}(\tilde{x}) \cap \overline{\mathbb{B}}\right) \leq L_r|x - \tilde{x}|.$$

   Finally, since it is possible to change the roles between x and $\tilde{x}$, there exists a constant $L_r(\alpha_g) > 0$ such that

$$|\alpha_g(x) - \alpha_g(y)| \leq L_r(\alpha_g)|x - y| \qquad \forall x, y \in \mathbb{B}(x, r) \cap \mathcal{M}_b.$$

□

7.4.3 The externally perturbed model

Recall that the main characteristic of the ordinary differential equations presented in this work is that there may exist some strata where no trajectory can slide for. However, in presence of external perturbations this feature may not be held because for any $i \notin \mathcal{I}_0$ and $j \in \mathcal{I}_0(i)$, it is possible to construct a continuous perturbation $\xi : \mathcal{M}_i \to \mathbb{R}^N$ such that

$$g_j(x) + \xi(x) \in \mathcal{T}_{\mathcal{M}_i}(x), \quad \forall x \in \mathcal{M}_i.$$

Hence, trajectories of an externally perturbed model may slide for $\mathcal{M}_i$, so an equation written as (**D**) may not have sense. To avoid this problem, one may replace the initial stratified dynamics by a set-valued map which takes into account all the *significative* directions of a SVF and it is defined on the whole state-space, namely

$$G^E(x) = \bigcup_{i\in\mathcal{I}_0(\imath(x))} \{g_i(y)\} \cap \mathcal{T}^B_{\mathcal{M}_i}(x), \quad \forall x \in \mathcal{K}.$$

Remark 7.4.1. *The idea of using a set-valued map of the essential directions has already been explored in others context. In particular for studying Hamilton-Jacobi-Bellman equation with discontinuous data we can quote the works of Barnard-Wolenski [17] and Rao-Zidani [106] and Rao-Siconolfi-Zidani [105].*

In this case, we write the perturbed equation as

$$\dot{y} \in G^E(x) + \sigma(x)\xi \quad \text{a.e. } t \in (0,T), \quad y(t) \in \mathcal{K}, \quad \text{for every } t \in [0,T], \tag{7.3}$$

where $\sigma : \mathcal{K} \to [0,+\infty)$ is continuous, $\xi : [0,T] \to \mathbb{R}^N$ is measurable.

Nonetheless, the initial formulation is still of concern and so, it would be interesting to know when a Carathéodory solution to (7.3) is also a solution in the stratified sense, as much as in the unperturbed equation (**D**). It turns out that a sort of maximality condition over the choice of the index $\mathcal{I}_0$ is required. To state the hypothesis, let us first introduce some notation, set $\alpha_j(x) = \alpha_{g_j}(x)$ for any $x \in \mathcal{K}$ and $j \in \mathcal{I}_0(x)$.

$$(\mathbf{H_3^7}) \qquad \begin{cases} \text{For any } i \in \mathcal{I} \text{ and } j \in \mathcal{I}_0(i): \\ i) \quad \text{The sign of } \alpha_j(\cdot) \text{ is constant all along } \mathcal{M}_i. \\ ii) \;\; \text{If } i \notin \mathcal{I}_0,\ \alpha_j(x) \neq 0,\ \forall x \in \mathcal{M}_i. \\ iii) \text{ If } i \in \mathcal{I}_0 \text{ and } \alpha_j(x) = 0 \text{ with } g_i(x) \neq 0,\ \forall x \in \mathcal{M}_i, \\ \qquad \text{then } g_j(x) = g_i(y)\ \forall x \in \mathcal{M}_i. \end{cases}$$

The first two points in ($\mathbf{H_3^7}$) are not sharp and they can be weakened. The main goal of the first one is to avoid trajectories that may switch infinitely many times between $\mathcal{M}_j$ and $\mathcal{M}_i$, and the second seeks to ensure that for any $j \in \mathcal{I}_0(i)$, $g_j(x) \in \mathcal{T}_{\mathcal{M}_i}(x)$ never happens. However, the third point seems to be an essential assumption as the next remark explains.

Remark 7.4.2. *Assume that* ($\mathbf{H_2^7}$) *holds. Let* $j \in \mathcal{I}_0$, $x \in \overline{\mathcal{M}}_j \setminus \mathcal{M}_j$ *and* $r > 0$ *so that*

$$\alpha_j(\tilde{x}) = 0 \textit{ and } g_j(\tilde{x}) \neq 0, \quad \forall \tilde{x} \in \overline{\mathcal{M}}_j \cap \mathbb{B}(x,r).$$

By Proposition 7.4.1, $g_j(\tilde{x}) \in rbd\left(\mathcal{T}^C_{\mathcal{M}_j}(\tilde{x})\right)$ *over* $\overline{\mathcal{M}}_j \cap \mathbb{B}(x,r)$. *Let* $\{x_n\} \subseteq \mathcal{M}_j$ *converging to* $x \in \mathcal{M}_i$ *and* y_n *a stratified solution of* (**D**) *with* $y_n(0) = x_n$ *and lying on* $\mathcal{M}_j$ *on an interval* $[0,T_n]$. *Suppose that* $T_n \geq T > 0$, $y_n \to y$ *uniformly on* $[0,T]$ *and* $y(t) \in \overline{\mathcal{M}}_j \setminus \mathcal{M}_j$ *on* $[0,T]$.

Under these circumstances, by continuity $\dot{y} = g_j(y)$ *on* $[0,T]$, *but since the stratification is locally finite, there exist* $i \preceq j$, $t \in (0,T)$ *and* $\varepsilon > 0$ *so that* $y(s) \in \mathcal{M}_i$ *for any* $s \in (t-\varepsilon, t+\varepsilon)$. *We readily notice that if* $g_i \neq g_j$, *then the limiting arc* y *can not be a stratified trajectory. In this sense, this hypothesis is a necessary condition for robustness. We recall that we have already pointed out this in Remark 7.2.1.*

On the other hand, the set of singularities enlisted in [30] for planar systems with affine control systems as well as the set of singularities found in the construction of the feedback in [63] satisfy this condition. Furthermore, this also implies that, if $g_i(y) \neq 0$, then

$$G^E(x) \cap \mathcal{T}_{\mathcal{M}_i}(x) = \begin{cases} \{g_i(y)\} & \text{if } i \in \mathcal{I}_0 \\ \emptyset & \text{if } i \notin \mathcal{I}_0, \end{cases} \quad \forall x \in \mathcal{M}_i. \tag{7.4}$$

The particular fashion how (7.3) have been stated was thought to provide an explicit upper bound for the size of the perturbation. Hereafter, we will use the notation

$$\begin{aligned}\mathcal{I}_i^+(x) &:= \{j \in \mathcal{I}_0(\imath(x)) : \ \alpha_j > 0 \text{ all along } \mathcal{M}_i\},\\ \mathcal{I}_i^-(x) &:= \{j \in \mathcal{I}_0(\imath(x)) : \ \alpha_j < 0 \text{ all along } \mathcal{M}_i\},\\ \mathcal{I}_i^0(x) &:= \{j \in \mathcal{I}_0(\imath(x)) : \ \alpha_j = 0 \text{ all along } \mathcal{M}_i\}.\end{aligned}$$

Then the bound for the size of the perturbation is given by

$$\sigma(x) \leq \frac{1}{2}\min\left\{\min_{j\in\mathcal{I}_i^+(x)} \alpha_j(x), \min_{j\in\mathcal{I}_i^-(x)} -\alpha_j(x)\right\}, \quad \forall x \in \mathcal{M}_i. \tag{7.5}$$

We now show that the perturbed system (7.3) is equivalent to stratified systems of ordinary differential equations, provided some types of singularities are ruled out.

Proposition 7.4.3. *Assume that* ($\mathbf{H_2^7}$) *and* ($\mathbf{H_3^7}$) *hold. Let* $\xi : [0,T] \to \overline{\mathbb{B}}$ *and* $\sigma : \mathcal{K} \to [0,+\infty)$ *be two given measurable functions. Suppose* $\sigma(\cdot)$ *satisfying* (7.5) *and* $y : [0,T] \to \mathcal{K}$ *is a Carathéodory solution to* (7.3)*. Set* $J_i = \{t \in [0,T] : \ y(t) \in \mathcal{M}_i\}$*, then*

i) $\forall i \notin \mathcal{I}_0$*, meas*$(J_i) = 0$*,*

ii) $\forall i \in \mathcal{I}_0$*,* $\dot{y}(t) = g_i(y(t)) + \sigma(y(t))\xi(t)$ *a.e. on* $J_i \cap \{t \in [0,T] : \ g_i(y(t)) \neq 0\}$*.*

Proof. let $i \in \mathcal{I}$ such that $\text{meas}(J_i) > 0$, by the Lebesgue Differentiation Theorem there exists $\tilde{J}_i \subseteq J_i$ measurable with $\text{meas}(\tilde{J}_i) = \text{meas}(J_i)$ such that

$$\lim_{\varepsilon\to 0^+} \frac{1}{\varepsilon}\int_t^{t+\varepsilon} \dot{y}(s)ds = \dot{y}(t) \in G^E(y(t)) + \sigma(y(t))\xi(t), \quad \forall t \in \tilde{J}_i.$$

Assume that $\tilde{J}_i$ does not contain isolated points. Since the stratification is locally finite, for any $t \in \tilde{J}_i$ there exist $j \succeq i$ with $j \in \mathcal{I}_0$, a sequence $\{t_n\} \subseteq \tilde{J}_i \setminus \{t\}$ with $t_n \to t$ so that

$$\frac{y(t_n) - y(t)}{t_n - t} \to \dot{y}(t) = g_j(y(t)) + \sigma(y(t))\xi(t).$$

Since, $y(t_n) \in \mathcal{M}_i$ for any $n \in \mathbb{N}$, $\dot{y}(t) \in \mathcal{T}^B_{\mathcal{M}_i}(y(t)) = \mathcal{T}_{\mathcal{M}_i}(y(t))$.

i) Suppose $i \notin \mathcal{I}_0$ and let $t \in \tilde{J}_i$. Hence, by Proposition 3.3.7, $\langle \dot{y}(t), \eta\rangle = 0$, for any $\eta \in \mathcal{N}^C_{\mathcal{M}_j}(y(t))$. This remark yields to

$$\langle g_j(y(t)), \eta\rangle - \sigma(y(t)) \leq \langle \dot{y}(t), \eta\rangle \leq \alpha_j(y(t)) + \sigma(y(t)), \quad \forall \eta \in \mathcal{N}^C_{\mathcal{M}_j}(y(t)) \text{ with } |\pi_{y(t)}(\eta)| = 1.$$

In particular, if $j \in \mathcal{I}_i^+(y(t))$, the leftthand side gives a contradiction with (7.5). If $j \in \mathcal{I}_i^-(y(t))$, the same occurs with righthand. Finally, by ($\mathbf{H_3^7}$), $\mathcal{I}_i^0(y(t)) = \emptyset$. Hence, $\text{meas}(J_i) = 0$ otherwise a contradiction is obtained.

ii) By (7.4), $G^E(y(t)) = \{g_i(y(t))\}$ for any $t \in \tilde{J}_i$.

So the conclusion follows. ∎

Notably, if ($\mathbf{H_3^7}$) holds and the perturbations are small enough, the following perturbed stratified system

$$(\mathbf{D}^\sigma) \qquad \dot{y} = g_i(y) + \sigma(y)\xi \quad \text{a.e. whenever } x \in \mathcal{M}_i, \quad y(t) \in \mathcal{K}, \quad t \geq 0$$

makes sense, and in fact, any solution to this model is a Caratheodory solution to (7.3). Solutions to this equation have the following form.

$$y(t) = x + \sum_{i \in I_0} \int_{J_i(t)} g_i(y(s))ds + \int_0^t \sigma(y(s))\xi(s)ds, \quad \forall t \in [0, T],$$

where $J_i(t) = \{s \in [0, t] : \ y(s) \in \mathcal{M}_i\}$.

7.4.4 Robustness result

The main result and principal motivation to write this paper can be summarized in the following theorem.

Theorem 7.4.1. *Assume that* ($\mathbf{H_2^7}$) *and* ($\mathbf{H_3^7}$) *hold. Let* $\{\xi_n\} \subseteq L^1([0, T]; \overline{\mathbb{B}})$ *be a sequence converging to* 0 *in* $L^1([0, T]; \overline{\mathbb{B}})$ *and consider* $\sigma : \mathcal{K} \to [0, +\infty)$ *a given continuous satisfying* (7.5)*. Suppose also that*

$$(\mathbf{H_4^7}) \qquad \begin{cases} \forall i \in \mathcal{I}_0, \ \text{if } i \preceq j \text{ then } j \in \mathcal{I}_0 \text{ and } \alpha_j(x) \geq 0, \ \forall x \in \mathcal{M}_i. \\ \forall i \notin \mathcal{I}_0, \ \text{if } j \in \mathcal{I}_0(j) \text{ then } \alpha_j(x) < 0, \ \forall x \in \mathcal{M}_i. \end{cases}$$

Let $y_n(\cdot)$ *be a stratified solution to*

$$(7.6) \qquad \dot{y} = g_i(y) + \sigma(x)\xi_n \quad \text{a.e. on } J_i(T), \quad y(t) \in \mathcal{K}, \quad \forall t \in [0, T], \quad y(0) = x_n \in \mathcal{K}.$$

Then, any accumulation point of $\{y_n\}$ *in the space of the continuous functions endowed with the uniform norm is a stratified solution to* ($\mathbf{D}$)*.*

To prove this theorem, some previous lemmas are required beforehand.

Lemma 7.4.1. *Assume that* ($\mathbf{H_2^7}$) *holds. Take a measurable map* $\xi : [0, T] \to \overline{\mathbb{B}}$ *and a continuous function* $\sigma : \mathcal{K} \to [0, +\infty)$ *that satisfies* (7.5)*. Let* y *be a stratified solution to* ($\mathbf{D}^\sigma$) *defined on* $[0, T]$ *and,* $i \in \mathcal{I}$ *and* $j \in \mathcal{I}_0$ *with* $i \succeq j$ *and* $\dim(\mathcal{M}_i) + 1 = \dim(\mathcal{M}_j)$*. Suppose that*

$$(7.7) \qquad \alpha_j(x) > 0 \quad \forall x \in \mathcal{M}_i.$$

and $y(t) \in \mathcal{M}_i \cup \mathcal{M}_j$ *for every* $t \in [0, T]$*. Then, for any* $x \in \mathcal{M}_i$ *there exists* $\rho_i > 0$ *such that*

$$\forall \rho \in (0, \rho_i), \exists \tau = \tau(\rho) \in (0, T] : \ y(0) \in \mathcal{M}_j \cap \mathbb{B}(x, \rho) \Rightarrow y(\tau) \in \mathcal{M}_i.$$

Furthermore, $\tau(\rho) \to 0$ *as* $\rho \to 0$ *and if* $i \in \mathcal{I}_0$ *then* $y(t) \in \mathcal{M}_i$ *on* $[\tau, T]$*.*

The preceding lemma has as goal to show that if $\alpha_j > 0$ around some point on a manifold $\mathcal{M}_i$, the if the trajectory starts sufficiently close, it will reach that manifold in finite time and moreover, once there, it can not pass into $\mathcal{M}_j$ directly from $\mathcal{M}_i$. In Figure 7.4 we exhibit a picture that represents the situation described above.

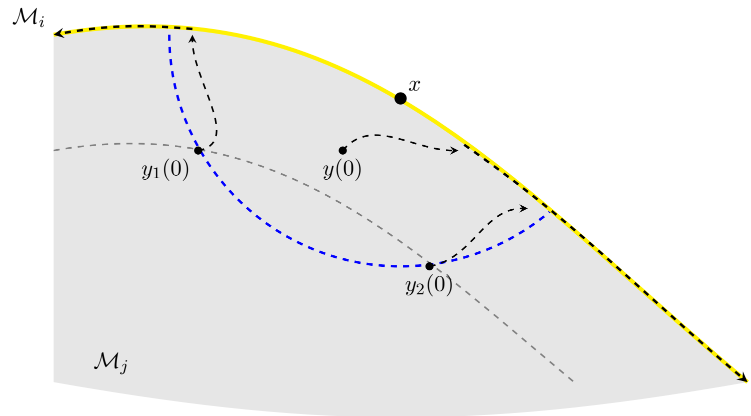


Figure 7.4: Illustration of Lemma 7.4.1

Proof of Lemma 7.4.1. Let $x \in \mathcal{M}_i$, then we can find $r > 0$ and a local defining map $h : \mathbb{B}(x, r) \to \mathbb{R}^d$ such that

$$\mathcal{N}_{\mathcal{M}_i}(\tilde{x}) = \operatorname{span}\{\nabla h_1(\tilde{x}), \ldots, \nabla h_d(\tilde{x})\}, \quad \tilde{x} \in \mathcal{M}_i \cap \mathbb{B}(x, r).$$

Due to the fact that $\overline{\mathcal{M}}_j$ is relatively wedged at x and $\dim(\mathcal{M}_i) + 1 = \dim(\mathcal{M}_j)$, there exists a pointed cone $\mathcal{N}_x$ and a $(d-1)$-dimensional vectorial space $X_{\mathcal{N}}(x) \subseteq \mathbb{R}^N$ so that $\mathcal{N}^C_{\overline{\mathcal{M}}_j}(x) = \mathcal{N}_x \oplus X_{\mathcal{N}}(x)$. Thereby, Proposition 3.3.7 implies that $\mathcal{N}^C_{\overline{\mathcal{M}}_j}(x) \subseteq \mathcal{N}_{\mathcal{M}_i}(x)$ and, using the Gram-Schmidt process we construct an orthonormal family of continuous vector fields $\eta_1, \ldots, \eta_d : \mathbb{B}(x, r) \to \mathbb{R}^N$ such that

$$\begin{aligned}
&\mathcal{N}_{\mathcal{M}_i}(\tilde{x}) = \operatorname{span}\{\eta(\tilde{x}), \ldots, \eta_d(\tilde{x})\}, \quad \tilde{x} \in \mathcal{M}_i \cap \mathbb{B}(x, r),\\
&\mathcal{N}_x = \operatorname{cone}\{\eta_1(x)\} \quad \text{and} \quad X_{\mathcal{N}}(x) = \operatorname{span}\{\eta_2(x), \ldots, \eta_d(x)\}.
\end{aligned}$$

Taking all this into account, one gets the following formula

$$\alpha_j(\tilde{x}) = \langle g_j(\tilde{x}), \eta_1(\tilde{x})\rangle, \qquad \tilde{x} \in \mathcal{M}_i \cap \mathbb{B}(x, r).$$

Assume that $\eta_1(x) = \sum_{n=1}^p \lambda_n \nabla h_n(x)$ with $\lambda_n \geq 0$ and $h_n(y) < 0$ for every $y \in \mathcal{M}_j \cap \mathbb{B}(\bar{x}, r)$. Otherwise, it is enough to replace h_n with $-h_n$.

Let $c_r > 0$ be an upper bound for $|g_j| + \sigma$ on $\mathbb{B}(x, r)$ and let $L_n^r > 0$ stand for the Lipchitz constant of ∇h_n on $\mathbb{B}(\bar{x}, r)$. Take $\tilde{r} \in (0, r)$ so that

$$\max\left\{|g_j(x) - g_j(\tilde{x})|, |\sigma(x) - \sigma(\tilde{x})|, \frac{\tilde{r}}{2L^r c_r}\right\} \leq \frac{1}{12}\alpha_j(\tilde{x}), \ \forall \tilde{x} \in \mathbb{B}(x, \tilde{r}).$$

Let $\rho > 0$ to be fixed and suppose $y(0) \in \mathcal{M}_j \cap \mathbb{B}(\bar{x}, \rho)$. Set $h(t) = \sum_{n=1}^p \lambda_n h_n(y(t))$ and note that $h(0) < 0$. Let

$$T_{\tilde{r}} = \sup\{t \in [0, T] : h(t) < 0 \text{ and } y(t) \in \mathbb{B}(x, \tilde{r})\}.$$

Hence for a.e. $t \in (0, T_{\tilde{r}})$ we get

$$\begin{aligned}\dot{h}(t) &= \sum_{n=1}^{p}\lambda_n\langle\nabla h_n(y(t)),\dot{y}(t)\rangle = \sum_{n=1}^{p}\lambda_n\langle\nabla h_n(y(t)),g_j(y(t))\rangle + \lambda_n\langle\nabla h_n(y(t)),\sigma(y(t))\xi(t)\rangle\\ &\geq \left\langle\sum_{n=1}^{p}\lambda_n\nabla h_n(y(t)),g_j(y(t))\right\rangle - \sigma(y(t))\left|\sum_{n=1}^{p}\lambda_n\nabla h_n(y(t))\right|\end{aligned}$$

Note that $|\sum_{n=1}^{p}\lambda_n\nabla h_n(y(t))| \leq 1 + L^r|y(t)-x|$ and

$$\begin{aligned}\left\langle\sum_{n=1}^{p}\lambda_n\nabla h_n(y(t)),g_j(y(t))\right\rangle &= \alpha_j(x) + \left\langle\sum_{n=1}^{p}\lambda_n\left(\nabla h_n(y(t))-\nabla h_n(x)\right),g_j(y(t))\right\rangle\\ &\quad \left\langle\sum_{n=1}^{p}\lambda_n\nabla h_n(x),g_j(y(t))-g_j(x)\right\rangle\\ &\geq \alpha_j(x) - L^r|y(t)-x||g_j(y(t)| - |g_j(y(t))-g_j(x)|.\end{aligned}$$

Consequently,

$$\dot{h}(t) \geq \alpha_j(x) - \sigma(x) - L^r c_r|y(t)-x| - |g_j(y(t))-g_j(x)| - |\sigma(y(t))-\sigma(x)| \geq \frac{1}{4}\alpha_j(x).$$

Where this last inequality comes from (7.5) and the choice of $\tilde{r}>0$. Therefore, $t\mapsto h(t)$ is strictly increasing and

$$h(t) \geq h(0) + \frac{1}{4}\alpha_j(x)t, \quad \forall t\in[0,T_{\tilde{r}}].$$

Hence, $T_{\tilde{r}} \leq -4h(0)\frac{1}{\alpha_j(x)}$ and due to each h_n is locally Lipschitz continuous, there exists a constant $\ell_{\tilde{r}}>0$ such that

$$\left|\sum_{n=1}^{p}\lambda_n h_n(\tilde{x})\right| \leq \ell_{\tilde{r}}\mathrm{dist}_{\mathcal{M}_i}(\tilde{x}) \leq \ell_{\tilde{r}}\rho, \quad \forall\tilde{x}\in\mathcal{M}_j\cap\mathbb{B}(x,\tilde{r}),$$

and so, $T_r \leq \frac{4\ell_{\tilde{r}}\rho}{\alpha_j(x)}$. Nevertheless, given that y is continuous with essentially bounded derivatives, the time needed to escape from the $\mathbb{B}(x,\tilde{r})$ should increase as long as ρ goes to 0. Thus, there exists $\rho_i>0$ such that

$$\inf\{t\in[0,T] : y(t)\in\mathcal{M}_j \text{ with } |y(t)-\bar{x}| = \tilde{r}\} > T_{\tilde{r}}$$

whenever $|y(0)-\bar{x}| \leq \rho \leq \rho_i$, and so $y(\tau)\in\mathcal{M}_i$ for any $\rho\leq\rho_i$ with $\tau = T_{\tilde{r}}$. Therefore, the first part of the statement holds.

Let us see the second part. Assume now that $i\in\mathcal{I}_0$ and $\tau<T$, for sake of simplicity, $\tau=0$ and so $y(0)\in\mathcal{M}_i$. Suppose there exists $t\in(0,T]$ such that $y(t)\in\mathcal{M}_j$. Since $\mathcal{M}_j$ is relatively open on $\mathcal{M}_i\cup\mathcal{M}_j$, there exists $t_0\in[0,t)$ such that $y(t_0)\in\mathcal{M}_i$ and $y(s)\in\mathcal{M}_j$ for every $s\in(t_0,t]$. To simply the notation, t_0 is set to be 0 and $\bar{x}=y(0)$. Let $\varepsilon\in\left(0,\frac{1}{2}\alpha_j(\bar{x})\right)$, then since σ and g_j are continuos, there exists $\delta>0$ such that

$$|\tilde{x}-\bar{x}|<\delta \quad \Rightarrow \max\{|\sigma(\tilde{x})-\sigma(\bar{x})|,|g_j(\tilde{x})-g_j(\bar{x})|\} \leq \frac{\varepsilon}{3}.$$

Take $\tilde{t} \in (0, t)$ so that $y(t) \in \mathbb{B}(\bar{x}, \delta)$ and

$$\max_{s\in[0,\tilde{t}]} \sigma(y(s)) \leq \sigma(\bar{x}) + \frac{\varepsilon}{3}, \ \max_{s\in[0,\tilde{t}]} |g_j(y(s)) - g_j(\bar{x})| \leq \frac{\varepsilon}{3}.$$

Note that for any s small enough,

$$0 > h(s) = \left\langle \sum_{n=1}^{p} \lambda_n \nabla h_n(\bar{x}), y(s) - \bar{x} \right\rangle + o(|y(s) - \bar{x}|^2).$$

So, without loss of generality

$$\left\langle \frac{y(s) - \bar{x}}{s}, \eta_1(\bar{x}) \right\rangle \leq \frac{1}{3}\varepsilon, \quad \forall s \in (0, \tilde{t}). \tag{7.8}$$

On the other hand, for any $s \in (0, \tilde{t})$ we have

$$\begin{aligned}
\left\langle \frac{y(s) - \bar{x}}{s}, \eta_1(\bar{x}) \right\rangle &= \frac{1}{s} \int_0^s \langle g_j(y(a)) + \sigma(y(a))\xi(a), \eta_1(\bar{x}) \rangle da \\
&\geq \alpha_j(\bar{x}) - \frac{1}{s} \int_0^s (|g_j(y(a)) - g_j(\bar{x})| + \sigma(y(a)))da \\
&\geq \alpha_j(\bar{x}) - \sigma(\bar{x}) - \frac{2}{3}\varepsilon,
\end{aligned}$$

where the last inequality is given by way how $\tilde{t} > 0$ has been selected. Therefore, by (7.5) and (7.8)

$$\varepsilon \geq \alpha_j(\bar{x}) - \frac{1}{2}\alpha_j(\bar{x}) = \frac{1}{2}\alpha_j(\bar{x}).$$

Which contradicts the choice of ε, so such t_0 can not exist. □

Remark 7.4.3. *The fact that* $\dim(\mathcal{M}_i) + 1 = \dim(M_j)$ *is crucial for the proof of the first part of the previous lemma. Indeed, if* $\dim(\mathcal{M}_i) + 1 < \dim(M_j)$ *the same conclusion of Lemma 7.4.1 does not hold, for example, consider stratified equation described by Figure 7.5. The stratification is* $\mathcal{M}_0 = \{(0, 0)\}$*,* $\mathcal{M}_1 = \{x + y = 0,\ x < 0\}$*,* $\mathcal{M}_2 = \{y = 0,\ x > 0\}$*,* $\mathcal{M}_3 = \{\min\{x + y, y\} < 0\}$ *and* $\mathcal{M}_4 = \{\min\{x + y, y\} > 0\}$*. The vector fields are* $g_1 = g_4 \equiv (1, 0)$ *and* $g_2 = g_3 \equiv (1, -1)$*.*

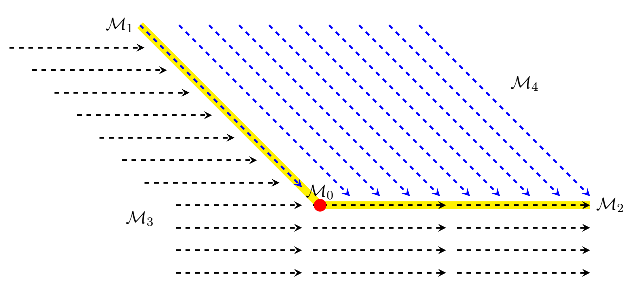


Figure 7.5: Example Remark 7.4.3

It is not difficult to see that $(\mathcal{M}_1, \mathcal{M}_3)$*,* $(\mathcal{M}_2, \mathcal{M}_4)$*,* $(\mathcal{M}_1, \mathcal{M}_0)$ *verifies the result stated in the lemma. Note also that* $\alpha_3(0, 0) = \frac{1}{\sqrt{2}} > 0$*. However, no trajectory of the perturbed system starting from* (x_1, x_2) *of norm arbitrarily small but with* $x_1 > 0$ *and* $x_2 < 0$ *reaches* $\mathcal{M}_0$*.*

A similar lemma can be stated when the sign of the outward-pointing modulus is negative. In this case, there is no restriction over the dimensions of the strata. In this case, we have a statement that implies that any trajectory of the perturbed dynamics that pass into a neighboring stratum can not come back to the initial stratum as long as the outward pointing modulus is strictly negative. This situation has been depicted in Figure 7.6.

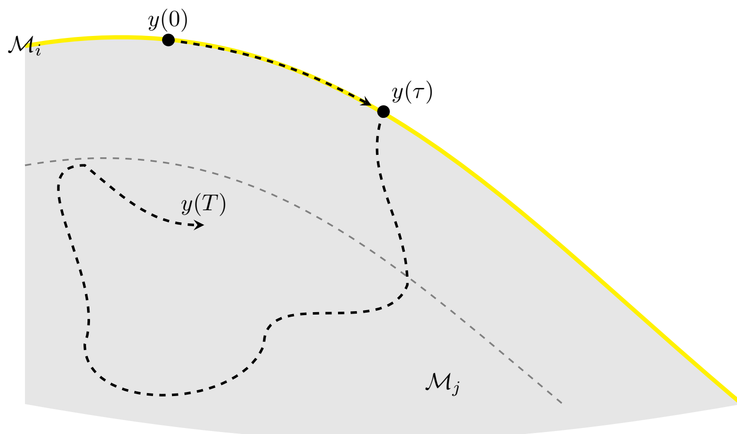


Figure 7.6: Illustration of Lemma 7.4.2

Lemma 7.4.2. *Assume that* $(\mathbf{H_2^7})$ *holds. Let* $\xi : [0,T] \to \mathbb{B}$ *be a measurable map and* $\sigma : \mathcal{K} \to [0,+\infty)$ *a continuous function that satisfies* (7.5). *Let* y *be a stratified solution of* $(\mathbf{D}^\sigma)$ *defined on* $[0,T]$. *Let* $i \in \mathcal{I}$ *and* $j \in \mathcal{I}_0$ *with* $i \preceq j$. *Assume that*

$$\alpha_j(x) < 0 \quad \forall x \in \mathcal{M}_i. \tag{7.9}$$

Suppose that $y(t) \in \mathcal{M}_i \cup \mathcal{M}_j$ *for every* $t \in [0,T]$ *with* $y(0) \in \mathcal{M}_i$. *Let*

$$\tau = \inf\{t \in [0,T] : y(t) \notin \mathcal{M}_i\}.$$

If $\tau < T$ *then* $y(t) \in \mathcal{M}_j$ *on* $(\tau, T]$.

Proof. Let $x \in \mathcal{M}_i$ and $r > 0$ as in the preceding lemma such that $y(0) \in \mathbb{B}(x,r)$. Without loss of generality, $\tau = 0$ and let $\tau_1 > 0$ be the maximal time such that $y(t) \in \mathcal{M}_j \cap \mathbb{B}(x,r)$ for every $t \in (0,\tau_1)$. By contradiction, suppose that $\tau_1 < T$. For any $\eta \in \mathcal{N}^C_{\mathcal{M}}(x)$ there exists a representation as

$$\eta = \sum_{n=1}^{p} \lambda_n \nabla h_n(x).$$

Setting $h(t) = \sum_{n=1}^{p} \lambda_n h_n(y(t))$ and using the similar estimations as in Lemma 7.4.1, we show right after that, $\dot{h}(t) < \frac{1}{4}\alpha_j(x)$ for almost all $t \in (0,\tau_1)$, and so, $t \mapsto h(t)$ is strictly decreasing. Note that $h(0) = 0$ for any η, thereby $y(\tau_1) \in \mathcal{M}_j$ with $|y(\tau_1) - \bar{x}| = r$ and the conclusion follows. $\square$

7.4.5 Proof of Theorem 7.4.1

With all this technical statements at hand, we can now turn our attention into the proof of the robustness result.

Proof (Thm. 7.4.1). By ($\mathbf{H_3^7}$), the sets of index $\mathcal{I}_i^+(x)$, $\mathcal{I}_i^-(x)$ and $\mathcal{I}_i^0(x)$ are independent of x, and for sake of notation the dependence with respect to x is suppressed. Note that the vector field $g(x) = g_l(x)$ whenever $x \in \mathcal{M}_l$ is continuous on

$$\mathcal{S}_i = \bigcup \{\mathcal{M}_j : \ j \in \mathcal{I}_i^0\}, \quad \text{whenever } i \in \mathcal{I}_0.$$

Note also that $\mathcal{S}_i$ is locally closed around each $x \in \mathcal{M}_i$.

To prove the statement of the theorem it is enough to show that for some $\tau > 0$, $y|_{[0,\tau]}$ is a stratified solution. For this purpose, let $i = \imath(x)$ and $i_n = \imath(x_n)$, where $x = y(0)$. Since the stratification is locally finite, $\{i_n\}$ is compact and so, for simplicity, it is assumed that $i_n = j \in \mathcal{I}$ for any $n \in \mathbb{N}$ with $i \preceq j$.

Suppose that $i \in \mathcal{I}_0$, then by ($\mathbf{H_4^7}$), $j \in \mathcal{I}_0$ as well. Let $R > 0$ such that $\mathcal{M}_i$ is the stratum of lower dimension on $\mathcal{K} \cap \overline{\mathbb{B}}(x, R)$ and let

$$\tau_n := \inf \left\{ t \in [0, T] : \ y_n(t) \notin \mathcal{K} \cap \overline{\mathbb{B}}(x, R) \right\}.$$

Note that, since the set of velocities associated with (7.6) can be bounded uniformly with respect to n on $\mathcal{K} \cap \overline{\mathbb{B}}(x, R)$, there exists $\tau > 0$ such that $\tau_n > \tau$ for any $n \in \mathbb{N}$. Recall that $x_n \to x$, so it can be as close as wanted of $\mathcal{S}_i$. Moreover, it is not difficult to see that by Lemma 7.4.1 and ($\mathbf{H_4^7}$) any arc y_n reaches $\mathcal{S}_i$ within time t_n, and even more, the sequence $\{t_n\}$ converges to zero as long as n goes to infinity. So, without loss of generality, $j \in \mathcal{I}_i^0$. Once again, by Lemma 7.4.1 and ($\mathbf{H_4^7}$), no trajectory of (7.6) can pass to a stratum of bigger dimension as long as the outward pointing modulus related to this stratum has positive sign. Therefore, $y_n(t) \in \mathcal{S}_i$ for any $t \in [0, \tau]$ and any $n \in \mathbb{N}$. Recall that g is continuous on each $\mathcal{S}_i$, so the conclusion follows by passing into the limit in

$$y_n(t) = x_n + \int_0^t \left[g(y_n) + \sigma(y_n)\xi_n \right] ds.$$

Assume that $i \notin \mathcal{I}_0$ and, suppose first that $j \in \mathcal{I}_0$, using the same arguments as in the previous part applied to j instead of i, it can be shown that there exists $\tau_n > 0$ such that $y_n(t) \in \mathcal{S}_j$ for any $t \in [0, \tau_n]$ and any $n \in \mathbb{N}$. Now, by ($\mathbf{H_4^7}$), $\alpha_j(x) < 0$ and by Lemma 7.4.2, τ_n can be uniformly bounded from below for a positive number, so it is possible to pass to the limit and get the conclusion.

Now consider the case $j \notin \mathcal{I}_0$, by ($\mathbf{H_4^7}$), $\mathcal{I}_0(j) = \mathcal{I}_j^-$ and $\mathcal{I}_0(i) = \mathcal{I}_i^-$, and by Lemma 7.4.1, each trajectory y_n can only dwell in strata whose indices belong to $\mathcal{I}_j^-$. So, by Lemma 7.4.2 there exists $k \in \mathcal{I}_j^-$ such that $y_n(t) \in \mathcal{M}_k$ in a maximal interval $(t_n, \tau_n]$ with $\tau_n > 0$. Since by ($\mathbf{H_4^7}$), $k \in \mathcal{I}_0$, Lemma 7.4.1 implies that the sequence $\{\tau_n\}$ is uniformly bounded from below by a positive number $\tau > 0$ and using the same argument as before one can pass to the limit and get the desired result. So the proof is completed. □

7.5 Discussion and perspectives

We start the discussion of the results obtained along the chapter by pointing out some similar works that have been addressed to discontinuous ordinary differential equations.

To the best of our knowledge, the paper of Marigo-Piccoli [90] is the closest to our setting. In that work the authors study the properties of a discontinuous ordinary differential equation starting from an axiomatic definition of stratified solutions. In particular, a notion of robustness that depends on a cone defined along trajectories was introduced.

Other works focused on qualitative analysis for 3-dimensional piecewise smooth dynamical systems and switching surfaces can be found in the literature; we refer for instance to Teixeira [128] and Jeffrey-Colombo [81]. We also mention that approaches considering generalized notions of solutions have been largely investigated in the last decades; see for example the manuscript of Filippov-Arscott [49], the papers authored to Brunoskỳ [34, 35], Hájek [64, 65], Honkapohja-Ito [73] and the references therein.

7.5.1 Contributions of the chapter

At the present exposition we have taken as initial point a given discontinuous vector field along with a stratification. These objects are compatible between each other by means of Definition 7.2.1, in particular the local and pointwise criterion for the existence of solutions presented in Theorem 7.3.2 and Theorem 7.3.3, respectively, are rather simple to verify if the SVF has been prescribed beforehand. Furthermore, the introduction of the outward pointing modulus, as seen on the analysis, provides a practicable way to measure the maximum size of the perturbations and also to identify some favorable classes of singularities that make the system stable with respect to external perturbations. We stress that the outward pointing modulus allows us to give explicit bounds for the size of errors that can be managed.

To summarize, the main contribution of the theory we have recently presented lies in the possibility to determine if a discontinuous ordinary differential equation admits classical solutions and if it is stable under external perturbations, using exclusively the initial data of the problem.

7.5.2 Some extensions

We end this section claiming some possible extensions for the results we have stated.

First of all, notice that, instead of considering a discontinuous vector field we might deal with a non regular set-valued maps. This could be the case if for example we work with set-valued feedbacks rather than single-valued ones. Or even more, if we have from the very beginning a stratified differential inclusion (as in [17]). Under such circumstances, Definition 7.2.1 has to be modified in the following way:

Definition 7.5.1. *Let $\mathcal{K} \subseteq \mathbb{R}^N$ be a stratifiable closed set with strata given by $\{\mathcal{M}_i\}_{i\in\mathcal{I}}$. Let $\mathcal{I}_0 \subseteq \mathcal{I}$ be a subset of index so that $\{\mathcal{M}_i\}_{i\in\mathcal{I}_0}$ is dense in $\mathcal{K}$. Then a stratified multifunction is a family of set-valued maps $G = \{G_i : \mathcal{M}_i \rightrightarrows \mathcal{T}_{\mathcal{M}_i}\}_{i\in\mathcal{I}_0}$ such that for each $i \in \mathcal{I}_0$*

$$G_i(x) \subseteq \mathcal{T}_{\mathcal{M}_i}(x), \quad \forall x \in \mathcal{M}_i.$$

In this situation, to guarantee the existence of solutions for the case $\mathcal{I}_0 \neq \mathcal{I}$ we have to require that G_i has closed graph on $\overline{\mathcal{M}}_i$ for every $i \in \mathcal{I}_0$ and use the generalization of the Nagumo's Theorem to this framework, which is usually referred in the literature as the Viability Theorem (cf. [11, Theorem 4.2.1]). Furthermore, the pointwise condition ($\mathbf{H_1^7}$) should now read as follows:

$$\begin{cases} \forall x \in \mathcal{K} \text{ with } \imath(x) \notin \mathcal{I}_0, \exists j \in \mathcal{I}_0(\imath(x)) \text{ such that:} \\ i)\ \ \overline{\mathcal{M}}_j \text{ is relatively wedged at } x \text{ with } G_j(x) \cap \mathrm{ri}\left(\mathcal{T}^C_{\overline{\mathcal{M}}_j}(x)\right) \neq \emptyset. \\ ii)\ \forall l \succeq \imath(x), l \neq j,\ \mathcal{M}_l \text{ has bounded curvature.} \end{cases}$$

Therefore, adapting the arguments of the chapter it is not difficult to see that a statement analogous to Theorem 7.3.3 can be obtained.

CHAPTER 8

Construction of suboptimal feedbacks

Abstract. In this chapter we address the question of the construction a continuous suboptimal feedback from an optimal synthesis. We show a procedure in order to do so. The construction we exhibit depends exclusively on the initial data obtained from the optimal feedback.

8.1 Introduction

Suppose that we can construct an optimal feedback law $U : \mathcal{K} \to \mathcal{U}$ for a given optimal control process with $\mathcal{K}$ as state-constraints. It is well-known that the trajectories associated with the closed-loop system

$$\dot{y}(s) = f(y(s), U(y(s))), \quad \text{a.e. on } [t, T]$$

may not be optimal, not to mention, stable with respect to perturbations; we refer to the discussion in Section 1.2.2 for more details.

Indeed, it may happen that the preceding ordinary differential equation generates too many solutions, several of them not even close of being optimal. Besides, the incorporation of perturbations into the system may cause the loss of the (sub)optimality due to possible Zeno effects. This *inconsistency* of optimal feedbacks is essentially due to the lack of continuity.

We mention that, in the case that no state-constraints are considered, some authors have shown that it is possible to construct suboptimal strategies that enjoy robustness properties without the need of finding the optimal synthesis first; see for example Ancona-Bressan [8] and Rowland-Vinter [117]. In the presence of state-constraints, the issue has also been addressed but imposing beforehand an inward pointing condition (IPC); we quote the contributions of Clarke-Rifford-Stern [43] and Ishii-Koike [79], and the preprint of Priuli [104]. To the best of our knowledge, there are no works that taking advantage of the optimal synthesis, propose a construction of suboptimal strategies consistent in the sense described earlier.

The purpose of this chapter is to point out that slightly modifying an optimal feedback law around some of its singularities we can obtain a suboptimal feedback that is locally Lipschitz continuous. The type of discontinuities we have in mind are those that occur in presence of a switching manifold (in the literature it is also referred as switching locus). These kind of singular sets are sometimes trajectories of the system, and so, they can also be seen as sliding manifolds. For this purpose, as done in Chapter 7, we assume that the set of singularities of the feedback admits a stratified structure.

The construction we propose does not require any type of IPC, and, as a matter of fact, it is rather simple because it is based on convex combinations between the feedback laws of one stratum $\mathcal{M}_{\text{ini}}$ and another stratum $\mathcal{M}_{\text{end}} \subseteq \overline{\mathcal{M}_{\text{ini}}} \setminus \mathcal{M}_{\text{ini}}$. To illustrate this idea, we evoke the optimal control problem exhibited in Example 1.2.4, that is, the minimum time problem defined below

$$\min T \quad \text{s.t. } \dot{y} = \begin{pmatrix} 1 - y_2 \dfrac{u+1}{2} \\ \\ (y_1+1)\dfrac{u+1}{2} \end{pmatrix}, \quad u(t) \in [-1,1] \quad \text{a.e. on } [0,T] \quad y(0)=x,\ y(T)=(0,0).$$

Let us include in the formulation of the problem the state-constraints $\mathcal{K} = \{x \in \mathbb{R}^2 \mid \ x_2 \leq 0\}$. Notice that under these circumstances, the IPC is not verified. Furthermore, it is not difficult to see that the optimal feedback for the state-constrained problem is given by Figure 8.1.

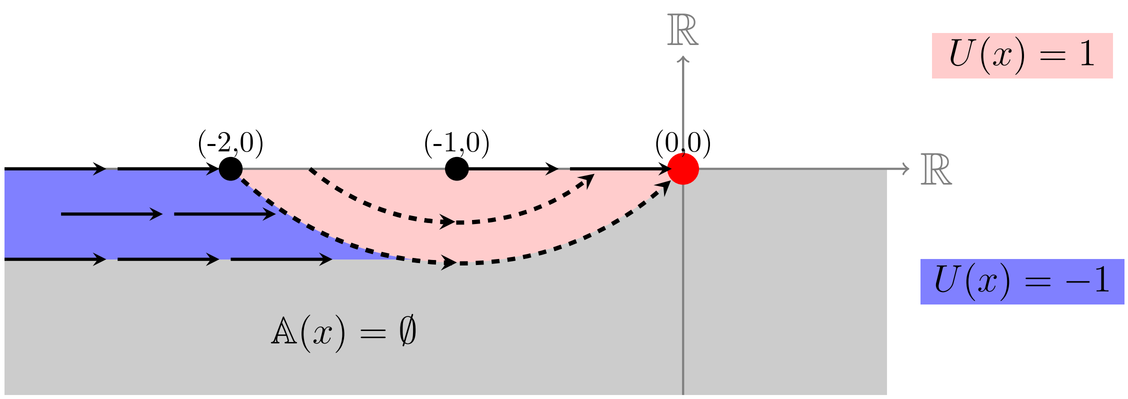


Figure 8.1: The optimal strategy for the problem of Example 1.2.4 with state-constraints.

The framework of the present chapter allows us to treat the singularity of the feedback at the points of $(-1,0) \times \{0\}$. The construction for this case might be focused on the strata

$$\mathcal{M}_{\text{ini}} = \mathbb{B}((-1,1), \sqrt{2}) \cap \text{int}(\mathcal{K}) \quad \text{and} \quad \mathcal{M}_{\text{end}} = (-1,0) \times \{0\}.$$

The procedure consists in modifying the feedback around $\mathcal{M}_{\text{end}}$ in such a way it changes in a continuous way. This has as result that the closed-loop systems is fully robust on $\mathcal{M}_{\text{ini}}$ (the light red zone in Figure 8.1). Moreover, we can also show that by doing such modification the time required to hit a neighborhood of the target is almost optimal.

8.2 Setting of the problem

The emphasis in this chapter is put on time-optimal processes to reach a certain closed target $\Theta \subseteq \mathcal{K}$ by means of a control-affine system

$$\dot{y} = f_0(y) + \sum_{i=1}^{m} u_i f_i(y).$$

Accordingly, when we assume all along this chapter that:

$$(H_f^8) \quad \begin{cases} (i) & \mathcal{U} = [-1,1]^m \text{ and } \exists f_0,\ldots,f_m : \mathbb{R}^N \to \mathbb{R}^N \text{ so that} \\ & f(x,u) = f_0(x) + \sum_{i=1}^m u_i f_i(x),\ \forall x \in \mathbb{R}^N, u \in \mathcal{U}. \\ (ii) & f_0,\ldots,f_m : \mathbb{R}^N \to \mathbb{R}^N \text{ are locally Lipschitz continuous.} \\ (iii) & \exists c_f > 0 \text{ such that } \ |f_i(x)| \le c_f(1+|x|),\ \forall i \in \{0,\ldots,m\}. \end{cases}$$

We recall that the minimum time problem of reaching Θ while being feasible on $\mathcal{K}$ can be written as

$$T^\Theta(x) := \inf\left\{T \ge 0 \mid \ u \in \mathbb{U}^T(x),\ y_x^u(T) \in \Theta\right\}, \quad \forall x \in \mathcal{K}, \tag{8.1}$$

where $\mathbb{U}^T(x)$ stands for the set of admissible measurable controls defined on $[0,T]$ for which the corresponding trajectory $y_x^u(\cdot)$ remains feasible on $\mathcal{K}$. Since Θ and $\mathcal{K}$ are closed, thanks to (H_f^8), the set of admissible trajectories starting at $x \in \mathcal{K}$ fixed is compact in the space of continuous functions and so, whenever $T^\Theta(x) < +\infty$ we can find a control $u_x \in \mathbb{U}^T(x)$ which realizes the infimum in (8.1); the proof is essentially the same as in Proposition 4.2.3. Hence, a time-optimal synthesis is a function $U : \mathrm{dom}(T^\Theta) \to \mathcal{U}$ that satisfies

$$U(y_x^{u_x}(t)) = u_x(t), \quad \text{whenever } T^\Theta(x) \in \mathbb{R} \text{ and for a.e. } t \in [0,T].$$

Notwithstanding the fact that feedback laws are usually discontinuous functions on the state, they are likely to have tame singularities, in the sense that in some regions of the state-space, the feedback is smooth and the notion of Caratéodory solutions can still be defined; we refer for example to the works of Hajek [63], Brunovskỳ [36, 37], Sussmann [126], Meeker [93] and by Boscain-Piccoli [30]. The set of points where an optimal strategy behaves like that is often an embedded manifold. The latter motivates the following definition. In connection with Chapter 7, we say that $\mathcal{M}$, an embedded manifold of $\mathbb{R}^N$, is a *sliding manifold* related to the strategy U provided

$$x \mapsto U|_{\mathcal{M}}(x) \text{ is locally Lipschitz continuous on } \mathcal{M} \text{ and } f(x,U(x)) \in \mathcal{T}_{\mathcal{M}}(x),\ \forall x \in \mathcal{M}.$$

Remark 8.2.1. *Notice that for any initial condition on a sliding manifold, there exist $T > 0$ and a unique smooth curve $y : [0,T) \to \mathcal{M}$ that verifies*

$$\dot{y} = f_0(y) + f(y, U(y)), \quad \text{on } [0,T). \tag{8.2}$$

This affirmation can be corroborated, for instance, with the arguments used to prove Theorem 7.3.1 in the previous Chapter.

From this point on, we assume that $U_0(\cdot)$ is a given optimal synthesis. As aforementioned, we are interested in the circumstances where there exist $\mathcal{M}_{\mathrm{ini}}$ and $\mathcal{M}_{\mathrm{end}}$, both being sliding manifolds associated with an optimal feedback and verifying

$$\Theta \cap \overline{\mathcal{M}_{\mathrm{end}}} \neq \emptyset, \quad \mathcal{M}_{\mathrm{end}} \subseteq \overline{\mathcal{M}_{\mathrm{ini}}} \quad \text{and} \quad \mathcal{M}_{\mathrm{ini}} \cup \mathcal{M}_{\mathrm{end}} \subseteq \mathrm{dom}\, T^\Theta.$$

In this setting, we are concerned with the cases in which the minimum time function to reach Θ starting from $\mathcal{M}_{\mathrm{ini}}$ can be decomposed in the following fashion:

$$T^\Theta(x) = \min_{\tau > 0}\left\{T^\Theta(y_x^{u_x}(\tau)) + \tau \,\middle|\, \begin{array}{ll} y_x^{u_x}(t) \in \mathcal{M}_{\mathrm{ini}}, & \forall t \in [0,\tau) \\ y_x^{u_x}(t) \in \mathcal{M}_{\mathrm{end}}, & \forall t \in [\tau, T^\Theta(x)) \end{array}\right\}, \quad \forall x \in \mathcal{M}_{\mathrm{ini}}. \tag{8.3}$$

In other words, the optimal strategy is the concatenation of two smooth feedbacks so that the path followed by a time-minimizing curve is contained in the corresponding sliding manifold; it starts at $\mathcal{M}_{\text{ini}}$, then reaches $\mathcal{M}_{\text{end}}$, and afterwards, it hits the target. This class of singularities is exactly the one described by Hajek in [63] for a synthesis around the origin for normal linear models and it also agrees with some of the generic singularities of a 2D system exhibited by Boscain and Piccoli in [30]. In this context, we refer to $\mathcal{M}_{\text{end}}$ as a *switching manifold*.

Now suppose that $\varepsilon > 0$ is given and set $\Theta_\varepsilon := (\Theta + \varepsilon\mathbb{B}) \cap \mathcal{K}$ an open neighborhood of Θ relative to $\mathcal{K}$. The main issue of this chapter is to investigate the possibility of constructing a set $\mathcal{K}_\varepsilon \subseteq \mathcal{K}$ that contains $\mathcal{M}_{\text{ini}}$ and function $U^\varepsilon : \mathcal{K} \setminus \Theta_\varepsilon \to \mathcal{U}$ such that it is a locally Lipschitz continuous in $\mathcal{K}_\varepsilon$ and

$$\tau_\varepsilon(x) := \min\{T \geq 0 \mid \; y_x^\varepsilon(T) \in \Theta_\varepsilon\} \leq T^\Theta(x) + \varepsilon, \quad \forall x \in \mathcal{M}_{\text{ini}} \setminus \Theta_\varepsilon,$$

where y_x^ε is the unique curve that solves (8.2) with $U = U^\varepsilon$ which remains in $\mathcal{K}_\varepsilon \setminus \Theta_\varepsilon$ and verifies $y_x^\varepsilon(0) = x$. In Figure 8.2 we show an illustration in order to give an idea of what is expected to happen; the optimal curve (drawn in black) hits Θ whereas the suboptimal one (the blue arc) does not reach the target but a neighborhood of it.

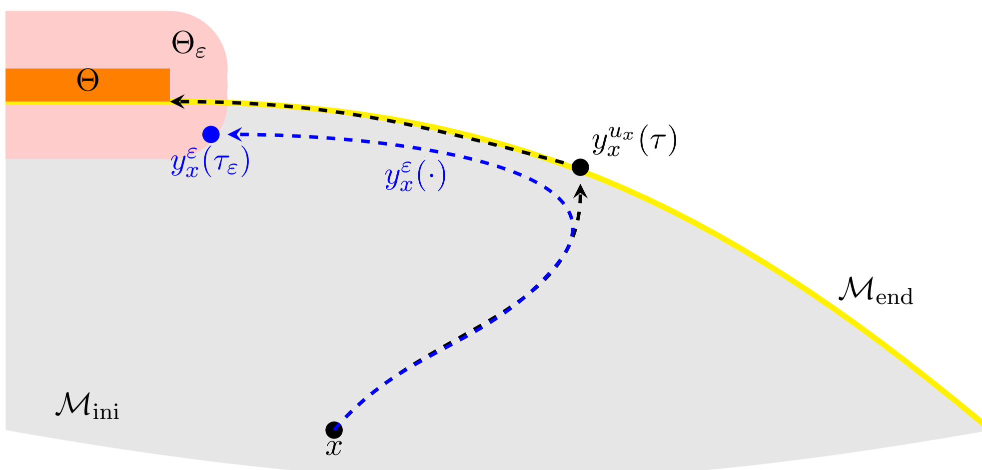


Figure 8.2: An illustration of the construction of a suboptimal feedback

The basic tool we use in our analysis is the Value Function itself, which we assume fulfills the following conditions:

$$(H_0^8) \qquad \begin{cases} i) \;\; \exists\ \mathcal{Q} \subseteq \mathbb{R}^N \text{ open with } \mathcal{M}_{\text{ini}} \cup \mathcal{M}_{\text{end}} \subseteq \mathcal{Q}. \\ ii) \; \exists\ \omega : \mathcal{Q} \to \mathbb{R} \text{ of class } \mathcal{C}^2 \text{ so that } \omega|_{\mathcal{M}_{\text{ini}}} = T^\Theta|_{\mathcal{M}_{\text{ini}}} \text{ on } \mathcal{M}_{\text{ini}}. \end{cases}$$

Moreover, since we are interested in the circumstances when the flows from $\mathcal{M}_{\text{ini}}$ are transversal to $\mathcal{M}_{\text{end}}$ we suppose in addition that if $U_0(\cdot)$ is an optimal synthesis, then

$$(H_1^8) \qquad \begin{cases} \exists U_{\text{ini}} : \overline{\mathcal{M}_{\text{ini}}} \to \mathcal{U} \text{ locally Lipschitz continuous so that} \\ U_{\text{ini}} = U_0|_{\mathcal{M}_{\text{ini}}} \text{ on } \mathcal{M}_{\text{ini}} \text{ and } f(x, U_{\text{ini}}(x)) \notin \mathcal{T}_{\mathcal{M}_{\text{end}}}(x), \ \forall x \in \mathcal{M}_{\text{end}}. \end{cases}$$

In the foregoing hypothesis, the existence of an extension of the feedback is immediately verified if the feedback is uniformly continuous on $\mathcal{M}_{\text{ini}}$. This is obtained by density arguments.

8.2.1 An example: the soft landing problem

Before presenting the theoretical development, we exhibit first an explicit example to enlighten the technique to be used in the rest of the chapter. We evoke from Example 1.2.1 the minimum time process of the soft landing problem:

$$\min T \quad \text{s.t. } \dot{y} = \begin{pmatrix} y_2 \\ u \end{pmatrix}, \quad u(t) \in \mathcal{U} = [-1, 1] \quad \text{a.e. on } [t, T], \quad y(0) = x, \quad y(T) = (0, 0).$$

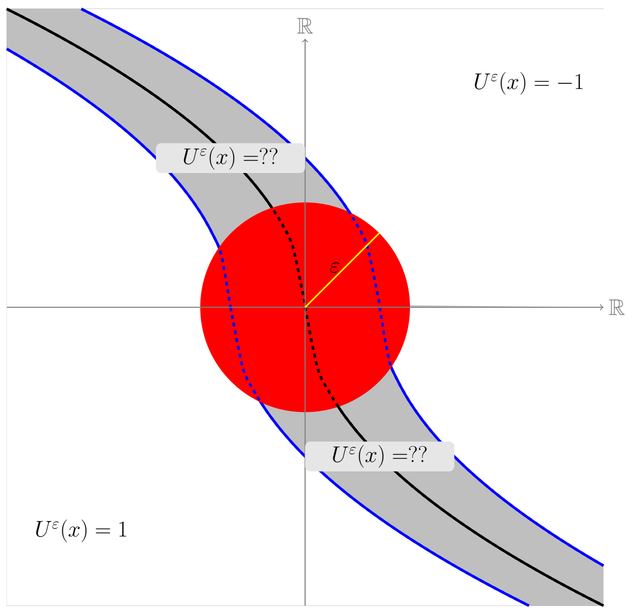


Figure 8.3: Soft landing example

In this example, the target is $\Theta = \{(0, 0)\}$ and the switching manifolds are contained in the curve $\{2x_1 + \text{sign}(x_2)x_2^2 = 0\}$. In Figure 8.3, this set is represented by the black curve and the red ball is the ε-neighbourhood of the origin we want to reach. Note that outside the gray zone the optimal policy is already locally Lipschitz continuous, so a nearly time-optimal continuous feedback U^δ only needs to differ from the optimal one in the gray zone.

We recall that in this situation, the minimum time function to reach the origin can be computed explicitly and it is given by

$$T^\Theta(x) = \begin{cases} -x_2 + \sqrt{2x_2^2 - 4x_1} & 2x_1 + \text{sign}(x_2)x_2^2 < 0, \\ x_2 + \sqrt{2x_2^2 + 4x_1} & 2x_1 + \text{sign}(x_2)x_2^2 > 0, \\ |x_2| & 2x_1 + \text{sign}(x_2)x_2^2 = 0. \end{cases}$$

Let us focus on the construction around the manifolds

$$\begin{aligned} \mathcal{M}_{\text{ini}} &= \{x \in \mathbb{R}^2 \mid \ 2x_1 + \text{sign}(x_2)x_2^2 > 0\} \\ \mathcal{M}_{\text{end}} &= \{x \in (0, +\infty) \times (-\infty, 0) \mid \ h(x) := 2x_1 - x_2^2 = 0\} \end{aligned}$$

We check that the (H_0^8) is verified with $\mathcal{Q} = \{2x_1 + x_2^2 > 0\}$ and $\omega(x) = x_2 + \sqrt{2x_2^2 + 4x_1}$.

Let $\varepsilon > 0$ given and take $\delta \in (0, \varepsilon)$ to be fixed. Consider the curve

$$\mathcal{M}_{\text{end}}^{\delta} = \{x \in (0, +\infty) \times (-\infty, 0) \mid \ h(x) = 2\delta\}.$$

The region of interest, where the optimal control is going to be modified is depicted in Figure 8.4. It tallies with the area between the curves $\mathcal{M}_{\text{end}}$ and $\mathcal{M}_{\text{end}}^{\delta}$. Outside of this zone, there is no real need to alter it, because, as aforementioned, the feedback is continuous outside of the switching curve. Therefore, we can set $\mathcal{K}_{\varepsilon} = \{x \in \mathbb{R}^2 \mid \ 2x_1 + \text{sign}(x_2)x_2^2 \geq 0\}$

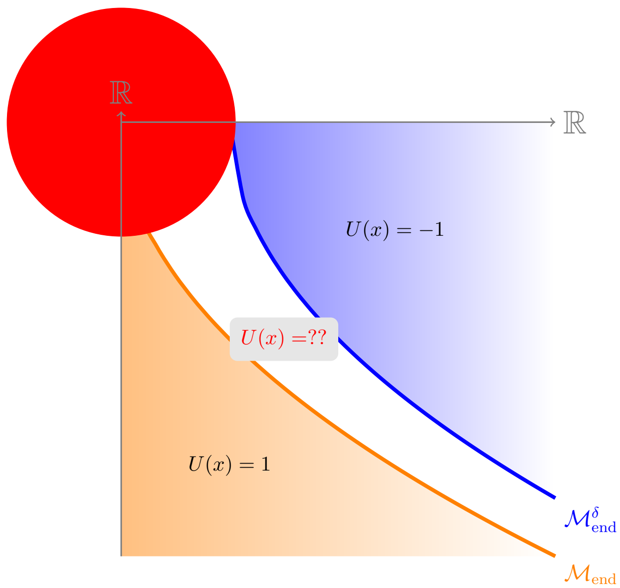


Figure 8.4: Zone of interest

Let Ω_δ be the zone where it is desired to modify the feedback, that is,

$$\Omega_\delta = \{x \in \mathcal{O} : \ 0 \leq h(x) \leq 2\delta\}, \quad \text{where } \mathcal{O} := \mathbb{R} \times (-\infty, 0).$$

We consider as well the locally Lipschitz continuous function $\lambda : \Omega_\delta \to [0, 1]$ defined via

$$\lambda(x) = \left(1 - \frac{1}{2\delta} h(x)\right), \quad x \in \Omega_\delta.$$

Notice that $\lambda(x) = 1$ if and only if $x \in \mathcal{M}_{\text{end}}$. Hence the prototype suboptimal strategy is

$$U^\delta(x) = \begin{cases} 1 & x \in \mathcal{O}, \ h(x) = 0, \\ -1 + 2\lambda(x) & x \in \text{int}(\Omega_\delta), \\ -1 & \text{otherwise}, \end{cases} \quad , \quad x \in \mathcal{K}_\varepsilon \setminus \mathbb{B}(0, \varepsilon). \tag{8.4}$$

Clearly, U^δ is continuous and therefore the next ordinary differential equation always admits solutions in the classical sense for any initial condition on $\mathcal{K}_\varepsilon \setminus \mathbb{B}(0, \varepsilon)$:

$$\begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \end{pmatrix} = \begin{pmatrix} y_2 \\ U^\delta(y) \end{pmatrix}. \tag{8.5}$$

Let y be a solution to (8.5) lying on Ω_δ with initial condition $x \in \operatorname{int}(\Omega_\delta)$. Let $[0,\tau)$ be the maximal interval of time for which y belongs to $\operatorname{int}(\Omega_\delta)$, that is

$$\tau = \inf\{t > 0 \mid \; y(t) \in \operatorname{int}(\Omega_\delta)\}.$$

Define $\rho(t) := h(y(t))$ for any $t \in [0,\tau)$ and note that this function is differentiable on $(0,\tau)$. Whereupon, setting $u = U^\delta(y)$ on $(0,\tau)$ we get:

$$\dot{\rho}(t) = 2y_2(t)(1-u(t)), \quad \forall t \in (0,\tau).$$

Therefore, as $y_2(t) < 0$ for any $t \in (0,\tau)$, the function $\rho(\cdot)$ is strictly decreasing. Using an argument of density, this affirmation can be extended to any curve that starts from $\mathcal{M}^\delta_{\text{end}}$.

Remark 8.2.2. *In the light of the information at hand, we claim that $\tau \in \mathbb{R}$. Actually, if it is not the case, we can assume that there is $\alpha \in (0,1)$ so that $u(t) \leq 1-\alpha$ for any $t \geq 0$. Otherwise, since $\rho(\cdot)$ is decreasing we would have that $\dot{y}_2(t) = u(t) > 1-\alpha$, which implies that τ is finite.*

We might also assume that $y_2(t) \leq -\alpha$ for any $t > 0$, and therefore $\dot{\rho}(t) \leq -2\alpha^2$. This inequality yields to a contradiction because for some $t > 0$, $\rho(t) = 0$ but $u(t) < 1$.

A simple computation shows that, since ρ is strictly decreasing on $(0,\tau)$, if $\tau_\varepsilon(x)$ stands for the time required to hit the target Θ_ε starting from x, the following estimate holds true

$$y_2(t) \leq -\alpha_\varepsilon(\delta), \quad \forall x \in \Omega_\delta,\; t \in [0,\tau_\varepsilon(x)],$$

where $\alpha_\varepsilon(\delta) := \sqrt{2}\sqrt{\sqrt{1+\varepsilon^2+2\delta}-(1+\delta)}$; the bound is obtained by finding the intersection point between $\mathcal{M}^\delta_{\text{end}}$ and the circle of radius ε.

Note also that $\tau > \tau_\varepsilon(x)$ is due to $\delta < \varepsilon$. Accordingly, thanks to Remark 8.2.2, $\tau_\varepsilon(x)$ is a finite number likewise τ. Furthermore, it is not difficult to see that

$$\dot{\rho}(t) = \frac{2}{\delta}y_2(t)\rho(t) = \frac{2}{\delta}\dot{y}_1(t)\rho(t), \quad \forall t \in (0,\tau),$$

which implies that

$$\rho(t) = h(x)\exp\left(\frac{2}{\delta}(y_1(t)-x_1)\right) \quad \forall t \in [0,\tau).$$

In particular, $\rho(t) > 0$ and $x_2(t) > 0$ for any $t \in [0,\tau]$, and so $y_2(\tau) = 0$. Indeed, any trajectory of the modified feedback that begins at $x \in \Omega_\delta$, belongs to the manifold $\mathcal{M}_x$ that has been portrayed in Figure 8.5 and whose analytic expression is

$$\mathcal{M}_x = \left\{\tilde{x} \in \mathcal{O} \mid \; h(\tilde{x}) = h(x)\exp\left(\frac{2}{\delta}(\tilde{x}_1 - x_1)\right)\right\},\; x \in \Omega_\delta.$$

On the other hand, on the interval $(0,\tau_\varepsilon(x))$ the next inequality holds:

$$\dot{\rho}(t) \leq -\frac{2\alpha_\varepsilon(\delta)}{\delta}\rho(t), \quad \forall t \in (0,\tau_\varepsilon).$$

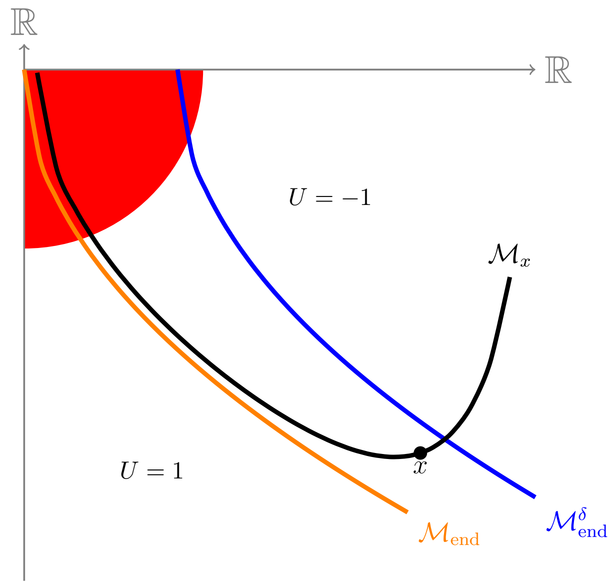


Figure 8.5: Manifold associated with the perturbed feedback.

So, $\rho(t) \leq h(x) \exp\left(-\frac{2\alpha_\varepsilon(\delta)}{\delta}t\right)$ for any $t \in [0, \tau_\varepsilon(x)]$. Whereupon,

$$\dot{y}_2(t) = 1 - \frac{1}{\delta}\rho(t) \geq 1 - \frac{h(x)}{\delta}\exp\left(-\frac{2\alpha_\varepsilon(\delta)}{\delta}t\right).$$

This yields, integrating the inequality between $t = 0$ and $t = \tau_\varepsilon(x)$, to

$$y_2(t) - x_2 \geq \tau_\varepsilon(x) - \frac{h(x)}{\delta}\int_0^{\tau_\varepsilon(x)} \exp\left(-\frac{2\alpha_\varepsilon(\delta)}{\delta}t\right) dt \geq \tau_\varepsilon(x) - \frac{h(x)}{2\alpha_\varepsilon(\delta)}.$$

Remark that $T^\Theta(x) = x_2 + \sqrt{4x_2^2 + 2h(x)}$, and so,

$$y_2(t) - x_2 \leq -\alpha_\varepsilon(\delta) + T^\Theta(x), \quad t \in [0, \tau_\varepsilon(x)].$$

Consequently, we have found out that

$$\frac{h(x)}{2\alpha_\varepsilon(\delta)} - \alpha_\varepsilon(\delta) + T^\Theta(x) \geq \tau_\varepsilon(x), \quad \forall x \in \Omega_\delta.$$

Besides, due to $h(x) \leq 2\delta$ we get

$$\frac{h(x)}{2\alpha_\varepsilon(\delta)} - \alpha_\varepsilon(\delta) \leq \frac{\delta - \alpha_\varepsilon(\delta)^2}{\alpha_\varepsilon(\delta)} \leq \frac{3\delta}{\sqrt{2}\sqrt{\sqrt{1+\varepsilon^2+2\delta} - (1+\delta)}}.$$

We readily see that the righthand side can be as close of zero as wanted, so we can find $\delta > 0$ small enough which makes the bound in the preceding inequality not greater than ε. In particular, we obtain the ensuing result.

Proposition 8.2.1. *For any $\varepsilon > 0$, there exists $\delta_0 \in (0, \varepsilon)$ which makes, for any $\delta \in (0, \delta_0]$, the feedback U^δ given by (8.4) suboptimal on $\mathcal{K}_\varepsilon = \{x \in \mathbb{R}^2 \mid 2x_1 + sign(x_2)x_2^2 \geq 0\}$, that is,*

$$T^\Theta(x) \leq \tau_\varepsilon(x) \leq T^\Theta(x) + \varepsilon, \quad \forall x \in \mathcal{K}_\varepsilon \setminus \mathbb{B}(0, \varepsilon).$$

A numerical test

From the analysis recently exposed we can see that, given $\varepsilon > 0$ if we pick $\delta_0 > 0$ to be a zero of the function $\chi_\varepsilon : [0, \varepsilon] \to [0, +\infty)$ defined via

$$\chi_\varepsilon(\delta) = 9\delta^2 - 2\varepsilon^2 \left(\sqrt{1 + \varepsilon^2 + 2\delta} - (1 + \delta) \right), \quad \forall \delta \in [0, \varepsilon],$$

then the feedback U^δ is suboptimal. In Figure 8.6 we have represented in blue the curve of zeros of the function $\varepsilon \mapsto \chi_\varepsilon$ for $\varepsilon \in [0, 1]$. We empirically observe that this function is of order $o(\varepsilon^2)$; in the same figure, the red curve portrays the function $\delta = \frac{1}{4}\varepsilon^2$.

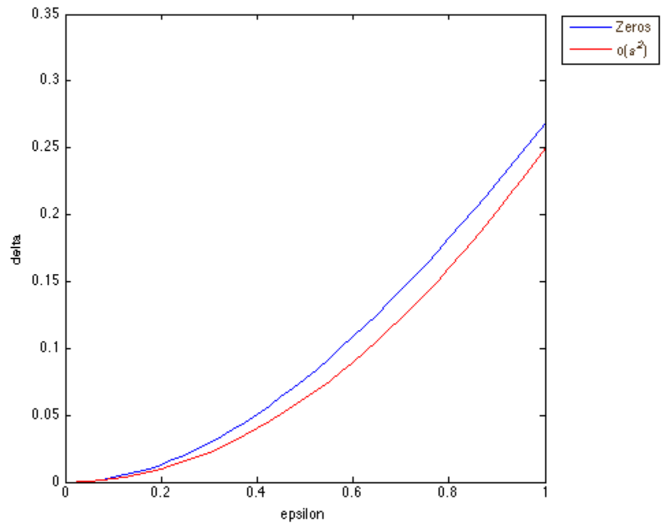


Figure 8.6: The zeros of the χ_ε.

Using the above-described fashion to choose δ_0 we have tested the feedback for the values $\varepsilon \in \{0.1, 0.05, 0.01\}$ from 100 random initial conditions lying on

$$\{x \in [0, 55] \times [-10, 0] \mid h(x) \geq 0\}.$$

Using the solver ode45 in Matlab we have obtained the following results for $T^\Theta(x) - \tau_\varepsilon(x)$

$T^\Theta(x) - \tau_\varepsilon(x)$	ε=0.1 δ_0=0.027	ε=0.05 δ_0=0.0016	ε=0.01 δ_0=0.0013
worst case	0.0960	0.0463	0.0070
best case	0.0999	0.0495	0.0099
average	0.0978	0.0476	0.0082

The last table provides an empirical support to the procedure we have exposed. Indeed, in any case, we have a much stronger result, that is, $T^\Theta(x) \geq \tau_\varepsilon(x)$. This fact can be explained by noticing that the optimal time to reach the target, from the circle of radius ε is of order ε as well. We also mention that, as it can also be inferred from the exposition, the choice of δ_0 is not at all sharp, which makes suitable to continue looking for better bounds related to the closed-loop control.

Further extensions

Instead of considering the particular choice of function $\lambda(x) = \left(1 - \frac{1}{2\delta}h(x)\right)$ we might consider any other the continuous functions verifying $\lambda : \Omega_\delta \to [0,1]$ and

$$\lambda|_{\mathcal{M}_{\text{end}}} \equiv 1 \quad \text{and} \quad \lambda|_{\mathcal{M}_{\text{ini}}}(x) = 0, \quad \text{as long as } h(x) = \delta. \tag{8.6}$$

Under these circumstances, we are able to prove the existence of a $\delta > 0$ which makes the strategy given by (8.4) suboptimal on compacts sets of $\mathcal{K}_\varepsilon$.

Proposition 8.2.2. *For any $\varepsilon > 0$ and $r > 0$, there exists $\delta_0 \in (0, \varepsilon)$ such that for any continuous functions $\lambda : \Omega_\delta \to [0,1]$ that verifies (8.6) with $\delta \in (0, \delta_0)$, the feedback U^δ given by (8.4) is suboptimal on*

$$\mathcal{K}_\varepsilon^r := \{x \in \mathbb{R}^2 \mid 2x_1 + sign(x_2)x_2^2 \geq 0\} \cap \overline{\mathbb{B}}(0, r).$$

Proof. Notice that since $\rho(t) := h(y_x^\varepsilon(t)) > 0$ for any $t \in (0, \tau)$, then T^Θ is differentiable along the arc $t \mapsto y(t) := y_x^\varepsilon(t)$ and so, for any $t \in (0, \tau)$

$$\begin{aligned}
\frac{d}{dt}T^\Theta(y(t)) &= -1 + 2\lambda(y_x^\varepsilon(t)) \left(1 + \frac{2y_2(t)}{\sqrt{4y_2^2(t) + 2\rho(t)}}\right) \\
&\leq -1 + 2\lambda(y_x^\varepsilon(t)) \left(\frac{\rho(t)}{4y_2^2(t) + 2\rho(t)}\right) \\
&\leq -1 + \frac{\delta}{\alpha_\varepsilon(\delta)^2}
\end{aligned}$$

Thereby, integrating between $t = 0$ and $t = \tau_\varepsilon(x)$ we get

$$-T^\Theta(x) \leq T^\Theta(y(\tau_\varepsilon(x))) - T^\Theta(x) \leq -\left(1 - \frac{\delta}{\alpha_\varepsilon(\delta)^2}\right)\tau_\varepsilon(x). \tag{8.7}$$

Since $x \mapsto T^\Theta(x)$ is continuous on $\mathcal{K}_\varepsilon^r$, the foregoing inequality implies that $\tau_\varepsilon(x)$ is uniformly bounded from above on $\mathcal{K}_\varepsilon^r$. Let t_r be its minimal upper bound and take $\delta_0 \in (0, \varepsilon)$ so that

$$\frac{\delta_0}{\alpha_\varepsilon(\delta_0)^2} = \frac{\delta_0}{2(\sqrt{1 + \varepsilon^2 + 2\delta_0} - (1 + \delta_0))} \leq \frac{\varepsilon}{t_r}.$$

The choice of δ_0 is possible inasmuch as $\delta \mapsto \frac{\delta}{\alpha_\varepsilon(\delta)^2}$ ranges between 0 and $+\infty$ on $(0, \varepsilon)$. Finally, we get the desired result from (8.7). $\square$

8.3 Control-affine systems

The motivation of the preceding section was to illustrate an explicit example in which the construction of a continuous nearly time-optimal feedback was plausible. In this section we look for akin constructions for a broader class of problems. We reckon with a control-affine system that verifies (H_f^8):

$$\dot{y} = f_0(y) + \sum_{i=1}^{m} u_i f_i(y).$$

Under these circumstances we assume that $\mathcal{M}_{\text{ini}}$ is an open set and $\mathcal{M}_{\text{end}}$ is a smooth surface of codimension 1 (at least of class $\mathcal{C}^2$). Since, the analysis we propose is merely local (on bounded sets), we can always find a local defining map for $\mathcal{M}_{\text{end}}$ whose domain is a neighborhood of $\mathcal{M}_{\text{end}} \setminus \Theta_\varepsilon$; this can be achieved by using a partition of the unity. Accordingly, for sake of simplicity we may rather assume that there is a continuous function $\rho : \mathcal{M}_{\text{end}} \to (0, +\infty)$ that makes $\mathcal{O} \subseteq \mathbb{R}^N$ a tubular neighborhood of $\mathcal{M}_{\text{end}}$; we Section 3.2.3 for further details. Therefore, the map $\pi_{\mathcal{M}_{\text{end}}} : \mathcal{O} \to \mathcal{M}_{\text{end}}$, the projection over $\mathcal{M}_{\text{end}}$ is well defined on $\mathcal{K}$ and locally Lipschitz continuous. In addition, we also suppose that we can find $h : \mathbb{R}^N \to \mathbb{R}$ continuous which is a $\mathcal{C}^k$ submersion on $\mathcal{O}$ so that

$$\mathcal{M}_{\text{end}} = \{x \in \mathcal{O} \mid h(x) = 0\} \text{ and } \mathcal{M}_{\text{ini}} \cap \mathcal{O} \subseteq \{x \in \mathcal{O} \mid h(x) > 0\}. \tag{8.8}$$

With a slight abuse of notation, let us write $\partial\mathcal{M}_{\text{end}}$ for $\overline{\mathcal{M}_{\text{end}}} \setminus \mathcal{M}_{\text{end}}$, and for any $r > 0$ and $\delta > 0$ we set

$$\Sigma^{r,\sigma} = \{x \in \mathcal{O} \mid |x| \leq r,\ \text{dist}_{\partial\mathcal{M}_{\text{end}}}(\pi_{\mathcal{M}_{\text{end}}}(x)) \geq \sigma\}\,.$$

These subsets of $\mathcal{M}_{\text{end}}$ are introduced in order to localize the area where the feedback is going to be modified. This plays the same role as the ball of radius r used in Section 8.2.1 but well-suited for the case $\mathcal{M}_{\text{end}}$ is bounded.

Let U_{ini} be the extension of $U_0|_{\mathcal{M}_{\text{ini}}}$ up to $\overline{\mathcal{M}_{\text{ini}}}$ given by (H_1^8) and consider as well

$$U_{\text{end}}(x) = U_0|_{\mathcal{M}_{\text{end}}}(\pi_{\mathcal{M}_{\text{end}}}(x)), \quad \forall x \in \mathcal{O}.$$

The main result of this section is described below.

Theorem 8.3.1. *Assume (H_f^8), (H_0^8) and (H_1^8) hold along with*

$$x \in \partial\mathcal{M}_{end} \;\Rightarrow \textit{Either } x \in \Theta \textit{ or } \exists \mu > 0\ f(x, U_{ini}(x)) = \mu f(x, U_{end}(x)). \tag{8.9}$$

Let $\varepsilon > 0$, $r > 0$, then, there exists $\delta_0 > 0$ such that for any $\delta \in (0, \delta_0)$ and any function $\lambda : h^{-1}([0, \delta]) \to [0, 1]$ locally Lipschitz continuous that satisfies $\lambda(x) = 0$ if $h(x) = \delta$ and $\lambda(x) = 1$ if $h(x) = 0$, the feedback control $U^\delta : \mathcal{M}_{ini} \cup \mathcal{M}_{end} \to \mathcal{U}$ defined as

$$U^\delta(x) = \begin{cases} U_{ini}(x) & h(x) \geq \delta \\ U_{ini}(x) + \lambda(x)(U_{end}(x) - U_{ini}(x)) & 0 < h(x) < \delta\,, \\ U_{end}(x) & h(x) = 0 \end{cases} \qquad \forall x \in \mathcal{M}_{ini} \cup \mathcal{M}_{end},$$

is continuous on an arbitrary large neighborhood of $(\mathcal{M}_{ini} \cup \mathcal{M}_{end}) \cap \mathbb{B}(0, r) \setminus (\partial\mathcal{M}_{end} + \varepsilon\mathbb{B})$ and suboptimal on $(\mathcal{M}_{ini} \cup \mathcal{M}_{end}) \cap \mathbb{B}(0, r)$.

8.3.1 Technical lemmas

The proof of the foregoing theorem is the outcome of several lemmas which we proceed to state from this point on. We set

$$\Omega_\delta^{r,\sigma} := h^{-1}([0, \delta]) \cap \Sigma^{r,\sigma}, \quad \forall r, \sigma, \delta > 0.$$

Lemma 8.3.1. *Let $r, \sigma, \varepsilon > 0$, if (H_f^8) and (H_1^8) are satisfied, then we can find $\delta, \beta > 0$ so that*

$$\langle \nabla h(x), f(x, U_{ini}(x)) \rangle \leq -\beta, \quad \forall x \in \Omega_\delta^{r,\sigma} \setminus \Theta_\varepsilon.$$

Proof. If the statement is not true, we can construct a sequence $x_n \in \Omega_\delta^{r,\sigma} \setminus \Theta_\varepsilon$ which converges to some $x \in \mathcal{M}_{\text{end}} \cap \Sigma^{r,\sigma} \setminus \Theta_\varepsilon$ that verifies

$$\langle \nabla h(x), f(x, U_{\text{ini}}(x)) \rangle \geq 0.$$

Let $\tilde{y}$ stand for the arc associated with U_{ini} which starts from some $\tilde{x} \in \mathcal{M}_{\text{ini}}$ and reaches x at time $\tau(\tilde{x}) = \inf\{t > 0 \mid \tilde{y}(t) \in \mathcal{M}_{\text{end}}\}$; $\tilde{y}$ is the backward curve emerging from x and by virtue of (8.3), it is well-defined. The Mean Value Theorem implies that $\exists t \in [0, \tau(\tilde{x})]$ for which:

$$0 > -\frac{h(\tilde{x})}{\tau(\tilde{x})} = \langle \nabla h(\tilde{y}(t)), f(\tilde{y}(t), U_{\text{ini}}(\tilde{y}(t))) \rangle.$$

The lefthand side is strictly negative and remains bounded as long as $\tilde{x} \to x$, this is because of the Gronwall's Lemma and the Mean Value Theorem imply

$$\begin{aligned} h(\tilde{x}) \leq & \sup_{s \in [0,1]} |\nabla h(x + s(\tilde{x} - x))| \int_0^{\tau(\tilde{x})} |\dot{\tilde{y}}(s)| ds \\ \leq & (1 + |\tilde{x}|)(e^{c_f \tau(\tilde{x})} - 1) \sup_{s \in [0,1]} |\nabla h(x + s(\tilde{x} - x))|. \end{aligned}$$

Hence, $\limsup_{\tilde{x} \to x} \frac{h(\tilde{x})}{\tau(\tilde{x})} \leq c_f(1 + |x|)|\nabla h(x)|$. In view of the initial supposition, the former inequality yields to

$$\langle \nabla h(x), f(x, U_{\text{ini}}(x)) \rangle = 0.$$

However, this final equation leads to a contradiction with (H_1^8). So, the conclusion follows. □

Lemma 8.3.2. *For any $r > 0$ and $\sigma > 0$, there are $\Delta > 0$ and $\varrho > 0$ so that*

$$dist_{\mathcal{M}_{end}}(x) \leq \Delta |h(x)|, \quad \forall x \in \Sigma^{r,\sigma} \cap (\mathcal{M}_{end} + \varrho \mathbb{B}).$$

Furthermore, $\Delta \inf\{|\nabla h(x)| \mid \Sigma^{r,\sigma} \cap \mathcal{M}_{end}\} \geq 1$.

Proof. By virtue of the Grave-Lyusternik Theorem (c.f. [40, Theorem 5.32]), for any $x \in \mathcal{M}_{\text{end}}$ there exist $\Delta_x > 0$ and $\varrho_x \in (0, \rho(x))$ so that

$$\text{dist}_{\mathcal{M}_{\text{end}}}(\tilde{x}) \leq \Delta_x |h(\tilde{x})|, \quad \forall \tilde{x} \in \mathbb{B}(x, \varrho_x).$$

Evaluating at $\tilde{x} = x + t\nabla h(x)$ with $t > 0$ we get

$$t|\nabla h(x)| \leq \Delta_x |h(x + t\nabla h(x)) - h(x)|.$$

Whereupon, dividing by t and letting $t \to 0$ we obtain that $|\nabla h(x)|\Delta_x \geq 1$.

Since $\Sigma^{r,\sigma} \cap \mathcal{M}_{\text{end}}$ is compact and can be covered by $\{\mathbb{B}(x, \varrho_x)\}_{x \in \mathcal{M}_{\text{end}}}$, we can take $x_1, \ldots, x_p \in \mathcal{M}_{\text{end}}$ so that $\{\mathbb{B}(x_i, \varrho_{x_i})\}_{i=1}^p$ covers $\Sigma^{r,\sigma} \cap \mathcal{M}_{\text{end}}$. Consequently, setting $\varrho = \min_{i=1\ldots,p} \varrho_{x_i}$ and $\Delta = \max_{i=1\ldots,p} \Delta_{x_i}$ we get the conclusion. □

Lemma 8.3.3. *Suppose that* (H_f^8), (H_0^8) *and* (H_1^8) *are verified, and let* $r, \sigma > 0$. *Then, there exist* $C > 0$ *and* $\varrho > 0$ *(the same as in Lemma 8.3.2) so that*

$$|\langle \nabla T^\Theta(x), f(x, U_{end}(x)) - f(x, U_{ini}(x))\rangle| \leq C|h(x)|,\ \forall x \in \Sigma^{r,\sigma} \cap (\mathcal{M}_{end} + \varrho\mathbb{B}).$$

Proof. Let ω be given by (H_0^8). Recall that the minimum time function is a classical solution of the Hamilton-Jacobi-Bellman equation on $\mathcal{M}_{\text{ini}}$, and so

$$-1 + H(x, \nabla\omega(x)) = 0, \quad x \in \mathcal{M}_{\text{ini}}$$

Due to the optimality of U_{ini} on $\mathcal{M}_{\text{ini}}$ we have, for any $x \in \mathcal{M}_{\text{ini}} \cap \mathcal{O}$

$$\langle \nabla\omega(x), f(x, U_{\text{ini}}(x))\rangle = -H(x, \nabla\omega(x)) \leq \langle \nabla\omega(x), f(x, U_{\text{end}}(x))\rangle.$$

Thus, by density we find out that

$$\alpha(x) := \langle \nabla\omega(x), f(x, U_{\text{end}}(x)) - f(x, U_{\text{ini}}(x))\rangle \geq 0, \quad \forall x \in \mathcal{M}_{\text{end}}.$$

On the other hand, by (8.3) we have that

$$T^\Theta(x) = T^\Theta_{\overline{\mathcal{M}_{\text{ini}}}}(x), \quad x \in \mathcal{M}_{\text{ini}} \cup \mathcal{M}_{\text{end}},$$

where $T^\Theta_{\overline{\mathcal{M}_{\text{ini}}}}$ is the minimum time function to reach the target Θ while being feasible on $\overline{\mathcal{M}_{\text{ini}}}$. Using the standard theory of Hamilton-Jacobi-Bellman with state-constraints, we can easily see that $T^\Theta_{\overline{\mathcal{M}_{\text{ini}}}}$ is a supersolution of the equation

$$-1 + H(x, \nabla\varphi(x)) = 0, \quad \forall x \in \overline{\mathcal{M}_{\text{ini}}}.$$

In particular, by the optimality of the feedback U_{end} and due to ω is an admissible test function ($\omega \equiv T^\Theta$ on $\mathcal{M}_{\text{ini}} \cup \mathcal{M}_{\text{end}}$) we have

$$-1 - \langle \nabla\omega(x), f(x, U_{\text{end}}(x))\rangle = -1 + H(x, \nabla\omega(x)) \geq 0, \quad \forall x \in \mathcal{M}_{\text{end}}.$$

By the same argument used earlier, we can show that $\alpha(x) \leq 0$ for any $x \in \mathcal{M}_{\text{end}}$. Hence, we find out that

$$\alpha(x) = 0, \quad \forall x \in \mathcal{M}_{\text{end}}.$$

Therefore, if $L > 0$ indicates the Lipschitz modulus of α on $\Sigma^{r,\sigma}$ we have that

$$|\alpha(x)| \leq L\text{dist}_{\mathcal{M}_{\text{end}}}(x), \quad \forall x \in \Sigma^{r,\sigma}.$$

By Lemma 8.3.2, the conclusion follows easily. □

8.3.2 Proof of the principal theorem

We are now in position to proof the main statement of the chapter.

Proof of Theorem 8.3.1. For sake of clarity, we split the proof in several steps. Let $\sigma = \frac{\varepsilon}{2}$ and $\tilde{r} \geq r$, consider $\tilde{\delta} > 0$ and $\varrho > 0$ given by Lemma 8.3.1 and 8.3.3 associated with ε, σ and $\tilde{r}$, respectively. Let $\rho_0 \in (0, \varrho)$ be a lower bound for $\rho(\cdot)$ on $\Sigma^{\tilde{r},\sigma}$ and $\Delta > 0$ given by Lemma 8.3.2. We set $\delta_0 = \min\{\tilde{\delta}, \frac{\varepsilon}{2\Delta}, \frac{\rho_0}{\Delta}\}$ and take $\delta \in (0, \delta_0)$ fixed but arbitrary.

Continuity of U^δ: First of all notice that by construction, the feedback law is locally Lipschitz continuous on $\Omega_\delta^{\tilde{r},\sigma}$ for any $\tilde{r} > 0$. Moreover, due to $\rho_0 \geq \Delta\delta_0$, we have that for any $x \in \mathcal{M}_{\text{end}}$ we can find $\sigma_x \in (0, \rho_0)$ so that $h(x + \sigma_x \nabla h(x)) = \delta$. By the Implicit Function Theorem we can also see that the function $x \mapsto \sigma_x$ is continuously differentiable on $\mathcal{M}_{\text{end}}$. Now, since $U^\delta(x) = U_{\text{ini}}(x)$ whenever $h(x) \geq \delta$ we have that $\sigma \mapsto U^\varepsilon(x + \sigma\nabla h(x))$ is continuous on $[0, \rho_0)$. Therefore, U^δ is continuous on $\Omega_\delta^{\tilde{r},\sigma} \cup h^{-1}([\delta, +\infty)) \cap \mathcal{M}_{\text{ini}}$. As a matter of fact, since U^δ is separately locally Lipschitz continuous in $\Omega_\delta^{\tilde{r},\sigma}$ and in $h^{-1}([\delta, +\infty)) \cap \mathcal{M}_{\text{ini}}$, it is necessarily locally Lipschitz continuous on the union of both sets.

Let $x \in \mathcal{M}_{\text{end}}$ with $\text{dist}_{\partial\mathcal{M}_{\text{end}}}(x) < \frac{\varepsilon}{2}$, then for any $s > 0$

$$\text{dist}_{\partial\mathcal{M}_{\text{end}}}(x + s\nabla h(x)) \leq \text{dist}_{\partial\mathcal{M}_{\text{end}}}(x) + \text{dist}_{\mathcal{M}_{\text{end}}}(x + s\nabla h(x)).$$

By Lemma 8.3.2 and the choice of δ_0, if $h(x + s\nabla h(x)) \leq \delta_0$ then we necessarily have that $\text{dist}_{\partial\mathcal{M}_{\text{end}}}(x + s\nabla h(x)) < \varepsilon$. In particular, since $\sigma = \frac{\varepsilon}{2}$ we obtain

$$(\mathcal{M}_{\text{ini}} \cup \mathcal{M}_{\text{end}}) \cap \mathbb{B}(0, r) \setminus (\partial\mathcal{M}_{\text{end}} + \varepsilon\mathbb{B}) \subseteq \Omega_\delta^{\tilde{r},\sigma} \cup \left[h^{-1}([\delta, +\infty)) \cap \mathcal{M}_{\text{ini}}\right].$$

Invariance of $\Omega_\delta^{\tilde{r},\sigma}$: Let $\beta > 0$ given by Lemma 8.3.1 and let y be the solution associated with the feedback U^δ given in the statement and whose initial condition is $x \in \text{int}\,\Omega_\delta^{r,\sigma}$. Let $\tau > 0$ be the escape time of y from $\Omega_\delta^{\tilde{r},\sigma}$. Thereby, setting $\rho := h \circ y$ we get for any $t \in (0, \tau)$

$$\dot{\rho}(t) = (1 - \lambda(y))\langle \nabla h(y), f(y, U_{\text{ini}}(y))\rangle + \lambda(y)\langle \nabla h(y), f(y, U_{\text{end}}(y))\rangle.$$

Recall that $\langle \nabla h(x), f(x, U_{\text{end}}(x))\rangle = 0$ on $\mathcal{M}_{\text{end}}$, so by (H_f^8) and Lemma 8.3.2 there exists a constant $\tilde{C} > 0$ so that

$$\langle \nabla h(x), f(x, U_{\text{end}}(x))\rangle \leq \tilde{C}|h(x)|, \quad \forall x \in \Sigma^{\tilde{r},\sigma} \cap (\mathcal{M}_{\text{end}} + \varrho\mathbb{B}).$$

Hence, by reducing δ_0 if necessary, we may assume that

$$\langle \nabla h(x), f(x, U_{\text{end}}(x))\rangle \leq \frac{\beta}{2}, \quad \forall x \in \Sigma^{\tilde{r},\sigma},$$

which leads to $\dot{\rho}(t) \leq -\frac{\beta}{2}$ on $(0, \tau)$. Furthermore, since the feedback is locally Lipschitz continuous on $\Omega_\delta^{\tilde{r},\sigma}$, $\rho(t) > 0$ for any $t \in (0, \tau)$; otherwise for some $x \in \mathcal{M}_{\text{end}}$ there are two backward solution, one reaching $\mathcal{M}_{\text{end}}$ and another remaining there. Consequently, by taking $\tilde{r}$ larger, we can assume that $\text{dist}_{\partial\mathcal{M}_{\text{end}}}(\pi_{\mathcal{M}_{\text{end}}}(y(\tau))) = \sigma$.

Reachability of the target: We claim that $y(\tau) \in \Theta_\varepsilon$. Indeed, let $z = \pi_{\mathcal{M}_{\text{end}}}(y(\tau))$ and suppose that $\text{dist}_\Theta(z) > \sigma = \frac{\varepsilon}{2}$, otherwise the affirmation does hold because, Lemma 8.3.2 leads to

$$\text{dist}_\Theta(y(\tau)) \leq \text{dist}_\Theta(z) + |z - y(\tau)| \leq \frac{\varepsilon}{2} + \Delta\delta < \varepsilon.$$

Remark that, since the optimal trajectory that starts from z reach the target without leaving $\mathcal{M}_{\text{end}}$, we have that

$$f(z, U_{\text{end}}(z)) \in \text{int}\left(\mathcal{T}^C_{\Sigma^{\tilde{r},\sigma}}(z)\right).$$

Accordingly, by reducing δ_0 once again if necessary and using the continuity of $f(\cdot, U_{\text{end}}(\cdot))$ on $\mathcal{O}$, we can assume that $f(y(\tau), U_{\text{end}}(y(\tau))) \in \text{int}(\mathcal{T}^C_{\Sigma^{\tilde{r},\sigma}}(y(\tau)))$ as well.

By (8.9), we can find $\bar{x} \in \partial\mathcal{M}_{\text{end}}$ and $\mu > 0$ so that $|\bar{x} - z| = \sigma$ with $f(\bar{x}, U_{\text{ini}}(\bar{x})) = \mu f(\bar{x}, U_{\text{end}}(\bar{x}))$. In particular, due to the continuity of the vector fields and to the fact that $f(y(\tau), U_{\text{end}}(y(\tau))) \in \text{int}(\mathcal{T}^C_{\Sigma^{\tilde{r},\sigma}}(y(\tau)))$ we can conclude that

$$f(y(\tau), U_{\text{ini}}(y(\tau))) \in \mathcal{T}^C_{\Sigma^{\tilde{r},\sigma}}(y(\tau)).$$

Therefore, by the control-affine structure of the dynamics, the convexity of the Clarke tangent cone and the Accessibility Lemma we get $f(y(\tau), U^\delta(y(\tau))) \in \text{int}(\mathcal{T}^C_{\Sigma^{\tilde{r},\sigma}}(y(\tau)))$ which is no possible because, since τ is a escaping time, we should have $-\dot{y}(\tau) \in \mathcal{T}^B_{\Sigma^{\tilde{r},\sigma}}(y(\tau)) = \mathcal{T}^C_{\Sigma^{\tilde{r},\sigma}}(y(\tau))$. Thus, in particular, $\tau > \tau_\varepsilon(x)$.

Moreover, by a density argument, since the dynamics is locally bounded, the same deduction is valid if the initial condition belongs to $\Omega^{r,\sigma}_\delta$.

Suboptimality of the feedback: Notice that T^Θ is differentiable along the arc $t \mapsto y(t)$ and so, in view of the control-affine structure of the dynamics, for any $t \in (0, \tau)$

$$\begin{aligned}
\frac{d}{dt} T^\Theta(y) &= \langle \nabla T^\Theta(y), f(y, U^\delta(y)) \rangle \\
&= \langle \nabla T^\Theta(y), f(y, U_{\text{ini}}(y)) \rangle \\
&+ \langle \nabla T^\Theta(y), f(y, U^\delta(y)) - f(y, U_{\text{ini}}(y)) \rangle \\
&= -1 + \lambda(y) \langle \nabla T^\Theta(y), f(y, U_{\text{end}}(y)) - f(y, U_{\text{ini}}(y)) \rangle \\
&\le -1 + 2C(\varepsilon, \tilde{r})\delta
\end{aligned}$$

The last inequality and $C(\varepsilon, \tilde{r})$ are due to Lemma 8.3.3. Additionally, by the same argument employed in Proposition 8.2.2, we can prove that $\tau_\varepsilon(x)$ is finite and bounded from above on any set $\Omega^{r,\sigma}_\delta$. Therefore, reducing δ_0 a last time if require, we might assume that $\tau_\varepsilon(x) C(\varepsilon, \tilde{r})\delta_0 \le \varepsilon$ so that

$$\tau_\varepsilon(x) \le T^\Theta(x) + \varepsilon, \quad \forall x \in \Omega^{r,\sigma}_\delta.$$

Finally, since outside $\Omega^{r,\sigma}_\delta$ the optimal control has not been changed, by (8.3) any trajectory starting at $x \in \mathcal{M}_{\text{ini}} \cup \mathcal{M}_{\text{end}} \setminus \Omega^{r,\sigma}_\delta$ reaches $\Omega^{r,\sigma}_\delta$ within finite time, $\tilde{\tau}_\varepsilon(x)$. Consequently, if y_x stands for the trajectory associated with the suboptimal feedback, we have

$$T^\Theta(x) = \tilde{\tau}_\varepsilon(x) + T^\Theta(y_x(\tilde{\tau}_\varepsilon(x))) \ge \tilde{\tau}_\varepsilon(x) + \tau_\varepsilon(y_x(\tilde{\tau}_\varepsilon(x))) - \varepsilon$$

So, since $\tilde{\tau}_\varepsilon(x) + \tau_\varepsilon(y_x(\tilde{\tau}_\varepsilon(x))) \ge \tau_\varepsilon(x)$ the conclusion follows. □

8.4 Discussion and perspectives

We finish the present chapter by discussing the contribution of the development exhibited and by indicating some possible extensions regarding the type of singularities that could be treated in future works.

Before going further, let us mention that in the literature there are papers dealing with the construction of *almost everywhere* continuous stabilizing feedbacks, that is, for the case in which there is no criterion to be minimized by the control system; we refer mainly to the works of Rifford [108, 109, 110].

8.4.1 Contributions of the chapter

In this chapter we have investigated the relation between optimal feedbacks with a stratified set of discontinuities and suboptimal continuous feedback. As reported in the introduction, this connection can be avoided if the optimal process at hand has no state-constraints involved in its formulation. However, for problems with restricted state-space, it seems to be a good strategy to proceed as we have done here. This is because, as we have drawn to attention in Chapter 4, or at least tried to do so, in many optimal control problems the boundary of the state-constraints is relevant and the pointing-like condition are not always satisfied.

Furthermore, we believe that the construction we have proposed is rather simple to be implemented once the optimal synthesis have been known. Furthermore, it yields automatically to full robustness around the area where the modification has taken place, which allows to eschew possible issues coming from inaccuracies in its implementation. For example, if the manifold $\mathcal{M}_{\text{end}}$ belongs to the boundary of the state-constraint, the suboptimal feedback we have given is such that none of its Carathéodory solutions will hit $\mathcal{M}_{\text{end}}$ but will remain close to it in order to reach finally a neighborhood of the target. Consequently, a discrete scheme with step-size sufficiently small will produce curves that track the suboptimal one and that stay in $\mathcal{M}_{\text{ini}}$. In contrast, if the optimal strategy is used directly, once close of the boundary, any discrete scheme will produce iterations that may lie outside the state-constraints, forcing the algorithm to project back over $\mathcal{K}$ and therefore producing the undesirable Zeno effect that could deteriorate the optimality of the curves associated with the discrete scheme.

In conclusion, the main contribution of this chapter is that we have pointed out that around some types of singularities the feedback can be modified in such a way it becomes considerably more regular than it was initially.

8.4.2 Further extensions

In the analysis we have exposed, it is important that the singularity of the feedback occurs at a switching manifold. However, it is not difficult to envisage other types of singularities that can be considered. For instance, if instead of reaching the manifold $\mathcal{M}_{\text{end}}$ we are allowed to leave at any point in a transversal way, then a similar analysis can be applied by using the backwards dynamics instead of the forwards.

We finally remark that in Theorem 8.3.1 the result was stated for an open set and a smooth surface of codimension 1, but a similar result can be stated if the dimension of both manifolds are smaller. Nevertheless, in that case, further hypotheses may be needed in order to make the suboptimal trajectories feasible on $\mathcal{K}$. This is because the following condition can not be automatically taken as granted:

$$f(x, U_{\text{ini}}(x) + \lambda(x)(U_{\text{end}}(x) - U_{\text{ini}}(x))) \in \mathcal{T}_{\mathcal{M}_{\text{ini}}}(x), \quad \forall x \in \mathcal{M}_{\text{ini}} \text{ near } \mathcal{M}_{\text{end}}.$$

PART IV

OPTIMAL CONTROL PROBLEMS ON NETWORKS

Abstract. In this part we deal with optimal control problems on networks. In this setting the dynamical system is no longer Lipschitz continuous and the dynamics may fail to have convex images. We show that the Value Function is still lower semicontinuous and that it can be identified as the unique solution to a Hamilton-Jacobi-Bellman equation which verifies suitable junction conditions. The formalism adopted in this part allows us to extend the results to generalized notions of networks where the junction is a manifold instead of a single point.

Resumé. Dans cette partie nous nous intéressons aux problèmes de commande optimale sur des réseaux. Dans ce cadre le système dynamique n'est plus Lipschitz continu et les images des dynamiques ne sont plus des ensembles convexes aux points de jonction. Nous démontrons que la Fonction Valeur est l'unique fonction semicontinue inferior qui résout l'equation de Hamilton-Jacobi-Bellman et qui satisfait des conditions de jonction adaptées à ce cadre. La façons comment nous abordons le cas classique de réseau nous permet d'étendre nos résultats à des notions généralisées de réseaux où la jonction est une variété lisse au lieu de un point isolé.

CHAPTER 9

Hamilton-Jacobi-Bellman Approach for Optimal Control on Networks

Abstract. In this chapter we study optimal control problems on networks without controllability assumption at the junctions. The Value Function associated with the control process is characterized as solution to a system of Hamilton-Jacobi-Bellman equations with appropriate junction conditions. The novel feature of the result lies in that the controllability conditions are not needed and the characterization remains valid even when the Value Function is not continuous. We present in addition an extension to the case in which the junction is not longer a single point but an embedded manifold of higher dimension.

9.1 Introduction

At present we are concerned with the Hamilton-Jacobi-Bellman (HJB) approach for control problems on networks. The latter are connected closed sets constituted by one-dimensional smooth curves or *branches* with some isolated intersections called *junctions*. As we have seen in Chapter 4, this is a special case of a more general setting of control problems where the admissible trajectories are constrained to stay in a stratified domain. We emphasis that the application shown in Section 4.2.4 was for the case in which the dynamics and cost are everywhere continuous. However, since the main motivation for control problems in networks comes from traffic flows, it is natural to impose different dynamics and costs on each branch of the network. Consequently, the resulting Hamiltonian is by nature discontinuous at the junction points, which poses several difficulties in applying the known results on HJB theory.

Control problems of this nature have attracted an increasing interest in the last years, and many authors have investigated the characterization of the Value Function under this framework; see for instance [1, 2, 75, 74]. In all these papers, a common controllability assumption has been considered at the junction points. More precisely, it is assumed that around the junction points, it is always possible to move backward-and-forward in each branch; see Figure 9.1a for an illustration. As a consequence of this assumption, the Value Function is continuous and can be characterized by means of a system of HJB equations posed on the branches with transmission conditions at the junctions.

In this chapter, we consider the situations where the controllability conditions are not necessarily satisfied. These include cases where the trajectories are constrained to move forward on the network without being allowed to stay on the junction and/or without having the possibility to move in both directions at the junctions as in Figure 9.1b or Figure 9.1c.

The main difficulties here come from the fact that the admissible set has an empty interior,

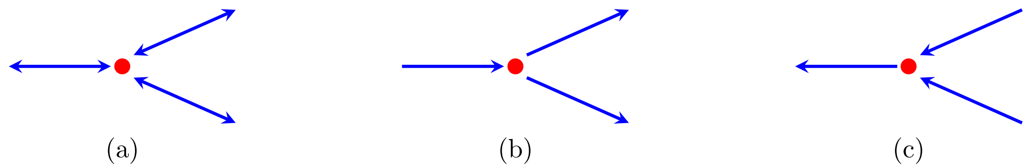


Figure 9.1: Different situations of transmissions conditions at a junction.

and the dynamics, as well as the distributed cost functions, are defined and continuous on each branch without being globally continuous everywhere on the network.

The results presented during the chapter can be adapted to generalized notions of topological networks in which the junctions are now embedded manifolds of $\mathbb{R}^N$, each one of them having the same dimension $d-1$, and the branches are also embedded manifolds of $\mathbb{R}^N$ but of dimension d. Under these circumstances further hypotheses are required to complete the analysis. This study is going to be addressed in details at the end of the chapter.

9.2 Setting of the problem

The formalism presented in this chapter is intended to treat the problem for the standard notion of topological network, that is, here the junctions are points and the branches are curves. Nevertheless, in order to facilitate the transition from this setting to generalized notions of networks, we prefer to treat the branches as embedded manifolds. In particular, the definition of generalized network and the basic assumptions are going to be simple modification of the introduced in this chapter. Moreover, by doing so, several of the statements presented in Chapter 4 and in Chapter 7 can be applied, which simplify the proof of the results exhibited hereafter.

9.2.1 The Mayer problem on networks

We evoke from Definition 4.2.3 that on this manuscript a *junction* is a point $o \in \mathbb{R}^N$ for which there exist $r > 0$ and a family $\{\mathcal{M}_1, \ldots, \mathcal{M}_p\}$ of connected pairwise disjoint $\mathcal{C}^k$-embedded manifolds of $\mathbb{R}^N$ ($k \in \mathbb{N} \cup \{+\infty\}$ and $p \geq 2$) such that

$$\{o\} = \left(\overline{\mathcal{M}}_i \setminus \mathcal{M}_i\right) \cap \mathbb{B}(o, r) \quad \text{and} \quad \dim(\mathcal{M}_i) = 1,\ \forall i \in \{1, \ldots, p\}.$$

We have used the notation $\mathcal{B}(o) = \{\mathcal{M}_1, \ldots, \mathcal{M}_p\}$ to indicate the set of branches associated with o. Moreover, according to Definition 4.2.4, a connected set $\mathcal{K} \subseteq \mathbb{R}^N$ is called a network provided there exist $\{o_j\}_{j\in\mathcal{J}}$, a locally finite and pairwise disjoint family of junctions and $\{\mathcal{M}_i\}_{i\in\mathcal{I}}$, a locally finite and pairwise disjoint family of branches, so that

$$\mathcal{K} = \bigcup_{j\in\mathcal{J}} \{o_j\} \cup \bigcup_{i\in\mathcal{I}} \mathcal{M}_i \quad \text{and} \quad \bigcup_{j\in\mathcal{J}} \mathcal{B}(o_j) = \bigcup_{i\in\mathcal{I}} \mathcal{M}_i.$$

We will say that the network is of class $\mathcal{C}^k$ if each branch is a $\mathcal{C}^k$-embedded manifold on $\mathbb{R}^N$. Moreover, we will use the notation $\mathcal{I}(j)$ to indicate the indices of the branches associated with the junction o_j, that is,

$$\mathcal{I}(j) = \{i \in \mathcal{I} \mid\ \mathcal{M}_i \in \mathcal{B}(o_j)\}, \quad \forall j \in \mathcal{J}.$$

In the first part of the chapter we assume $\mathcal{K}$ is a (one-dimensional) network and its branches are smooth enough:

(H_0^9) $\exists k \geq 2$ so that $\mathcal{K}$ is a closed $\mathcal{C}^k$-network on $\mathbb{R}^N$.

In Figure 9.2 we have illustrated an example of a network embedded in $\mathbb{R}^2$ with 4 branches and a single junction.

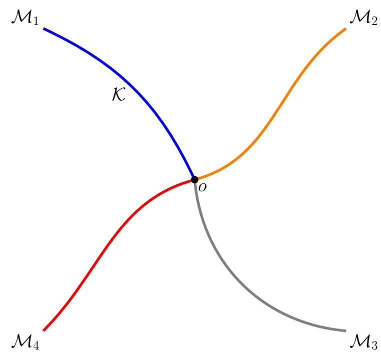


Figure 9.2: An example of a network in $\mathbb{R}^2$.

For sake of simplicity, we are going to consider exclusively the Mayer problem and so the dynamical constraint is written as a differential inclusion; anyhow the results are easily transferred to other types of control problems with fixed final time. For this purpose we consider that for each branch $\mathcal{M}_i$ there exists a set-valued map $F_i : \mathbb{R}^N \rightrightarrows \mathbb{R}^N$ that verifies

$$F_i(x) \neq \emptyset \quad \text{and} \quad F_i(x) \subseteq \mathcal{T}_{\mathcal{M}_i}(x), \quad \forall x \in \mathcal{M}_i.$$

This assumption allows us to give sense to tangent trajectories on each branch $\mathcal{M}_i$. Notice that the velocities of the curves starting from a junction o_j are determined by the dynamics of its surrounding branches. The latter motivates the introduction of the multifunction $F : \mathcal{K} \rightrightarrows \mathbb{R}^N$ defined via

$$F(x) := \begin{cases} F_i(x) & x \in \mathcal{M}_i, \\ \displaystyle\bigcup_{i \in \mathcal{I}(j)} F_i(o_j) & x = o_j, \end{cases} \qquad \forall x \in \mathcal{K}.$$

Consequently, the dynamical constraint of the control system is written as follows

$$\dot{y}(s) \in F(y(s)), \quad \text{a.e. on } [t,T], \quad y(t) = x, \quad y(s) \in \mathcal{K}, \quad \forall s \in [t,T].$$

As usual, we denote by $\mathbb{S}_t^T(x)$ the set of admissible curves of the foregoing differential inclusion. Moreover, all along the chapter we assume that for each branch there exists a final cost $\psi_i : \mathbb{R}^N \to \mathbb{R} \cup \{+\infty\}$ whose values can be completely different from branch to branch. We only require that they are regular on the corresponding $\mathcal{M}_i$.

(H_ψ^9) For any $i \in \mathcal{I}$, $\psi_i : \mathbb{R}^N \to \mathbb{R} \cup \{+\infty\}$ is continuous and real-valued on $\overline{\mathcal{M}_i}$.

Similarly as done for the dynamics, we define the global final cost as

$$\psi(x) := \begin{cases} \psi_i(x) & x \in \mathcal{M}_i, \\ \inf\{\psi_i(x) \mid \ i \in \mathcal{I}(j)\} & x = o_j, \end{cases} \qquad \forall x \in \mathcal{K}.$$

Hence, the Mayer problem of concern in this chapter is given by

$$\vartheta(t,x) := \inf\{\psi(y(T)) \mid \ y \in \mathbb{S}_t^T(x)\}, \quad \forall (t,x) \in [0,T] \times \mathcal{K}.$$

Notice that ψ is lower semicontinuous provided that (H_ψ^9) holds. Accordingly, without any further assumptions, the Value Function $\vartheta(\cdot)$ is merely lower semicontinuous as well.

9.2.2 Structural assumptions

We stress that under the present framework the dynamics are likely to differ from one branch to another. Therefore, it is possible that for some junction o_j we have that

$$\bigcup\{F_i(o_j) \mid \ i \in \mathcal{I}(j)\} \text{ is not a convex subset of } \mathbb{R}^N.$$

This yields to work with (optionally) nonconvex-valued dynamics, because by imposing the convexity of $F(x)$ at every $x \in \mathcal{K}$ we risk to exclude several situations of interest. For example, by doing so the case exhibited in Figure 9.3 can not be treated; notice that the convex hull of $F(o)$ contains the zero vector notwithstanding $F(o)$ does not.

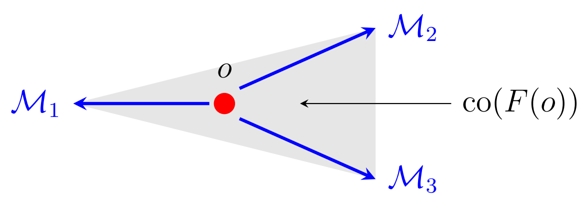


Figure 9.3: A case excluded by the convexity assumption.

This issue is the most important difference between the setting at hand and the state-constraints problem studied in Chapter 4; in the latter it is relatively standard suppose the convexity of the dynamics all along the set of constraints. In the context of networks, is more appropriate to impose conditions over the dynamics on each branch. We rather assume that

$$(H_F^9) \qquad \begin{cases} i) \ \text{For each } i \in \mathcal{I},\ F_i \text{ is locally Lipschitz continuous and has nonempty} \\ \quad\ \text{compact convex images on } \overline{\mathcal{M}}_i, \text{ and } F_i(x) \subseteq \mathcal{T}_{\mathcal{M}_i}(x),\ \forall x \in \mathcal{M}_i. \\ ii) \ \exists c_F > 0 \text{ so that } \forall i \in \mathcal{I},\ \max\{|v| \mid \ v \in F_i(x)\} \leq c_F(1+|x|),\ \forall x \in \overline{\mathcal{M}}_i. \end{cases}$$

On the other hand, in a network framework it is natural to require that the whole set is a viable domain, meaning that for any $x \in \mathcal{K}$ there exists $y \in \mathbb{S}_t^T(x)$ a trajectory of the control system. Since we are not imposing any controllability condition at the junctions, some further assumptions need to be considered in order to ensure this property. Recall that we are concerned with the cases in which it may not be allowed to remain at the junctions.

In the framework we have posed the problem, we are essentially facing a differential inclusion that is not well-posed in the standard setting of that theory; cf. Aubin-Cellina [11] or Clarke et al. [41]. Nonetheless since the main difficulties are basically at the junctions, it is not difficult to provide some criterion for the viability of the network. To do so, we mainly used the results for stratified ordinary differential equations reported in Chapter 7. Accordingly, from now on we suppose the next hypothesis:

$$(H_1^9) \qquad \forall j \in \mathcal{J},\ \exists i \in \mathcal{I}(j) \text{ so that } F(o_j) \cap \mathcal{T}^C_{\overline{\mathcal{M}_i}}(o_j) \neq \emptyset.$$

Remark 9.2.1. *First of all, notice that since $0 \in \mathcal{T}^C_{\overline{\mathcal{M}_i}}(o_j)$ for any $j \in \mathcal{J}$ and $i \in \mathcal{I}(j)$, then (H_1^9) is weaker than usual controllability assumption at the junctions found in the literature; we refer for instance to [1, 2, 75, 74]. Furthermore, it is somehow the minimal requirement we can ask to a network in order to well define solutions of the dynamical system.*

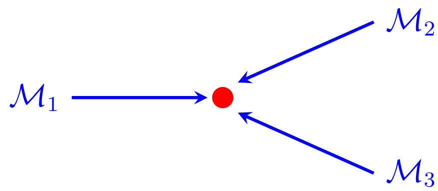
To illustrate the importance of (H_1^9) for the viability property, let us consider the network whose branches are

$$\mathcal{M}_1 = (-\infty, 0] \times \{0\}, \quad \mathcal{M}_2 = \{(s,s) \mid s \geq 0\}, \quad \textit{and} \quad \mathcal{M}_3 = \{(s,-s) \mid s \geq 0\}.$$

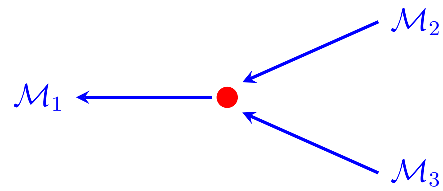
Consider the dynamics given by (see Figure 9.4a)

$$F_1(x) = \{(1,0)\}, \quad F_2(x) = \{(-1,-1)\}, \quad \textit{and} \quad F_3(x) = \{(-1,1)\}, \quad \forall x \in \mathcal{K}.$$

We can see that (H_1^9) does not hold and that no trajectory of the control systems can emerge from the junction point $(0,0)$. However, it suffices to change the orientation of the dynamics at one of the branches in order to get the existence of solutions starting from $(0,0)$; in Figure 9.4b we have illustrated the case in which $F_1(x)$ has bee replaced with $\{(-1,0)\}$.



(a)



(b)

Figure 9.4: Some situations for the dynamics at a junction.

On the other hand, in order to ruled out some pathological cases, for instance bounded branches having infinite length, we suppose all through this chapter that the branches are embedded manifold up to its boundary. In other words, we consider the following:

$$(H_2^9) \qquad \begin{cases} \text{For each } i \in \mathcal{I} \text{ there is a } \mathcal{C}^k\text{-embedded manifold } \mathcal{M}_i^{\text{ext}} \text{ of } \mathbb{R}^N \text{ so that} \\ \overline{\mathcal{M}}_i \subseteq \mathcal{M}_i^{\text{ext}} \text{ and } \forall x \in \mathcal{M}_i,\ \exists r > 0 \text{ so that } \mathcal{M}_i \cap \mathbb{B}(x,r) = \mathcal{M}_i^{\text{ext}} \cap \mathbb{B}(x,r). \end{cases}$$

Remark 9.2.2. *One model of network often studied in the literature is when the branches are contained in half-lines; see for instance [1, 2, 75, 74]. Clearly, under these circumstances, (H_2^9) holds immediately with $\mathcal{M}_i^{ext}$ being the prolongation of the corresponding half-line.*

It is not difficult to see that $\mathcal{M}_i^{\text{ext}}$ has the same dimension than $\mathcal{M}_i$; this is because $\mathcal{M}_i^{\text{ext}}$ and $\mathcal{M}_i$ coincide in a local sense around each point of $\mathcal{M}_i$. Consequently, $\mathcal{M}_i^{\text{ext}}$ can be seen as an extension of $\mathcal{M}_i$ to a neighborhood of it. This fact has some consequences that are summarized in the next statement.

Lemma 9.2.1. *Suppose that $\mathcal{K}$ is a network that verifies (H_0^9) and (H_2^9). Then, for each $j \in \mathcal{J}$ and $i \in \mathcal{I}(j)$ there exists $\eta_i^j \in \mathcal{T}_{\mathcal{M}_i^{ext}}(o_j) \setminus \{0\}$ for which*

$$\mathcal{T}^C_{\overline{\mathcal{M}}_i}(o_j) = \mathcal{T}^B_{\overline{\mathcal{M}}_i}(o_j) = \{-\lambda \eta_i^j \mid \lambda \geq 0\}, \quad \forall j \in \mathcal{J},\ \forall i \in \mathcal{I}(j). \tag{9.1}$$

Furthermore, for each $i \in \mathcal{I}$ we have that $\overline{\mathcal{M}}_i$ is relatively wedged (see Definition 3.4.1).

Proof. Notice that since $\mathcal{K}$ is a closed network and the branches are pairwise disjoints, $\overline{\mathcal{M}}_i \backslash \mathcal{M}_i$ is nonempty and contains only junctions. Let o_j be a junction such that $i \in \mathcal{I}(j)$ and let $\mathcal{M}_i^{\text{ext}}$ be the manifolds that extends $\mathcal{M}_i$ given by (H_2^9).

Since $\dim(\mathcal{M}_i^{\text{ext}}) = \dim(\mathcal{M}_i) = 1$, there exists $\eta_i^j \in \mathbb{R}^N \setminus \{0\}$ so that

$$\mathcal{T}_{\mathcal{M}_i^{\text{ext}}}(o_j) = \text{span}\{\eta_i^j, -\eta_i^j\}.$$

Let $r_j > 0$ and consider $h : \mathbb{B}(o_j, r_j) \to \mathbb{R}^{N-1}$, a local defining map for $\mathcal{M}_i^{\text{ext}}$ around o_j. We claim that, by taking a smaller $r_j > 0$ if necessary, o_j is the unique solution on $\mathbb{B}(o_j, r_j)$ of

$$\Psi(x) = 0, \quad \text{where } \Psi(x) = \begin{pmatrix} h(x) \\ \langle x - o_j, \eta_i^j \rangle \end{pmatrix}, \quad \forall x \in \mathbb{B}(o_j, r_j).$$

Indeed, given that $\eta_i^j \in \ker(d_x h|_{x=o_j}) \setminus \{0\}$, we have that $d_x \Psi|_{x=o_j}$ is a nonsingular linear operator from $\mathbb{R}^N$ into $\mathbb{R}^N$. Hence, by the Inverse Function Theorem the preceding claim holds true and in particular we have that

$$\{o_j\} = \{x \in \mathcal{M}_i^{\text{ext}} \cap \mathbb{B}(o_j, r_j) \mid \langle x - o_j, \eta_i^j \rangle = 0\}.$$

On the other hand, by virtue of (H_2^9), we have that

$$\mathcal{M}_i \cap \mathbb{B}(o_j, r_j) = \{x \in \mathcal{M}_i^{\text{ext}} \cap \mathbb{B}(o_j, r_j) \mid \langle x - o_j, \eta_i^j \rangle < 0\} \cup \{x \in \mathcal{M}_i^{\text{ext}} \cap \mathbb{B}(o_j, r_j) \mid \langle x - o_j, \eta_i^j \rangle > 0\}.$$

Since the sets on the righthand side can be taken to be disjoint (by reducing r_j once again), without loss of generality we can assume that

$$\overline{\mathcal{M}}_i \cap \mathbb{B}(o_j, r_j) = \{x \in \mathcal{M}_i^{\text{ext}} \cap \mathbb{B}(o_j, r_j) \mid \langle x - o_j, \eta_i^j \rangle \leq 0\}.$$

A simple computation shows that the last expression yields to (9.1), which ends the proof. □

Under the assumptions we have done so far we can prove that the Value Function is a real-valued lower semicontinuous map. The next statement is similar to Proposition 4.3.1. However, the main and most important difference is that the convexity of the dynamics may fail. Therefore, to prove this result we need to adapt the classical arguments to our setting.

Proposition 9.2.1. *Suppose that (H_0^9), (H_ψ^9) and (H_F^9) hold along with (H_1^9) and (H_2^9). Then, for every $(t, x) \in [0, T) \times \mathcal{K}$ there exists an optimal trajectory $y \in \mathbb{S}_t^T(x)$ for the Mayer problem. Furthermore, $\vartheta : [0, T] \times \mathcal{K} \to \mathbb{R}$ is lower semicontinuous.*

Remark 9.2.3. *We stress that in the current literature, rather strong controllability conditions around the junctions are imposed; see for instance [1, 2, 75, 74]. These assumptions ensure that the Value Function is continuous everywhere on the network. Nonetheless, in our framework, these hypotheses are not required, and so, the Value Function is likely to have jumps in its values at the junctions.*

Proof of Proposition 9.2.1. For sake of simplicity we split the proof into three steps.

Step 1 (viability): Notice first that $\{o_j\}_{j\in\mathcal{J}} \cup \{\mathcal{M}_i\}_{i\in\mathcal{I}}$ is a W_a-stratification of $\mathcal{K}$; this is is due to the fact that each $\{o_j\}$ is a 0-dimensional embedded manifold and so its tangent space agrees with $\{0\}$. Moreover, by (H_2^9) each $\mathcal{M}_i$ has bounded curvature around each junction point. So, we might assume without loss of generality that each $\mathcal{M}_i$ has bounded curvature globally. The idea of the proof consists in selecting a stratified vector field (Definition 7.2.1) from the dynamical system that governs the optimal control problem at hand, and afterwards, use Theorem 7.3.3 in order to state the existence of solutions for any $(t,x)\in[0,T)\times\mathcal{K}$.

By virtue of the Michael's Selection Theorem (Proposition 2.2.4) and (H_F^9), for each $i\in\mathcal{I}$ we can construct a continuous selection of F_i, that is, a vector field $g_i : \overline{\mathcal{M}_i}\to\mathbb{R}^N$ which in addition verifies $g_i(x)\in\mathcal{T}_{\mathcal{M}_i}(x)$ for any $x\in\mathcal{M}_i$

Let $\mathcal{J}_0\subseteq\mathcal{J}$ be the set of junction indices defined as follows

$$\mathcal{J}_0 := \{j\in\mathcal{J} \mid \ 0\in F(o_j)\}.$$

By virtue of (H_1^9), for any $j\in\mathcal{J}\setminus\mathcal{J}_0$ we can find $i\in\mathcal{I}(j)$ so that $F(o_j)\cap\mathcal{T}^C_{\mathcal{M}_i}(o_j)\neq\emptyset$. Since $0\notin F(o_j)$ then we necessarily have that there is $v_i\in F(o_j)\cap\operatorname{ri}\left(\mathcal{T}^C_{\mathcal{M}_i}(o_j)\right)$. The latter is because of (9.1) in Lemma 9.2.1 implies that

$$\mathcal{T}^C_{\mathcal{M}_i}(o_j) = \{0\}\cup\operatorname{ri}\left(\mathcal{T}^C_{\mathcal{M}_i}(o_j)\right) \quad \text{and} \quad \{0\}\cap\operatorname{ri}\left(\mathcal{T}^C_{\mathcal{M}_i}(o_j)\right) = \emptyset.$$

Note that the selection $g_i(\cdot)$ described earlier can be taken in such a way $g_i(o_j) = v_i$. This is because $v_i\in F(o_j)$. Consequently, defining for each $j\in\mathcal{J}_0$ the trivial vector field $g_j(o_j) = 0\in\mathbb{R}^N$, we get that $G = \{g_j\}_{j\in\mathcal{J}_0}\cup\{g_i\}_{i\in\mathcal{I}}$ is a stratified vector field on the network $\mathcal{K}$ for the (W_a)-stratification $\{o_j\}_{j\in\mathcal{J}}\cup\{\mathcal{M}_i\}_{i\in\mathcal{I}}$. Thanks to (H_F^9), this stratified vector field has linear growth. Furthermore, by construction $(\mathbf{H_1^7})$ is verified and thus, thanks to Theorem 7.3.3 we have that for each $(t,x)\in[0,T)\times\mathcal{K}$ there exists an absolutely continuous curve $y:[t,+\infty)\to\mathcal{K}$ satisfying $y(t)=x$ and

$$\dot{y}(s) = \begin{cases} g_i(y(s))\in F(y(s)) & \text{whenever } y(s)\in\mathcal{M}_i, \\ g_j(y(s)) = 0\in F(y(s)) & \text{whenever } y(s) = o_j,\ j\in\mathcal{J}_0, \end{cases} \qquad \text{for a.e. } s\in[t,+\infty).$$

So the network is a viable domain.

Step 2 (existence of optimal trajectories): We claim that $\vartheta(t,x)\in\mathbb{R}$ for any $(t,x)\in[0,T)\times\mathcal{K}$. Indeed, from the previous step (viability) the Value Function is bounded from above. Moreover, by (H_F^9) and the Gronwall's Lemma (Proposition 2.4.1) we have

$$y(s)\in\mathbb{B}(x,r(t,x)), \quad \forall s\in[t,T] \quad \text{where } r(t,x) = (1+|x|)(e^{c_F(T-t)}-1), \quad \forall y\in\mathbb{S}_t^T(x).$$

In particular $\vartheta(t,x)\geq\inf_{i\in\mathcal{I}}\{\psi_i(\tilde{x}) \mid \tilde{x}\in\overline{\mathcal{M}_i}\cap\mathbb{B}(x,r(t,x))\}$. Since the number of branches is locally finite and each ψ_i is continuous on $\overline{\mathcal{M}_i}$, the righthand side is finite and so, as claimed earlier, $\vartheta(t,x)\in\mathbb{R}$. In particular, for each $(t,x)\in[0,T)\times\mathcal{K}$ we can take a minimizing sequence $\{y_n\}\subseteq\mathbb{S}_t^T(x)$ for the problem at issue.

The Gronwall's Lemma and a compactness argument ([11, Theorem 0.3.4] for instance) yield to assert that (passing into a subsequence if necessary) $\{y_n\}$ converges uniformly to an absolutely continuous arc $y : [t, T] \to \mathcal{K}$ and in addition, $\dot{y}_n$ converges weakly in $L^1([0,T];\mathbb{R}^N)$ to its weak derivative $\dot{y}$. Thanks to the Convergence Theorem (Proposition 2.4.4) we actually have that y is a trajectory of the convexified dynamical systems, that is,

$$\dot{y}(s) \in \operatorname{co}(F(y(s))), \quad \text{a.e. on } [t, T].$$

Since $\{y_n\}$ is a minimizing sequence and ψ is lower semicontinuous, it is not difficult to see that $\vartheta(t,x) \geq \psi(y(T))$. Consequently, we only need to show that $y \in \mathbb{S}_t^T(x)$ to conclude that $y(\cdot)$ is an optimal trajectory. To do so, remark that (H_F^9) implies that $\operatorname{co}(F(\tilde{x})) = F_i(\tilde{x})$ for any $i \in \mathcal{I}$ and $\tilde{x} \in \mathcal{M}_i$. Thus in particular

$$\dot{y}(s) \in F_i(y(s)), \quad \text{a.e. on } [t, T] \text{ whenever } y(s) \in \mathcal{M}_i.$$

On the other hand, it is not difficult to see that

$$\dot{y}(s) = 0, \quad \text{a.e. on } [t, T] \text{ whenever } y(s) = o_j \text{ for some } j \in \mathcal{J}.$$

Hence, to conclude it only remains to show that, if $0 \in \operatorname{co}(F(o_j)) \setminus F(o_j)$, then $y(\cdot)$ can not stay at a junction $\{o_j\}$ for a set of times of positive measure. In other words, we need to show:

> **Claim A:** Suppose there is $j \in \mathcal{J}$ so that $0 \in \operatorname{co}(F(o_j)) \setminus F(o_j)$. If $y(s) = o_j$ for some $s \in [t, T]$, then, there exists $\delta > 0$ so that $y(\tau) \neq o_j$ for any $\tau \in (s, s+\delta)$.

To prove the claim, we follow a similar idea as in Proposition 7.4.3. We begin with noticing that if $j \in \mathcal{J}$ is as in Claim A, then we necessarily have that

$$F_i(o_j) \subseteq \operatorname{ri}(\mathcal{T}^C_{\mathcal{M}_i}(o_j)) \quad \text{or well} \quad -F_i(o_j) \subseteq \operatorname{ri}(\mathcal{T}^C_{\mathcal{M}_i}(o_j)), \quad \forall i \in \mathcal{I}(j).$$

So, by Lemma 9.2.1, there exists

$$\sup_{v \in F_i(o_j)} \langle v, \eta_i^j \rangle < 0 \quad \text{or well} \quad \inf_{v \in F_i(o_j)} \langle v, \eta_i^j \rangle > 0, \quad \forall i \in \mathcal{I}(j). \tag{9.2}$$

Let us assume by contradiction that there is a sequence $\{s_n\} \subseteq (s, T)$ with $s_n \to s$ and $y(s_n) = o_j$. Suppose first that for some $n \in \mathbb{N}$ large enough, s_n and s_{n+1} are consecutive switching times, that is, $y(\tau) \neq o_j$ on (s_n, s_{n+1}). We are going to prove that, thanks to (9.2), this situation never happens.

By the network structure, there exists $i \in \mathcal{I}(j)$ so that $y(\tau) \in \mathcal{M}_i$ on (s_n, s_{n+1}). Therefore

$$\inf_{v \in F_i(y(\tau))} \langle v, \eta_i^j \rangle \leq \langle \dot{y}(\tau), \eta_i^j \rangle \leq \sup_{v \in F_i(y(\tau))} \langle v, \eta_i^j \rangle, \quad \text{for a.e.} \tau \in (s_n, s_{n+1}). \tag{9.3}$$

By (H_F^9), F_i is locally Lipschitz continuous, so if $n \in \mathbb{N}$ is large enough we can assume that

$$\frac{1}{2} \inf_{v \in F_i(o_j)} \langle v, \eta_i^j \rangle \leq \inf_{v \in F_i(y(\tau))} \langle v, \eta_i^j \rangle \quad \text{and} \quad \sup_{v \in F_i(y(\tau))} \langle v, \eta_i^j \rangle \leq \frac{1}{2} \sup_{v \in F_i(o_j)} \langle v, \eta_i^j \rangle, \quad \forall \tau \in (s_n, s_{n+1}).$$

Integrating (9.3) between s_n and s_{n+1}, and using the preceding affirmation, we find out that

$$\frac{s_{n+1} - s_n}{2} \inf_{v \in F_i(o_j)} \langle v, \eta_i^j \rangle \leq 0 = \langle y(s_{n+1}) - y(s_n), \eta_i^j \rangle \leq \frac{s_{n+1} - s_n}{2} \sup_{v \in F_i(o_j)} \langle v, \eta_i^j \rangle.$$

Notice that, in any case, this contradicts (9.2), which means that no s_n can be a switching time (from $n \in \mathbb{N}$ large enough). This implies in particular that the curve $y(\cdot)$ can not chatter between o_j and it branches.

Accordingly, if the statement of Claim A does not hold, the only option that remains is that $y(\tau) = o_j$ for any $\tau \in [s, T]$. However, this case can not occur. To see this notice that the analysis done earlier for y is also valid for any y_n. Particularly, no y_n chatters between o_j and it branches. Furthermore, since $0 \notin F(o_j)$, y_n can not remain at o_j (only pass through), but this contradicts the fact that y_n converges uniformly to y. So, the conclusion follows

Step 3 (lower semicontinuity): To prove this we use the same arguments as for the state-constraints case. If $\{(t_n, x_n)\} \subseteq \operatorname{dom} \vartheta$ converges to (t, x), we take $y_n \in \mathbb{S}^T_{t_n}(x_n)$ optimal and use the same compactness arguments as in the preceding step to prove that y_n has a subsequence that converges to an element of $\mathbb{S}^T_t(x)$. The conclusion follows by the definition of ϑ.

□

The formalism we have described in this section may look at first glance rather technical for treating the one-dimensional networks. However, as we will see at the end of this chapter, this approach will facilitate the transition from the usual notion of network to its generalizations to higher dimensional networks.

9.3 Characterization of the Value Function

We are now in position to state the main theorem of this section. This result shows that the system of inequalities introduced in Theorem 4.3.1 characterizes as well the Value Function when an additional junction condition (equation (9.10) below) is required in the framework of networks with discontinuous dynamics.

Similarly as done in Chapter 4 we denote by

$$F^\sharp(x) = F(x) \cap \mathcal{T}^B_{\mathcal{K}}(x), \quad \forall x \in \mathcal{K}.$$

Notice that on the branches, $F^\sharp$ coincides with F, this is because $\mathcal{T}^B_{\mathcal{K}}(x) = \mathcal{T}_{\mathcal{M}_i}(x)$ and $F_i(x) \subseteq \mathcal{T}_{\mathcal{M}_i}(x)$ whenever $x \in \mathcal{M}_i$. In particular,

$$F^\sharp(x) = F_i(x), \quad \forall i \in \mathcal{I},\ \forall x \in \mathcal{M}_i.$$

Theorem 9.3.1. *Suppose (H^9_0), (H^9_1) and (H^9_2) hold along with (H^9_ψ) and (H^9_F). Then the Value Function of the Mayer problem on the network $\mathcal{K}$ is the unique lower semicontinuous function on $[0, T] \times \mathcal{K}$ which is $+\infty$ outside $[0, T] \times \mathcal{K}$ and that verifies:*

- **The HJB equation:**

$$-\theta + \max_{v \in F_i(x)} \langle -v, \zeta \rangle = 0, \quad \forall i \in \mathcal{I},\ \forall (t, x) \in (0, T) \times \mathcal{M}_i,\ \forall (\theta, \zeta) \in \partial_V \vartheta(t, x). \tag{9.4}$$

- **The final time conditions:**

$$-\theta + \max_{v \in F_i(x)} \langle -v, \zeta \rangle \leq 0, \quad \forall i \in \mathcal{I},\ \forall x \in \mathcal{M}_i,\ \forall (\theta, \zeta) \in \partial_V \vartheta(T, x), \tag{9.5}$$

$$\vartheta(T, x) = \psi(x), \qquad x \in \mathcal{K}. \tag{9.6}$$

- **The initial time condition:**

$$-\theta + \max_{v \in F_i(x)} \langle -v, \zeta \rangle \geq 0, \quad \forall i \in \mathcal{I},\ \forall x \in \mathcal{M}_i,\ \forall (\theta, \zeta) \in \partial_V \vartheta(0, x). \tag{9.7}$$

- **The junction conditions for** $j \in \mathcal{J}_0 := \{j \in \mathcal{J} \mid 0 \in F(o_j)\}$**:**

$$-\theta + \max_{v \in F^\sharp(o_j)} \langle -v, \zeta \rangle \geq 0, \quad \forall j \in \mathcal{J}_0,\ \forall t \in [0, T),\ \forall (\theta, \zeta) \in \partial_V \vartheta(t, o_j), \tag{9.8}$$

$$\theta \geq 0, \quad \forall j \in \mathcal{J}_0,\ \forall t \in (0, T],\ \forall \theta \in \partial_V \vartheta_j(t), \tag{9.9}$$

where $t \mapsto \vartheta_j(t) := \vartheta(t, o_j)$.

- **The junction conditions for** $j \in \mathcal{J} \setminus \mathcal{J}_0$**, that is** $0 \notin F(o_j)$**:**

$$\begin{cases} \forall j \in \mathcal{J} \setminus \mathcal{J}_0,\ \exists i \in \mathcal{I}(j) \text{ so that } F_i(o_j) \subseteq \mathcal{T}^{C}_{\mathcal{M}_i}(o_j) \text{ and} \\ -\theta + \max_{v \in F_i(o_j)} \langle -v, \zeta \rangle \geq 0, \quad \forall t \in [0, T),\ \forall (\theta, \zeta) \in \partial_V \vartheta(t, o_j). \end{cases} \tag{9.10}$$

The proof of the above-stated result relies on the monotone properties of the Value Function along trajectories stated in Section 4.3. Notice that in our setting no hypothesis implies that the trajectories of the convexified dynamics $\mathrm{co}(F)$ agrees with original one. When this happens, the analysis is rather standard and we can use most of the conclusions of Section 4.3. In any other case, a specialized study should be carried on; this explains in part why the junction conditions depends on whether $j \in \mathcal{J}_0$ or not.

Remark 9.3.1. *To the best of out knowledge* (9.10) *(or a similar junction condition) has never been reported before in the literature. This is due to the fact that, as we have already discussed, the usual controllability assumptions done at the junctions imply that* $\mathcal{J}_0 = \mathcal{J}$.

On the other hand, similarly as for the result described for the state-constraints problem in Section 4.2.4, in the continuous case, (9.8) *agrees with one of the standard junctions conditions of [1, 2] , however,* (9.9) *seem to be weaker than the required in the quoted works.*

Proof of Theorem 9.3.1. We begin by noticing that the Value Function of the Mayer problem (on a network setting) satisfies as well the Dynamic Programming Principles

$$\vartheta(t, x) = \inf \left\{ \vartheta(s, y(s)) \mid \quad y \in \mathbb{S}_t^T(x) \right\}, \quad \forall x \in \mathcal{K}, \quad \forall 0 \leq t \leq s \leq T.$$

As a matter of fact, for any $(t, x) \in [0, T] \times \mathcal{K}$, the infimum is realized by a trajectory of the dynamical system. This is a direct consequence of Proposition 9.2.1.

Therefore, for each $(t, x) \in [0, T] \times \mathcal{K}$ the Value Function verifies:

- $\exists y \in \mathbb{S}_t^T(x)$ for which $\vartheta(s, y(s)) \leq \vartheta(t, x)$ for any $s \in [t, T]$.
- $\forall y \in \mathbb{S}_t^T(x)$ we have $\vartheta(s, y(s)) \geq \vartheta(t, x)$ for any $s \in [t, T]$.

In other words, ϑ is weakly decreasing and strongly increasing along trajectories of $\mathbb{S}^T$, respectively. This implies that, by the same arguments of Proposition 4.3.2 and Proposition 4.3.3, the Value Function verifies (9.4), (9.5), (9.7) and (9.9). Besides, by definition the Value Function satisfies (9.6).

Let us show that ϑ satisfies (9.8); this proof is slightly different from that of Proposition 4.3.2, the essential difference is described next. Let $j \in \mathcal{J}$ be fixed but arbitrary. By the weakly decreasing property, there is $y \in \mathbb{S}_t^T(o_j)$ so that

$$\vartheta\big(s, y(s)\big) \leq \vartheta(t, o_j) \quad \text{for all } t \leq s \leq T.$$

Now choose any sequence $\{s_n\} \subseteq (t, T]$ with $s_n \to t$ and $v_n := \frac{y(s_n)-x}{s_n - t} \to v$. By the network structure, we can chose the s_n in such a ways $y(s_n) \in \overline{\mathcal{M}}_i$ for some $i \in \mathcal{I}(j)$. The same arguments used in Proposition 4.3.2 can be applied in this situation to prove that

$$v \in \big(F_i(o_j) + \overline{\mathbb{B}}(0,\varepsilon)\big) \cap \mathcal{T}_{\mathcal{K}}^B(o_j), \quad \forall \varepsilon > 0.$$

This is because $F_i(o_j)$ has convex nonempty images. Hence, letting $\varepsilon \to 0$ we get that $v \in F_i(o_j) \cap \mathcal{T}_{\mathcal{K}}^B(o_j) \subseteq F^\sharp(o_j)$. Now, picking up the proof of Proposition 4.3.2, we can easily prove that ϑ verifies (9.8). Furthermore, let us point out that, as seen in the proof of Proposition 9.2.1, if $j \notin \mathcal{J}_0$, then $y(\cdot)$ can be chosen in such a way $y(s) \in \mathcal{M}_i$ for any $s \in (t, T)$ close enough of t. Since in this case $i \in \mathcal{I}(j)$ can be taken independently of $t \in [0, T)$ and $(\theta, \zeta) \in \partial_V \varphi(t, o_j)$, it is not difficult to see that (9.10) also holds.

Let us now focus on the uniqueness. To do so, suppose that φ is another solution of (9.4)-(9.10) and that its domain is $[0, T] \times \mathcal{K}$.

Notice that (9.4), (9.5) and (9.9) imply that (4.19), the strongly increasing criterion of Proposition 4.3.3, is verified with the stratification $\{o_j\}_{j\in\mathcal{J}} \cup \{\mathcal{M}_i\}_{i\in\mathcal{I}}$. This is because, if for any $j \in \mathcal{J}$ we set $F_j = \{v \in F(o_j) \mid v = 0\}$, then

$$\sup_{v\in F_j} \langle -v, \zeta\rangle = \begin{cases} 0, & \text{if } j \in \mathcal{J}_0, \\ -\infty & \text{otherwise,} \end{cases} \qquad \forall \zeta \in \mathbb{R}^N.$$

In addition, the Lipschitz assumption (H_F^4) is a direct consequence of (H_F^9) and the controllability assumption (H_3^4) holds trivially, because it only matters on the junctions.

Remark that the convexity of the dynamics $F(\cdot)$ does play any role in the proof of Proposition 4.3.3; the principal role is played by the tangent dynamics to each stratum. Hence we might repeat the arguments used there in order to prove that φ is strongly decreasing along trajectories of $\mathbb{S}^T$. Consequently, by Lemma 4.4.1 and (9.6) we get that $\varphi \leq \vartheta$ on $[0, T] \times \mathcal{K}$.

Therefore, to conclude it only remains to show that $\varphi \geq \vartheta$ on $[0, T] \times \mathcal{K}$. Let us assume by contradiction that there is $(t, x) \in [0, T) \times \mathcal{K}$ so that $\varphi(t, x) < \vartheta(t, x)$.

For sake of clarity, we divide the rest of the proof into three steps, each one of them aiming to prove a determined claim.

Step 1:

> **Claim B:** If x is not a junction point, that is there is $i \in \mathcal{I}$ so that $x \in \mathcal{M}_i$. Then, there exist $\tau > t$ and $y \in \mathbb{S}_t^\tau(x)$ so that
>
> $$\varphi(s, y(s)) \leq< \vartheta(t, x), \quad \forall s \in [t, \tau) \quad \text{and if } \tau \in \mathbb{R}, \quad y(\tau) = o_j, \quad \text{for some } j \in \mathcal{J}.$$

To prove this, we only need to show that there exist $\tau \in (t, T)$ and $y \in \mathbb{S}_t^\tau(x)$ so that

$$\varphi(s, y(s)) \le \varphi(t, x).$$

To do so, we are going to use the weak invariance criterion exhibited in Chapter 2 on the branch $\mathcal{M}_i$. Let us first state that the dynamics F_i can be defined on a neighborhood of $\mathcal{M}_i$.

By (H_0^9) we have that $\mathcal{M}_i$ is at least of class $\mathcal{C}^2$ and so, there exist an open set $\mathcal{O} \subseteq \mathbb{R}^N$ containing $\mathcal{M}_i$ and a locally Lipschitz continuous set-valued maps $\hat{F}_i : \mathbb{R}^N \rightrightarrows \mathbb{R}^N$ that extends F_i and has nonempty compact convex images on $\overline{\mathcal{O}}$; thanks to Proposition 3.2.9 the extension can be defined on a tubular neighborhood $\mathcal{O}$ of $\mathcal{M}_i$ by means of the projection over $\mathcal{M}_i$, then by density it can be redefined up to $\overline{\mathcal{O}}$.

Recall that $\partial_P\varphi(t, x) \subseteq \partial_V\varphi(t, x)$. In particular, in the light of Proposition 2.3.9, it is not difficult to see that (9.4) and (9.7) imply that for any $(t, x) \in [0, T) \times \mathcal{M}_i$

$$\min_{v\in F_i(x)} \big\langle (1, v, 0), (\theta, \zeta, -1)\big\rangle \le 0 \quad \forall\, (\theta, \zeta, -1) \in \mathcal{N}_{\mathcal{S}}^P(t, x, \varphi(t, x)), \tag{9.11}$$

where $\mathcal{S} := \operatorname{epi}\varphi$. Consider the set-valued map $\Gamma_i : \mathbb{R} \times \mathbb{R}^N \times \mathbb{R} \rightrightarrows \mathbb{R} \times \mathbb{R}^N \times \mathbb{R}$ defined via

$$\Gamma_i(t, x, z) = \left\{(1, v, 0) \mid\ v \in \hat{F}_i(x)\right\}, \quad \forall (t, x, z) \in \mathbb{R} \times \mathbb{R}^N \times \mathbb{R}.$$

Let $\mathcal{Q} := \mathbb{R}\times\mathcal{O}\times\mathbb{R}$, notice that $\mathcal{S}\cap\mathcal{Q} \subseteq [0, T]\times\mathcal{M}_i\times\mathbb{R}$. Then Γ_i is upper semicontinuous with locally bounded images and it has nonempty compact convex images on $\mathcal{S}\cap\mathcal{Q}$. We assert that $(\Gamma, \mathcal{S})$ is weakly invariant in $\mathcal{Q}$ in terms of Definition 2.4.1. By (9.11) and adapting the arguments of the sufficiency part in Proposition 4.2.5, we can prove that:

$$\min_{\nu\in\Gamma_i(t,x,z)} \langle w, \eta\rangle \le 0 \quad \forall\, \eta \in \mathcal{N}_{\mathcal{S}}^P(t, x, z), \quad \forall (t, x, z) \in \mathcal{S} \cap \mathcal{Q}.$$

Consequently, by Proposition 2.4.5 we have that $(\Gamma_i, \mathcal{S})$ is weakly invariant in $\mathcal{Q}$, which means that, since $(t, x, \varphi(t, x)) \in \mathcal{S}$, we can find $\tau \in (t, T]$ and a curve $\gamma : [t, \tau) \to \mathbb{R} \times \mathbb{R}^N \times \mathbb{R}$ that solves $\dot\gamma \in \Gamma_i(\gamma)$ a.e. on $[t, \tau)$, lives in $\mathcal{S}$ on $[t, \tau)$ and verify $\gamma(t) = (t, x, \varphi(t, x))$. By the nature of Γ_i, we see that

$$\gamma(s) = (s, y(s), \varphi(t, x)), \quad \forall s \in [t, \tau)$$

where $y \in \mathbb{S}_t^\tau(x)$. Since $\gamma(s) \in \mathcal{S}$ for any $s \in [t, \tau)$, we have that $\varphi(s, y(s)) \le \varphi(t, x)$ for each $s \in [t, \tau)$. Moreover, if finite, τ is such that $\gamma(s)$ approaches the boundary $\mathcal{Q}$ as $s \to \tau$. But this means that $y(\tau) \in \overline{\mathcal{M}}_i \setminus \mathcal{M}_i$. Notice that since $\mathcal{K}$ is a closed network and the branches are pairwise disjoints, $\overline{\mathcal{M}}_i \setminus \mathcal{M}_i$ is contains only junctions. Given that the junctions are isolated points, the conclusion follows.

Notice that if x is as in Claim B and $\tau \ge T$ we get a contradiction, because by (9.6)

$$\psi(y(T)) \le \varphi(T, y(T)) < \vartheta(t, x).$$

But, this is not possible because $y|_{[t,T]} \in \mathbb{S}_t^T(x)$. Therefore, we can rule out this situation of the contradiction analysis.

On the other hand, if $\tau < T$, then $y(\tau) = o_j$ for some $j \in \mathcal{J}$ and, since the Value Function is strongly increasing along trajectories of $\mathbb{S}^T$ and φ is lower semicontinuous, we have

$$\varphi(\tau, o_j) = \varphi(\tau, y(\tau)) \le \varphi(t, x) < \vartheta(t, x) \le \vartheta(\tau, y(\tau)) = \vartheta(\tau, o_j).$$

The preceding reasoning shows that we can assume without loss of generality that x is a junction point, which we do from this point on. Let us begin with the case where $j \in \mathcal{J}_0$.

Step 2:

> **Claim C:** Suppose $x = o_j$ for some $j \in \mathcal{J}_0$. Then, we can find $\tau > t$ and a trajectory $y \in \mathbb{S}_t^\tau(x)$ so that
>
> $$\varphi(s, y(s)) < \vartheta(t, x), \quad \forall s \in [t, \tau) \quad \text{and if } \tau \in \mathbb{R} \quad y(\tau) = o_l, \quad \text{for some } l \in \mathcal{J} \setminus \{j\}.$$

The proof of this assertion is simpler than for $j \notin \mathcal{J}_0$. The reason is that the dynamics of the convexified problem and the original one always coincide around the junction if $j \in \mathcal{J}_0$.

Let $\mathcal{O}_j \subseteq \mathbb{R}^N$ be an open subset such that o_j is the unique junction contained in $\mathcal{K} \cap \mathcal{O}_j$ and $\overline{\mathcal{M}}_i \subseteq \overline{\mathcal{O}}$ for any $i \in \mathcal{I}(j)$; this is always possible to construct because $\mathcal{I}(j)$ is a finite set. We are going to prove that (9.4), (9.7) and (9.8) imply that $(\operatorname{epi} \varphi, \Gamma)$ is weakly invariant in $\mathcal{Q} := \mathbb{R} \times \mathcal{O}_j \times \mathbb{R}$, where Γ is a suitable extension of $\{1\} \times \operatorname{co}(F) \times \{0\}$. By showing this and proceeding as in the preceding claim, the conclusion will be easily reached.

Let us begin with the existence of such extension. Notice that the set-valued map defined on $\mathcal{K}$ via $x \mapsto \{1\} \times \operatorname{co}(F(x)) \times \{0\}$ is upper semicontinuous with nonempty compact convex images. Furthermore, by the linear growth condition over F, there exists $c_j > 0$ so that

$$\max\{|v| \mid v \in \{1\} \times \operatorname{co}(F(x)) \times \{0\}\} \leq c_j, \quad \forall x \in \mathcal{K} \cap \overline{\mathcal{O}}_j.$$

Hence, by [119, Theorem 2.6] there is an upper semicontinuous extension of the multifunction

$$\Gamma_0(t, x, z) := \{1\} \times \operatorname{co}(F(x)) \times \{0\}, \quad \forall (t, x, z) \in [0, T] \times \mathcal{K} \cap \overline{\mathcal{O}}_j$$

up to $\mathbb{R} \times \mathbb{R}^N \times \mathbb{R}$, which has convex closed images and whose elements are bounded (by the same constant c_j). Besides, by construction the extension given by [119, Theorem 2.6] has nonempty images all along $\mathbb{R} \times \mathbb{R}^N \times \mathbb{R}$. Let Γ stands for this extension of Γ_0.

We set once again $\mathcal{S} = \operatorname{epi} \varphi \subseteq [0, T] \times \mathcal{K} \times \mathbb{R}$. Let us point out that in Claim B we have shown that (9.4) and (9.7) yield to

$$\min_{\nu \in \Gamma(t,x,z)} \langle w, \eta \rangle \leq 0 \quad \forall \eta \in \mathcal{N}_{\mathcal{S}}^P(t, x, z), \quad \forall (t, x, z) \in \mathcal{S} \cap \mathcal{Q}, \ x \in \mathcal{M}_i \text{ for any } i \in \mathcal{I}(j).$$

Consequently, to establish the weakly invariance of the system (by means of Proposition 2.4.5), it only remains to show that (9.8) implies that

$$\min_{\nu \in \Gamma(t,o_j,z)} \langle w, \eta \rangle \leq 0 \quad \forall \eta \in \mathcal{N}_{\mathcal{S}}^P(t, o_j, z), \quad \forall (t, o_j, z) \in \mathcal{S} \cap \mathcal{Q}. \tag{9.12}$$

To see this, it is enough to note that

$$\max_{v \in F^\sharp(o_j)} \langle -v, \zeta \rangle \leq \max_{v \in \operatorname{co}(F(o_j))} \langle -v, \zeta \rangle, \quad \forall t \in [0, T), \ \forall (\theta, \zeta) \in \partial_V \vartheta(t, o_j).$$

Thus, the arguments of Proposition 4.3.2 can be used at present to prove that 9.12 holds. So, by Proposition 2.4.5 the system $(\operatorname{epi} \varphi, \Gamma)$ is weakly invariant in $\mathcal{Q}$.

In particular, by the same reasoning of the preceding claim, there exist $\tau > 0$ and an absolutely continuous curve $y : [t, \tau) \to \mathcal{K}$ with $y(t) = x$, for which

$$\varphi(s, y(s)) \leq \varphi(t, x), \quad \forall s \in [t, \tau) \quad \text{and} \quad \dot{y} \in \mathrm{co}(F(y(s))), \quad \text{for a.e. } s \in [t, \tau).$$

Moreover, we can easily see that

$$\dot{y}(s) \in \begin{cases} F_i(y(s)) & \text{whenever } y(s) \in \mathcal{M}_i, \\ \{0\} & \text{whenever } y(s) = o_j, \end{cases} \qquad \text{for a.e. } s \in [t, \tau).$$

But, since $j \in \mathcal{J}_0$ we have that $0 \in F(o_j)$, from where we get that $y(\cdot)$ is a trajectory of the original control system, and so the conclusion follows easily.

The same reasons used for the conclusion of Claim B show that, if x is as in Claim C and $\tau \geq T$ we get a contradiction. Consequently, we can as well rule out this situation of the contradiction analysis.

Notice that the only case that remains to dismiss is when $x = o_j$ for $j \notin \mathcal{J}_0$. Indeed, if x is as in Claim C but $\tau_0 := \tau < T$, then $y(\tau_0) = o_{j_1}$ for some $j_1 \in J$. If $j_1 \in \mathcal{J}_0$, then applying Claim C to this new junction we find some $\tau_1 > \tau_0$ so that

$$\varphi(s, y(s)) \leq \varphi(t, x) < \vartheta(t, x) \leq \vartheta(s, y(s)), \quad \forall s \in [\tau_0, \tau_1).$$

If $\tau_1 < T$ it means that $y(\tau_1) = o_{j_2}$ for some $j_2 \in \mathcal{J} \setminus \{j_1\}$. It is clear that this process can continue until some τ_k is greater than T; this is due to the fact that by (H_F^9) and the Gronwall's Lemma, the difference $\tau_k - \tau_{k-1}$ can be uniformly bounded from below on a compact set containing all the possible trajectories emerging from x. Furthermore, if at each step we have that $j_k \in \mathcal{J}_0$ then by the preceding arguments we find a contradiction.

Therefore, to finish the proof we just have to prove that the conclusion of Claim C holds as well in the case $j \notin \mathcal{J}_0$.

Step 3:

> **Claim D:** Suppose $x = o_j$ for some $j \in \mathcal{J} \setminus \mathcal{J}_0$. Then, we can find $\tau > t$ and a trajectory $y \in \mathbb{S}_t^\tau(x)$ so that
>
> $$\varphi(s, y(s)) < \vartheta(t, x), \quad \forall s \in [t, \tau) \quad \text{and if } \tau \in \mathbb{R} \quad y(\tau) = o_l, \quad \text{for some } l \in \mathcal{J} \setminus \{j\}.$$

Let $i \in \mathcal{I}(j)$ be given by (9.10), there exists $i \in \mathcal{I}(j)$ Moreover, since $0 \notin F_i(o_j)$ we have that $F_i(o_j) \subseteq \mathrm{ri}\left(\mathcal{T}_{\mathcal{M}_i}^C(o_j)\right)$. Let us show first that

$$-\theta + \max_{v \in F_i(o_j)} \langle -v, \zeta \rangle \geq 0, \quad \forall t \in [0, T), \ \forall (\theta, \zeta) \in \partial_P \varphi_i(t, o_j), \tag{9.13}$$

where $\varphi_i = \varphi$ on $[0, T] \times \overline{\mathcal{M}}_i$ and is $+\infty$ elsewhere. To do this, let us fix $(\theta, \zeta) \in \partial_P \varphi_i(t, o_j)$. By the Sum Rule for the proximal subdifferential (see for instance [41, Theorem 1.8.3]), we can construct the following sequence:

- $\{(t_n, x_n)\} \in [0, T) \times \mathcal{K}$ with $(t_n, x_n) \to (t, 0_j)$ and $\varphi(t_n, x_n) \to \varphi(t, 0_j)$.

- $\{(\theta_n, \zeta_n)\} \in \mathbb{R} \times \mathbb{R}^N$ with $(\theta_n, \zeta_n) \in \partial_P \varphi(t_n, x_n)$ for any $n \in \mathbb{N}$.
- $\{(\tilde{x}_n, \eta_n)\} \in \overline{\mathcal{M}}_i \times \mathbb{R}^N$ with $\eta_n \in \mathcal{N}^P_{\mathcal{M}_i}(\tilde{x}_n)$ for any $n \in \mathbb{N}$.

Furthermore, these sequences also verify that $\theta_n \to \theta$ and $\zeta_n + \eta_n \to \zeta$ as long as $n \to +\infty$. Suppose that there exists a subsequence of $\{x_n\}$ that lies in $\mathcal{M}_i$, then, avoiding relabeling the subsequence, we have that (9.4) and (9.7) imply that

$$-\theta_n + \max_{v \in F_i(x_n)} \langle -v, \zeta_n \rangle \geq 0, \quad \forall n \in \mathbb{N}.$$

Since F_i has compact images, for each $n \in \mathbb{N}$ there exists $v_n \in F_i(x_n)$ so that $\theta_n + \langle v_n, \zeta_n \rangle \leq 0$. Moreover, since F_i is upper semicontinuous and uniformly bounded around o_j, we can assume that $\{v_n\}$ converges to some $v \in F_i(o_j)$. Besides, since F_i is locally Lipschitz continuous and $F_i(o_j) \subseteq \operatorname{ri}\left(\mathcal{T}^C_{\mathcal{M}_i}(o_j)\right)$, there exists $\{\varepsilon_n\} \subseteq (0,1)$ so that $\varepsilon_n \to 0$ and $\langle v_n, \eta_n \rangle \leq \varepsilon_n$. So, gathering the information we find out that

$$\theta_n + \langle v_n, \zeta_n + \eta_n \rangle \leq \varepsilon_n, \quad \forall n \in \mathbb{N}.$$

Letting $n \to +\infty$, we get (9.13) after a few algebraic steps.

On the other hand, if such subsequence does exist we might assume that $x_n = o_j$ for any $n \in \mathbb{N}$. However, in this case (9.10) yields to

$$-\theta_n + \max_{v \in F_i(o_j)} \langle -v, \zeta_n \rangle \geq 0, \quad \forall n \in \mathbb{N}.$$

Hence, using the same arguments as above we can easily prove that (9.13) holds as well.

Finally, notice that setting $\mathcal{S} = \operatorname{epi} \varphi_i$ it is not difficult to see that (9.13) implies that

$$\min_{\nu \in \Gamma_i(t,x,z)} \langle w, \eta \rangle \leq 0 \quad \forall \eta \in \mathcal{N}^P_{\mathcal{S}}(t,x,z), \quad \forall (t,x,z) \in \mathcal{S} \cap \mathcal{Q},$$

where Γ_i is an upper semicontinuous extension up to $\mathbb{R} \times \mathbb{R}^N \times \mathbb{R}$ of the set-valued map $(t,x,z) \mapsto \{1\} \times F_i(x) \times \{0\}$, which is only defined on $\mathbb{R} \times \overline{\mathcal{M}}_i \times \mathbb{R}$, and $\mathcal{Q}$ is an open set that contains $\mathbb{R} \times \mathcal{M}_i \cup \{o_j\} \times \mathbb{R}$. Notice that, as explained earlier, Γ_i can be taken as to have nonempty convex and compact images on $\mathcal{Q}$.

Finally, using Proposition 2.4.5, the conclusion follows by the same arguments used in the previous claims. This also completes the proof of the theorem.

□

9.4 A generalized notion of d-dimensional network

Continuing the aim of the chapter, we investigate the Hamilton-Jacobi-Bellman (HJB) approach for control problems on networks. However, at present we focus our attention on a generalized notion network where the branches are d-dimensional manifolds. The main motivation in doing so is to understand the underlying role played by the junctions, hidden in the structure of zero-dimensional ones.

Let us mention that a particular case of generalized N-dimensional network on $\mathbb{R}^N$ is the case in which there are two open set $\mathcal{M}_1$ and $\mathcal{M}_2$ separated by a smooth surface Υ. This case has been addressed in the literature by many authors; see for instance Barles-Briani-Chasseigne [15, 16], Rao-Zidani [106] and Rao-Siconolfi-Zidani [105]. All this works are oriented to study the case where the Value Function is continuous. Consequently, strong types of controllability are imposed around the surface Υ (not only on the surface).

In this exposition, we also require a controllability assumption but only for the tangential dynamics to each junction. This hypothesis is done in order to manage trajectories that may chatter between a junction and its branches. Let us stress that we do not demand any other type of controllability on the branches nor on a neighborhood of the junctions (only at the junctions). We also emphasis that the controllability assumption done here is immediately verified for the one-dimensional case, which explains why we introduce it now and not before.

9.4.1 Basic definitions and main hypotheses

Recall we have defined the notion of junction on a network as a single point $o \in \mathbb{R}^N$, which is the local intersection of a finite number of branches $\{\mathcal{M}_1, \ldots, \mathcal{M}_p\}$, that is, smooth curves of $\mathbb{R}^N$. This definition allows us to extend the idea of junction to higher dimensions in a rather natural way. To do so, we allow the dimension of the branches to be larger.

Definition 9.4.1. *A connected $(d-1)$-dimensional embedded manifold Υ of $\mathbb{R}^N$ is called a d-dimensional junction if there exist $r > 0$ and a finite collection $\{\mathcal{M}_1, \ldots, \mathcal{M}_p\}$ of connected pairwise disjoint $\mathcal{C}^k$-embedded manifolds of $\mathbb{R}^N$, ($k \in \mathbb{N} \cup \{+\infty\}$, $p \geq 2$ and $d \geq 1$) for which*

$$\Upsilon = \left(\overline{\mathcal{M}}_i \setminus \mathcal{M}_i\right) \cap \mathbb{B}(x, r),\ \forall x \in \Upsilon \quad \textit{and} \quad \dim(\mathcal{M}_i) = d,\ \forall i \in \{1, \ldots, p\}.$$

Similarly as done before, we use the notation $\mathcal{B}(\Upsilon) = \{\mathcal{M}_1, \ldots, \mathcal{M}_p\}$ to indicate the set of branches associated with the junction Υ. In Figure 9.5 we have portrayed an example of a 2-dimensional junction embedded in $\mathbb{R}^3$.

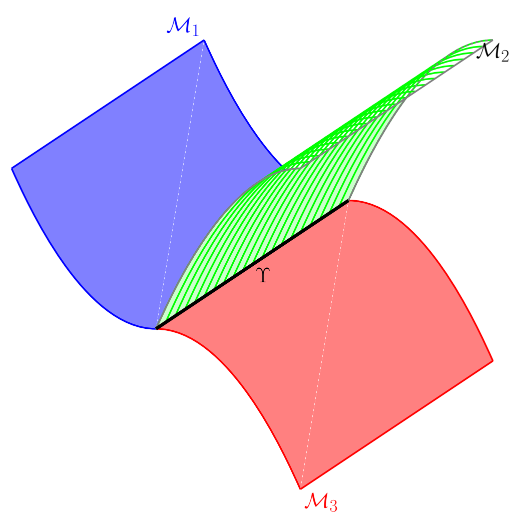


Figure 9.5: A 2-dimensional junction embedded in $\mathbb{R}^3$.

In analogy with Definition 4.2.4 we define a generalized d-dimensional network as follows.

Definition 9.4.2. *A connected set $\mathcal{K} \subseteq \mathbb{R}^N$ is called a generalized d-dimensional network provided there exist $\{\Gamma_j\}_{j\in\mathcal{J}}$, a locally finite and pairwise disjoint family of d-dimensional junctions so that*

$$\mathcal{K} = \bigcup_{j\in\mathcal{J}} \left(\Upsilon_j \cup \bigcup_{\mathcal{M}\in\mathcal{B}(\Gamma)} \mathcal{M} \right).$$

We will say that the generalized d-dimensional network is of class $\mathcal{C}^k$ if each junction and each branch is a $\mathcal{C}^k$-embedded manifold on $\mathbb{R}^N$.

For convenience of notation, we introduced an index $\mathcal{I}$ that lists, in a pairwise disjoint way, all the branches associated with a generalized d-dimensional network, that is,

$$\forall i \in \mathcal{I},\ \exists j \in \mathcal{J} \text{ so that } \mathcal{M}_i \in \mathcal{B}(\Upsilon_j) \text{ and } \mathcal{M}_i \cap \mathcal{M}_l = \emptyset,\ \forall l \in \mathcal{I} \setminus \{i\}.$$

We also write $\mathcal{I}(j) = \{i \in \mathcal{I} \mid \ \mathcal{M}_i \in \mathcal{B}(\Upsilon_j)\}$ for each $j \in \mathcal{J}$.

In the rest of the chapter we assume that $\mathcal{K}$ is a generalized network with sufficiently regular junctions and branches. More precisely,

$(H^9_{0,G})$ $\quad$ $\exists k \geq 2$ so that $\mathcal{K}$ is a generalized d-dimensional $\mathcal{C}^k$-network on $\mathbb{R}^N$.

Notice that the formalism used at the beginning can be easily adapted to this new setting. Indeed, if we consider that for each branch $\mathcal{M}_i$ there exists a set-valued map $F_i : \mathbb{R}^N \rightrightarrows \mathbb{R}^N$ that verifies

$$F_i(x) \neq \emptyset \quad \text{and} \quad F_i(x) \subseteq \mathcal{T}_{\mathcal{M}_i}(x),\ \forall x \in \mathcal{M}_i,$$

then the dynamics that governs the control system is the map $F : \mathcal{K} \rightrightarrows \mathbb{R}^N$ defined by

$$F(x) := \begin{cases} F_i(x) & x \in \mathcal{M}_i, \\ \displaystyle\bigcup_{i\in\mathcal{I}(j)} F_i(x) & x \in \Gamma_j, \end{cases} \qquad \forall x \in \mathcal{K}.$$

Consequently, the dynamical constraint of the control system is written as follows

$$\dot{y}(s) \in F(y(s)), \quad \text{a.e. on } [t,T], \quad y(t) = x, \quad y(s) \in \mathcal{K}, \quad \forall s \in [t,T].$$

We still write $\mathbb{S}^T_t(x)$ for the set of admissible curves of the foregoing differential inclusion. Accordingly, if for each branch there is a final cost $\psi_i : \mathbb{R}^N \to \mathbb{R} \cup \{+\infty\}$, the Mayer problem in this setting is written in the same way as for the one-dimensional networks, that is,

$$\vartheta(t,x) := \inf_{y\in\mathbb{S}^T_t(x)} \psi(y(T)), \quad \forall x \in \mathcal{K}, t \in [0,T],$$

where the global final cost now is given by

$$\psi(x) := \begin{cases} \psi_i(x) & x \in \mathcal{M}_i, \\ \inf\{\psi_i(x) \mid \ i \in \mathcal{I}(j)\} & x \in \Upsilon_j, \end{cases} \qquad \forall x \in \mathcal{K}.$$

We assume the same basic hypotheses (H^9_ψ) and (H^9_F) as before. Notice that, since the dimension of the branches $\mathcal{M}_i$ is not involved in the statement of the assumptions, they are automatically adapted to this new setting. Moreover, this is also the case for the hypothesis (H^9_2).

Therefore, whenever required, it should be understood that $\mathcal{M}_i$ and $\mathcal{M}_i^{\text{ext}}$ are d-dimensional embedded manifolds.

On the other hand, the structural assumptions can also be easily suited to this framework. To state the viability property on a generalized network, the assumption (H_1^9) has to be slightly modified in the following way

$$(H_{1,G}^9) \qquad \forall j \in \mathcal{J},\ \exists i \in \mathcal{I}(j) \text{ so that } F(x) \cap \mathcal{T}^C_{\mathcal{M}_i}(x) \neq \emptyset, \forall x \in \Upsilon_j.$$

Consequently, the corresponding version of Lemma 9.2.1 reads as follows.

Lemma 9.4.1. *Suppose that $\mathcal{K}$ is a generalized network that verifies ($H_{0,G}^9$) and (H_2^9). Then $\{\Upsilon_j\}_{j\in\mathcal{J}} \cup \{\mathcal{M}_i\}_{i\in\mathcal{I}}$ is a (W_a)-stratification of $\mathcal{K}$. Additionally, for each $j \in \mathcal{J}$ and $i \in \mathcal{I}(j)$ there exists $\eta_i^j(\cdot)$, a continuous selection of $\mathcal{T}_{\mathcal{M}_i^{ext}}(\cdot)$, for which $\eta_i^j(x) \neq 0$ for each $x \in \Gamma_j$ and*

$$\mathcal{T}^C_{\mathcal{M}_i}(x) = \mathcal{T}^B_{\mathcal{M}_i}(x) = \mathcal{T}_{\Upsilon_j}(x) \oplus \{-\lambda \eta_i^j(x) \mid\ \lambda \geq 0\}, \quad \forall j \in \mathcal{J},\ \forall i \in \mathcal{I}(j),\ \forall x \in \Upsilon_j. \tag{9.14}$$

This also implies that for each $i \in \mathcal{I}$ we have that $\overline{\mathcal{M}}_i$ is relatively wedged.

Proof. First of all, note that thanks to (H_2^9), for any $j \in \mathcal{J}$ and $i \in \mathcal{I}(j)$, the Whitney (a)-condition is verified by $(\Upsilon_j, \mathcal{M}_i)$. Hence, in particular, $\{\Upsilon_j\}_{j\in\mathcal{J}} \cup \{\mathcal{M}_i\}_{i\in\mathcal{I}}$ is a (W_a)-stratification of $\mathcal{K}$.

Furthermore, if $\mathcal{M}_i^{\text{ext}}$ stands for the extension given by (H_2^9), then, for each $x \in \Upsilon_j$, we also have that $\mathcal{T}_{\mathcal{M}_i^{\text{ext}}}(x) = \mathcal{T}_x \oplus \mathcal{T}_{\Upsilon_j}(x)$, with $\mathcal{T}_x$ being a one-dimensional vectorial space. This is because $\dim(\mathcal{M}_i^{\text{ext}}) = d$ and $\dim(\Upsilon_j) = d-1$.

Let $x \in \Upsilon_j$, then there are $r_x > 0$ and a local defining map $h : \mathbb{B}(x, r_x) \to \mathbb{R}^{N-d+1}$ so that

$$\Upsilon_j \cap \mathbb{B}(x, r_x) = \{\tilde{x} \in \mathbb{B}(x, r_x) \mid\ h(\tilde{x}) = 0\}.$$

Without loss of generality, we can assume that $\mathcal{M}_i \cap \mathbb{B}(x, r_x) \subseteq \{\tilde{x} \in \mathbb{B}(x, r_x) \mid\ h(\tilde{x}) < 0\}$, where $h(\tilde{x}) < 0$ means the strict inequality for each component of h. If h^{ext} is a local defining map for $\mathcal{M}_i^{\text{ext}}$ on $\mathbb{B}(x, r_x)$, we can easily show that

$$\overline{\mathcal{M}}_i \cap \mathbb{B}(x, r_x) = \{\tilde{x} \in \mathbb{B}(x, r_x) \mid\ h(\tilde{x}) \leq 0,\ h^{\text{ext}}(\tilde{x}) = 0\}.$$

From the preceding local algebraic expression for $\overline{\mathcal{M}}_i$ we get that there exists $\eta^j_{i,x} \in \mathcal{T}_x \setminus \{0\}$ so that

$$\mathcal{T}^C_{\mathcal{M}_i}(x) = \mathcal{T}_{\Upsilon_j}(x) \oplus \{-\lambda \eta^j_{i,x} \mid\ \lambda \geq 0\}.$$

This is due to the fact that, after a rotation, we can chose each $\nabla h_n^{\text{ext}}(x)$ to be a linear combination $\{\nabla h_1(x), \ldots, \nabla h_{N-d}(x)\}$. In particular, $\overline{\mathcal{M}}_i$ is relatively wedged at x.

Finally, by the Michael's Selection Theorem (Proposition 2.2.4), we can find a continuous selection of $\mathcal{T}_{\mathcal{M}_i^{\text{ext}}}(x)$, which is always lower semicontinuous (Proposition 3.2.4), so that $\eta_i^j(x) = \eta^j_{i,x}$. By continuity we can assume that so that $\eta_i^j(\tilde{x}) \neq 0$ for any $\tilde{x} \in \Upsilon_j \cap \mathbb{B}(x, r_x)$. Hence, using a partition of the unity subordinated to the covering $\{\mathbb{B}(x, r_x)\}$ we can easily construct a continuous selection that verifies 9.14, So the proof is completed. ☐

In contrast with the one-dimensional case, the trajectories remaining at the junctions are not trivial, that is, they can move all along the junction. This is the essential difference between the generalized case and the one-dimensional one. For this reason an extra compatibility assumption need to be considered. For this purpose we introduce the tangent dynamics to the junctions as follows:

$$F_j(x) = F(x) \cap \mathcal{T}_{\Upsilon_j}(x), \quad \forall j \in \mathcal{J},\ \forall x \in \Upsilon_j.$$

Similarly as done in Chapter 4, we suppose that each F_j is regular, meaning that

$$(H_3^9) \qquad \begin{cases} i) \text{ Each } F_j \text{ is locally Lipschitz continuous on } \Upsilon_j \text{ for the Hausdorff distance.} \\ ii) \text{ If } \operatorname{dom}(F_j) \neq \emptyset \text{ then } F_j(x) = \operatorname{co}(F(x)) \cap \mathcal{T}_{\Upsilon_j}(x) \text{ for each } \Upsilon_j. \end{cases}$$

Remark 9.4.1. *Notice that in the case of one-dimensional networks, (H_3^9) is immediately satisfied because the images of F_j are either empty or $\{0\}$.*

The foregoing hypothesis implies that either trajectories can remain for arbitrary long periods of times at the junction Υ_j or they can only pass through it. Hence, we might also distinguish between the junction where trajectories can slide for and where they can not. To do so, we introduce the following notation

$$\mathcal{J}_0 = \{j \in \mathcal{J} \mid \ \operatorname{dom}(F_j) \neq \emptyset\}.$$

We are now in position to provide a proposition that asserts the viability of a generalized network as well as the lower semicontinuity of the Value Function. The next statement is analogous to Proposition 9.2.1 and its prove is similar, for this reason we only detail the points that differ from one case to the other.

Proposition 9.4.1. *Suppose that ($H_{0,G}^9$), (H_ψ^9) and (H_F^9) hold along with ($H_{1,G}^9$), (H_2^9) and (H_3^9). Then, for every $(t,x) \in [0,T) \times \mathcal{K}$ there exists an optimal trajectory $\bar{y} \in \mathbb{S}_t^T(x)$ for the Mayer problem. Furthermore, $\vartheta : [0,T] \times \mathcal{K} \to \mathbb{R}$ is lower semicontinuous.*

Proof. Since each $\overline{\mathcal{M}}_i$ is relatively wedged and ($H_{1,G}^9$) holds, whenever $j \notin \mathcal{J} \setminus \mathcal{J}_0$ we have

$$F_i(x) \subseteq \operatorname{ri}(\mathcal{T}^C_{\overline{\mathcal{M}}_i}(x)), \quad \forall x \in \overline{\mathcal{M}}_i. \tag{9.15}$$

By the same arguments used in Proposition 9.2.1 we can easily construct a stratified vector field on the stratification $\{\Upsilon_j\} \cup \{\mathcal{M}_i\}_{i \in \mathcal{I}}$ with sliding manifolds determined by the index $\mathcal{J}_0 \cup \mathcal{I}$; this is because of ($H_3^9$) and the Michael's Selection Theorem (Proposition 2.2.4). Furthermore, by (9.15), this stratified vector field can be taken as to verify the assumptions of Theorem 7.3.3. So, the viability of the network follows from that result.

On the other hand, (9.15) also implies that no trajectory of the control system can remain at a junction whose index does not belong to $\mathcal{J}_0$. In any other case, the trajectories of the convexified dynamics agree with the ones of the original dynamics. This is thanks to point (ii) in (H_3^9). Therefore, noticing this and using the arguments of Proposition 9.2.1, we easily get that optimal trajectories do exists and that the Value Function is lower semicontinuous. □

9.4.2 Characterization of the Value Function

From now on we focus our attention on proving the characterization of the Value Function of the Mayer problem on a generalized d-dimensional network. The difference between this case and with the one-dimensional one lies in the junction conditions, which are now inequalities that need to be verified all along the corresponding junction. However, the proof of the next statement is as well a modification of the one provided for Theorem 9.3.1.

To prove the next theorem, a controllability assumption is required on the junctions.

(H_4^9) $\quad \forall j \in \mathcal{J}$ with $\operatorname{dom}\Upsilon_j \neq \emptyset$, $\exists \rho_j > 0$ so that $\mathbb{B}(x,\rho_j) \cap \mathcal{T}_{\Upsilon_j}(x) \subseteq F_j(x), \forall x \in \Upsilon_j$.

Remark 9.4.2. *The foregoing controllability assumption is not the sharpest possible, however it is suitable for the framework we are considering. It has already been considered in the work addressed to multi domains; see for instance [15, 16, 105].*

Therefore, the main result of the section can be stated as follows.

Theorem 9.4.1. *Suppose* $(H_{0,G}^9)$, $(H_{1,G}^9)$, (H_2^9), (H_3^9) *and* (H_4^9) *hold along with* (H_ψ^9) *and* (H_F^9). *Then the Value Function of the Mayer problem on the generalized network* $\mathcal{K}$ *is the unique lower semicontinuous function whose domain is contained in* $[0,T] \times \mathcal{K}$ *verifying:*

- **The HJB equation:**

$$-\theta + \max_{v \in F_i(x)} \langle -v, \zeta \rangle = 0, \quad \forall i \in \mathcal{I},\ \forall (t,x) \in (0,T) \times \mathcal{M}_i,\ \forall (\theta,\zeta) \in \partial_V \vartheta(t,x). \tag{9.16}$$

- **The final time conditions:**

$$-\theta + \max_{v \in F_i(x)} \langle -v, \zeta \rangle \leq 0, \quad \forall i \in \mathcal{I},\ \forall x \in \mathcal{M}_i,\ \forall (\theta,\zeta) \in \partial_V \vartheta(T,x), \tag{9.17}$$

$$\vartheta(T,x) = \psi(x), \qquad x \in \mathcal{K}. \tag{9.18}$$

- **The initial time condition:**

$$-\theta + \max_{v \in F_i(x)} \langle -v, \zeta \rangle \geq 0, \quad \forall i \in \mathcal{I},\ \forall x \in \mathcal{M}_i,\ \forall (\theta,\zeta) \in \partial_V \vartheta(0,x). \tag{9.19}$$

- **The junction conditions for** $j \in \mathcal{J}_0 = \{j \in \mathcal{J} \mid \textbf{dom}\, F_j \neq \emptyset\}$**:**

$$-\theta + \max_{v \in F^\sharp(x)} \langle -v, \zeta \rangle \geq 0, \quad \forall j \in \mathcal{J}_0,\ (t,x) \in [0,T) \times \Upsilon_j,\ \forall (\theta,\zeta) \in \partial_V \vartheta(t,o_j), \tag{9.20}$$

$$-\theta + \max_{v \in F_j(x)} \langle -v, \zeta \rangle \leq 0, \quad \forall j \in \mathcal{J}_0,\ \forall (t,x) \in (0,T] \times \Upsilon_j,\ \forall \theta \in \partial_V \vartheta_j(t,x), \tag{9.21}$$

where $t \mapsto \vartheta_j(t,x) := \vartheta(t,x)$ *if* $x \in \Upsilon_j$ *and* $+\infty$ *otherwise.*

- **The junction conditions for** $j \in \mathcal{J} \setminus \mathcal{J}_0$**, that is** $0 \notin F(o_j)$**:**

$$\begin{cases} \forall j \in \mathcal{J} \setminus \mathcal{J}_0,\ \exists i \in \mathcal{I}(j) \text{ so that } F_i(x) \subseteq \mathcal{T}_{\mathcal{M}_i}^C(x) \text{ for any } x \in \Upsilon_j \text{ and} \\ -\theta + \max_{v \in F_i(x)} \langle -v, \zeta \rangle \geq 0, \quad \forall (t,x) \in [0,T) \times \Upsilon_j,\ \forall (\theta,\zeta) \in \partial_V \vartheta(t,x). \end{cases} \tag{9.22}$$

Proof. As in the one-dimensional case, the subsolution principle is a rather direct consequence of the theory developed in Chapter 4. Indeed, recall that the controllability assumption (H_4^9) implies in particular the controllability assumption (H_3^4) . Therefore, it is not difficult to see that the arguments of Chapter 4 can be adapted to this new setting. In particular, we have that if a lower semicontinuous function φ verifies (9.16), (9.17) and (9.21) in addition to (9.18), then $\vartheta(t,x) \geq \varphi(t,x)$ on $[0,T] \times \mathcal{K}$.

Consequently, we only need to check that $\vartheta(t,x) \leq \varphi(t,x)$ on $[0,T] \times \mathcal{K}$.

Notice that, as for Theorem 9.3.1, thanks to the equations (9.16) and (9.19) together with standard arguments of weakly invariance of dynamical systems, we can restrict our attention to the situation in which x belongs to a generalized junction.

In the case where $x \in \Upsilon_j$ with $j \in \mathcal{J}_0$, by (H_F^9), the trajectories of the convexified dynamical system match with the curves of the initial dynamics. In any other case, we have already discussed that no trajectory can remain at a junction related to an index $j \notin \mathcal{J}_0$.

Therefore, reproducing the arguments of Theorem 9.3.1 and using the preceding remarks, we easily reach the conclusion of the theorem.

□

9.5 Discussion and perpectives

The result we have exposed in this chapter allows to treat optimal control problem on network in a rather general way. In particular, the analysis is carried on without required any type of controllability assumption around the junction. This fact is the most important contribution of the analysis recently exposed.

Let us point out that problems on networks have been widely investigated in others fields of applied mathematics. We particularly mention the study associated with traffic flow on networks and conservations laws, which can be considered as the main motivation to query about optimal control problems on networks. We refer for more details on traffic flow problems to the book of Garavello-Piccoli [57].

9.5.1 Junction conditions

We stress that, since no controllability assumption is required, the junction conditions depend on whether trajectories can remain at the junction or not. Apparently, this has not been observed before in the literature, we refer in particular to condition (9.10). The reason is that, as aforesaid, the usual controllability assumption yield to consider exclusively the case in which $J_0 = \mathcal{J}$, or in other words,

$$0 \in F(o_j), \quad \forall j \in \mathcal{J}.$$

If the preceding condition is met, it is rather simple to verify that the curves of the convexified dynamical systems (usually referred to as *relaxed trajectories*) agree with the arc of the original control system; this is basically due to the network structure. However, if $0 \notin F(o_j)$, the question of whether the relaxed trajectories are arc of the dynamical system relies upon the geometry of the network, and so, for two equivalent networks, we can give different answers to this question. For example, consider the cases exposed in Figure 9.6. In the case described in Figure 9.6a (the same as in Figure 9.3), we can see that $0 \in \text{co}(F(o))$ and so, the arc $y(s) = o$ is a relaxed trajectory notwithstanding that it is not a curve of the dynamical system. Nevertheless, the network portrayed in Figure 9.6b has the same type of behavior at the junction as that of Figure 9.6a, so topologically speaking are the same, but $0 \notin \text{co}(F(o))$ and furthermore, the relaxed trajectories are arcs of the control system.

(a) (b)

Figure 9.6: Two equivalent networks.

The previous reasoning implies in particular that the condition $0 \in \text{co}(F(o))$ is not a topological invariant for networks. For this reason we have chosen to distinguish between the junction where it is possible to remain and where it is not.

We also emphasis that it is important that the condition (9.9) is verified only for $j \in \mathcal{J}_0$. To clarify this, let us consider the following example. Suppose that $\mathcal{K} = \mathbb{R}$ and $o = 0$ is the unique junction of the network; see Figure 9.7. Consider $T = 1$, the final cost $\varphi_i(x) = \frac{1}{2}x^2$ for all $x \in \mathbb{R}$ and the dynamics

$$F_1(x) = -1, \quad \forall x < 0 \quad \text{and} \quad F_2(x) = 1, \quad \forall x > 0$$

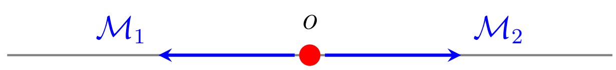


Figure 9.7: An example of network in $\mathbb{R}$.

It is not difficult to see that the Value Function is given by

$$\vartheta(t,x) = \begin{cases} \frac{1}{2}(x - t + 1)^2 & x \geq 0, \\ \frac{1}{2}(x + t - 1)^2 & x < 0, \end{cases} \quad \forall (t,x) \in [0,1] \times \mathbb{R}.$$

Therefore, if ϑ_o stands for $t \mapsto \vartheta(t,0)$, we have that $\theta \in \partial_V \vartheta_o(t)$ if and only if $\theta = t - 1$. If we take $t \in (0,1)$ we readily see that ϑ does not verify (9.9). This is because $F(o) = \{1, -1\}$ and so, it is not possible to stay at the junction.

9.5.2 Necessary conditions

We close this chapter addressing a few words on some interesting issues that can still be studied on the setting of optimal control on networks.

Along the exposition we have studied the HJB approach for this type of problem, but we have not said anything about *necessary condition of optimality*. This is a challenging question that certainly deserves some attention.

From the optimization point of view, an optimal process on a network is a constrained problem. However, the global dynamical system is not locally Lipschitz continuous, which might precludes to apply directly one of the available theorems found in the literature; see for instance [131, 40]. Furthermore, if we would have posed the problem under a locally Lipschitz setting (as in Section 4.2.4) the necessary conditions are usually written in a general way, not really adapted to the structure of the state-constraints.

As we have seen earlier, the network structure has some particularities that allow to extend classical results, such as the HJB approach, to these discontinuous dynamical systems. Hence, it is natural to wonder if this can also be done for the Pontryagin Maximum Principles, that is, write the necessary condition of optimal control problem especially adapted to a network. The main interest in doing so is the possibility of incorporating the measure that appears in the adjoint vector in the structure of the conditions.

We finally mention that, in the case that an optimal trajectory does not chatter around a junction, then the necessary conditions can be written in the light of the hybrid maximum principle; see for instance Sussmann [127] or Garavello-Piccoli [56]. The real challenge is the case in which chattering at junctions can occur.

9.5.3 Generalized networks

We have shown that the arguments used to treat one-dimensional networks can be extended in a rather simple way to the case in which the junction is a manifold, and not a isolated point. The main contribution of this study is to point out the underlying difficulties that appears when considering sets that are the intersection of hyper surfaces.

The central hypothesis required in the analysis is that the convexified dynamics should agree with the dynamics at the junctions where trajectories can remain therein, that is,

$$F(x) \cap \mathcal{T}_{\Upsilon_j}(x) = \mathrm{co}(F(x)) \cap \mathcal{T}_{\Upsilon_j}(x), \quad \forall x \in \Upsilon_j, \ \forall j \in \mathcal{J}_0.$$

If this hypothesis is not satisfied, then the Value Function might not be lower semicontinuous nor optimal trajectories may exist. This is due to the fact that the standard compactness arguments might fail. In the one-dimensional case, this assumption is automatically verified; this is because the tangential dynamics are either empty-valued or $\{0\}$. In the generalized instance, since we are allowing to trajectories to move along the junctions, it is important to impose this compatibility assumption, in order to guarantee that the dynamics of the neighboring branches do not create arcs whose velocities are not part of the control system.

We finally stress that in the analysis, the Value Function is not necessarily continuous because discontinuities can happen at the generalized junctions. In order to have the continuity, we might have to impose a controllability condition all around the generalized junction that allows to construct trajectories linking any two points around a junction. Anyhow, it seems that, continuity when restricted to each branch and each junction is a generic property of Value Function on (generalized) networks, but this is still to be proven.

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Index

Title: Optimal Control Problems on Well-structured Domains and Stratified Feedback Controls.

Abstract: The aim of this dissertation is to study some issues in Control Theory of ordinary differential equations. Optimal control problems with tame state-constraints and feedback controls with stratified discontinuities are of special interest. The techniques employed along the manuscript have been chiefly taken from control theory, nonsmooth analysis, variational analysis, tame geometry, convex analysis and differential inclusions theory.

The first part of the thesis is devoted to provide general results and definitions required for a good understanding of the entire manuscript. In particular, a strong invariance criterion adapted to manifolds is presented. Moreover, a short insight into manifolds and stratifications is done. The notions of relatively wedged sets is introduced and in addition, some of its properties are stated.

The second part is concerned with the characterization of the Value Function of an optimal control problem with state-constraints. Three cases have been taken into account. The first one treats stratifiable state-constraints, that is, sets that can be decomposed into manifolds of different dimensions. The second case is focused on linear systems with convex state-constraints, and the last one considers convex state-constraints as well, but from a penalization point of view. In the latter situation, the dynamics are nonlinear and verify an absorbing property at the boundary.

The third part is about discontinuous feedbacks laws whose singularities form a stratified set on the state-space. This type of controls yields to consider stratified discontinuous ordinary differential equations, which motivates an analysis of existence of solutions and robustness with respect to external perturbation for these equations. The construction of a suboptimal continuous feedback from an optimal one is also addressed in this part.

The fourth part is dedicated to investigate optimal control problems on networks. The main feature of this contribution is that no controllability assumption around the junctions is imposed. The results can also be extended to generalized notions of networks, where the junction is not a single point but a manifold.

Keywords: Optimal Control Problems with state-constraints, Hamilton-Jacobi-Bellman Equations and invariance theory, Stratified feedback laws, Tame state-constraints, Optimal Control Problem on networks, Optimality principles in control theory.

Titre: Problèmes de commande optimale sur des domaines structurés et lois de commandes en boucles fermées stratifiées.

Résumé: Cette thèse porte sur la théorie de la commande optimale. Les problèmes de contrôle optimale sous contraintes d'état bien structurées et les lois de feedback stratifiées sont considérés. Les techniques utilisées dans ce manuscrit concernent principalement la théorie de la commande, l'analyse non lisse, l'analyse variationnelle, la géométrie modérée, l'analyse convexe et les inclusions différentielles.

La première partie de la thèse est consacrée à donner des résultats et définitions généraux mais nécessaires pour mieux comprendre les parties suivantes de la thèse. En particulier, un critère d'invariance forte est présenté. De plus, un bref aperçu sur les variétés lisses et les ensembles stratifiés est exposé. La notion d'ensemble relativement wedged est introduite et de plus, quelques de ses propriétés sont aussi analysées.

La deuxième partie est concernée à caractériser la Fonction Valeur d'un problème de contrôle optimal sous contraintes d'état. Trois situations ont été considérées. Le premier cas traite les contraintes d'état qui sont également des ensembles stratifiés, c'est-à-dire ceux qui peuvent être décomposé en une collection de variétés de différents dimensions. La deuxième situation se concentre sur les systèmes linéaires sous contraintes d'état convexes. Le dernier cas considère aussi les contraintes d'état qui sont ensembles convexes mais avec une technique de pénalisation. Dans cette dernière situation, les dynamiques sont non linéaires et absorbants sur la frontière de l'ensemble de contraintes.

La troisième partie se focalise sur les lois de feedback discontinues dont les ensembles de points singuliers ont une structure stratifiée par rapport à l'espace d'état. Ces contrôles produisent des équations différentielles ordinaires stratifiées, ce qui motive une étude sur l'existence des solutions et sur la robustesse par rapport aux perturbations externes de ses équations. La construction de lois de feedback continues mais sous-optimaux à partir de l'information fourni par les contrôles optimaux est aussi traitée dans cette partie.

La quatrième partie est dédiée à l'étude des problèmes de contrôle optimale sur des réseaux. La principale contribution de cette étude est qu'il n'y a pas de hypothèse de contrôlabilité autour des jonctions. Les résultats sont étendus aux réseaux généralisés dont les jonctions ne sont plus de points isolés mais de variétés.

Mots-clés: Problèmes de contrôle optimal sous contraintes d'état, Équations de Hamilton-Jacobi-Bellman et théorie de l'invariance, Contrôles en boucle fermée stratifiés, Contraintes d'état avec une structure modérée, Problèmes de Contrôle Optimal sur des réseaux, Principes d'optimalité en commande optimal.